relatively prime. Thus, we conclude

$$\frac{Y + 5X^2 + 1}{2} = u^4, \quad \frac{Y - 5X^2 - 1}{2} = 5v^4,$$

which yields

$$u^4 - 5v^4 = 5X^2 + 1, \; uv = X, \text{ i.e., } u^4 - 5u^2v^2 - 5v^4 = 1.$$

This equation may be written $(2u^2 - 5v^2)^2 - 45v^4 = 4$. By Corollary 2, this equation holds only for $v = 0$ and 1, but only $v = 0$ yields a solution, namely $u = 1$. Thus, the only solution of (4) is $x = 3$, $y = 0$. So if n is even and $F_n = x^2 + 1$, then $n = 2$.

In conclusion, we note that Williams [4] has shown that the complete solution of (1) implies that the only integer solutions of the equation $(x - y)^7 = x^5 - y^5$ with $x > y$ are (1, 0) and (0, -1).

REFERENCES

1. J. H. E. Cohn. "Square Fibonacci Numbers, Etc." *The Fibonacci Quarterly* 2 (1964):109-113.
2. J. H. E. Cohn. "Lucas and Fibonacci Numbers and Some Diophantine Equations." *Proc. Glasgow Math. Soc.* 7 (1965):24-28.
3. R. Finkelstein. "On Fibonacci Numbers which Are One More Than a Square." *J. für die reine und angew. Math.* 262/263 (1973):171-178.
4. H. C. Williams. "On Fibonacci Numbers of the Form $k^2 + 1$. *The Fibonacci Quarterly* 13 (1975):213-214.

*****

FIBONACCI SEQUENCE CAN SERVE PHYSICIANS AND BIOLOGISTS

ROBERT J. KINNEY

Albuquerque, New Mexico

PART I: SOME OF THEIR SPECIAL NEEDS

All of earth's living beings can react suitably to a range of circumstances. They react to certain stimuli. How much they react is related to how much is the stimulus. Biologists have found that over a wide range of intensity, a proportionate change in stimulus calls forth the same change in response.

Two examples of this relationship are:

1. the nervous system of an animal recognizes as increase in stimulus the same proportional change across most of the range of stimulus;
2. the immune system of an animal responds to the same proportional change in challenge across most of a very wide range.

From (1) derives the Weber-Fechner law of sensory perception. It is a generalization from a wealth of data. Thus, a certain person feels as heavier as little as 11 ounces compared to 10 ounces, and 11 grams compared to 10 grams, and 11 pounds compared to 10 pounds, as well as 110 pounds compared to 100 pounds. Across a range of five-thousandfold, that person distinguishes the same proportionate difference of 10 percent. Note that the basic distinction is not 1 gram nor 1 pound nor 10 pounds, but remains 10 percent.

From (1) likewise derives that some person can hear one musical note as sharper or flatter than another when it is as little as 0.5 percent sharper or flatter, whether the note is tested at the basso's CC, or the coloratura's ccc, which is five octaves higher with soundwaves vibrating 32 times faster than CC. Indeed, the person's range of perception of musical notes may extend beyond two-hundredfold, much more than 32.

With example (2), the immune system, the range often stretches beyond a millionfold. Throughout that range, the amount of offending protein (antigen) that elicits a given rise in the body's immune substance (antibody) stays proportional to the amount of antibody already present.

So in the workaday world of (1) a biologist measuring changes of taste-sensitivity, or of (2) a physician treating a patient's allergic disorder by regular periodic injections of a solution of an offending protein allergen, either scientist should seek to lay out a schedule of ever-increasing strength of solutions, the increase usually being at a constant proportional rate.

Either scientist, therefore, must use a very long geometric series of increments of strength of solution. They measure out the amounts of solution and of diluent with standard laboratory glassware that is calibrated at arithmetic intervals.

