Identity 7: $9\sum_{0}^{2n-1}(-1)^k H_{k+1}H_{k+2}H_{k+4}H_{k+5} = H_{2n+5}^4 - 5H_{2n+4}^4 - 14H_{2n+3}^4 + H_{2n+2}^4 + 3e^2 + D,$

where $D = q(4p^3 + 6p^2q + 4pq^2 + q^3)$.

The proof of Identities 1-7 follow along the same lines as in [1], hence the details are omitted here.

Some more identities that are easily verifiable by induction follow:

(a) $2\sum_{0}^{n}(-1)^r H_{m+3r} = (-1)^n H_{m+3n+1} + H_{m-2}$ $\quad m = 2, 3, \ldots;$

(b) $3\sum_{0}^{n}(-1)^r H_{m+4r} = (-1)^n H_{