A mathematician can solve the problem that has arisen, which is this. How should they measure out the usual long geometric series using only an arithmetic scale on their glassware tools? They cannot measure out, for instance, ten steps of 5 percent increase with their tools, for it would call for measuring out a series,

$$1,\ 1.05,\ (1.05)^2 \text{ which is } 1.1025,\ \ldots,\ (1.05)^{10} \text{ which is longer than } 1.628{,}895,\ \ldots\ .$$

Yet the needs of their experiment or of their treatment of a patient may make them aim to use about ten steps of about 5 percent increase.

Here we may stop to ponder an example of serious error that was made in tools and schedules used to treat patients for allergic disorders. For many years, a fine old drug firm made high-quality extracts of protein allergens that physicians need and often use. The extracts were supplied sterile in a multiple-dose syringe, but the results of the treatment seemed poor despite the finest of syringes and extracts. The reason was finally found to be the build-in Procrustean schedule of dosages that was based on erroneous mathematics, so that the unhappy patient was sure never to get the right dose as needed. The syringes came calibrated in ten equal divisions. Schedule of dosage was naively planned as 1 such unit first, next 2 units, then 3 units, lastly 4 units from the first syringe; from the second syringe that contained a tenfold concentrate of the solution in the first, the same arithmetic scale of dosages was calibrated for. Each successive syringe contained a tenfold concentrate of the solution in the last former one used.

Increments of dosage thus ran 100 percent, 50 percent, $33\frac{1}{3}$ percent, 250 percent, then again 100 percent, 50 percent, and so on. Increments grew on an arithmetic scale during use of each syringe. They only grew on a geometric scale from one syringe to the next, the step always being tenfold. Mathematicians will at once note that to increase doses tenfold over four even steps of increment, each increment should remain $\sqrt[4]{10} - 1$, which = 77.95... percent.

Besides its nonproportionate increments, the system had another major built-in fault. It was inflexible. It made John Doe's dosages all the same as Jane Doe's and as Richard Roe's. Of course, in real life, each would do best on his or her own schedule; and at special times that schedule should vary, as when John Doe is having a chest cold, or after Richard Roe moves away from an area low in ragweed pollen into an area medium-high in both ragweed and tumbleweed pollen. The faults of the system could have been corrected by proper calibrations of the syringes along with correct mathematical design and directions. Instead, the drug firm's accountants' balances led it to abandon the business, which otherwise had provided unexcelled quality of syringe materials and of allergen extracts.

During the past three decades, biologists and physicians among other scientists have grown to accept that they routinely need to seek the skills and insights of statisticians, both early when designing their work and later when drawing conclusions from their efforts. We write this and some follow-up articles to show that on these and other occasions, certain biologists and physicians need an expert in mathematics to plan and to adapt a schedule of dosages of allergen for a patient, and to plan mathematical details for measuring sensory thresholds of taste.

In this first article, let us peek ahead at coming attractions. Let us watch a medical case being treated.

The patient, a woman of 24, had married an American soldier while he was serving in her homeland in Europe, and had immigrated to the U.S. A. when he finished his military service. They moved to his home state in the upper mid-West. She took ill during her first summer in the United States with sneezing, itching eyes, stuffed and drizzling nostrils, loss of sleep, failing appetite, loss of weight, and so on. The physician's tests of allergens in skin and eyes showed marked hypersensitivity to ragweed pollens that were infesting the air throughout the state and region. He planned treatment that included visiting a cousin who lived in ragweed-free Arizona. After heavy frosts of late autumn cleared the ragweed pollens from the ambient air, she returned as planned to her home already well scrubbed and with contents well laundered. She remained free of distressing symptoms. To prepare her resistance and to lower her hypersensitivity toward the ragweed pollens of the next year's late summer and early autumn, the physician treated her by a long series of injections, one every seven days, of extract containing the proteins of ragweed pollens.

Initial dosage contains only one millionth of the estimated final effective dosage. Dosages grow each step at a rate of about 62 percent. (We shall see later that this is larger than most patients usually need as increment.) The constant increment is pared down near the end of the months of treatment so that the last eight injections increase only by the amount that four of the earlier doses did. All doses were measured out in a standard syringe of the "tuberculin" type, which contains one milliliter (formerly styled "cubic centimeter") and is calibrated at hundredths, 0.01, 0.02, ..., 0.99, 1.00 ml. The first five injections were administered one every seven days beginning later November through December. The first five amounts measured were 0.08, 0.13, 0.21, 0.34, and 0.55. The greatest and least increments between any of these doses were 62.5 percent and 61.5385... percent. Q.E.D.

For another patient, a sequence of dosages growing stronger at a rate of approximately 27 percent might be appropriate. Such a sequence could be given using two solutions, A and B, with B approximately 27 percent stronger than A and alternating the dosages between the solutions as follows:

0.08A, 0.08B, 0.13A, 0.13B, 0.21A, 0.21B, ...

We shall follow up the progress of these patients and shall review other patients' allergic problems in a later paper.

So far, we have considered the mathematical problems met in sensory biology and in allergy-immunology but not often solved by nonmathematicians. We have touched upon the special limitations imposed by our tools that measure out amounts of liquids, and have groped toward adapting these tools for best results. We shall aim to get results that can be safe, simple, and on the mark. Our quest will lead us through continued fractions, and sometimes through Fibonacci-ratio fractional approximations.

*****

VALUES OF CIRCULANTS WITH INTEGER ENTRIES

H. TURNER LAQUER

Student, University of New Mexico, Albuquerque, NM 87131

It is well known that the differences of squares $m = x^2 - y^2$, with x and y integers, are the integers satisfying $2 \nmid m$ or $4 \mid m$. It is not difficult to show that the integers m of the form $x^3 + y^3 + z^3 - 3xyz$, with x, y, and z integers, are those integers satisfying $3 \nmid m$ or $9 \mid m$. This paper generalizes on those results.

Let $C_n(x_1, \ldots, x_n)$ be the determinant of the circulant matrix (a_{ij}) in which $a_{ij} = x_k$ when $j - i + 1 \equiv k \pmod{n}$. Note that $C_2(x, y) = x^2 - y^2$ and $C_3(x, y, z) = x^3 + y^3 + z^3 - 3xyz$.

Let V_n be the set of values of C_n when the domain is the set of all ordered n-tuples $(x_1, \ldots, x_n)$ with integer entries x_k. We will show below that, for odd primes p, V_p consists of the integers m with either $p \nmid m$ or $p^2 \mid m$, and that V_{2p} consists of the integers m satisfying either $p \nmid m$ or $p^2 \mid m$ and also satisfying either $2 \nmid m$ or $4 \mid m$, i.e.,

$$V_{2p} = [\{m : p \nmid m\} \cup \{m : p^2 \mid m\}] \cap [\{m : 2 \nmid m\} \cup \{m : 4 \mid m\}].$$

1. GENERAL N

In this section, the x_k may be any complex numbers. It is well known (see [1]) that

$$C_n(x_1, \ldots, x_n) = \prod_{h=0}^{n-1}\left(\sum_{k=1}^{n} x_k \exp[2\pi h(k-1)i/n]\right). \tag{1.1}$$

We use this to establish the following.

<u>*Theorem 1*</u>: $C_n(x_1 + a, x_2 + a, \ldots, x_n + a) = R \cdot C_n(x_1, x_2, \ldots, x_n)$ where $R = (na + x_1 + x_2 + \cdots + x_n)/(x_1 + x_2 + \cdots + x_n)$.

<u>*Proof*</u>:

$$\begin{aligned} C_n(x_1 + a, \ldots, x_n + a) &= \prod_{h=0}^{n-1}\left(\sum_{k=1}^{n} (x_k + a) \exp[2\pi h(k-1)i/n]\right) \\ &= \prod_{h=0}^{n-1}\left(\sum_{k=1}^{n} x_k \exp[2\pi h(k-1)i/n] + a\sum_{k=1}^{n}\exp[2\pi h(k-1)i/n]\right). \end{aligned}$$

Now

$$\sum_{k=1}^{n}\exp[2\pi h(k-1)i/n] = \begin{cases} n & \text{for } h = 0 \\ \dfrac{1 - \exp(2\pi hni/n)}{1 - \exp(2\pi hi/n)} = 0 & \text{for } 1 \le h \le n - 1. \end{cases}$$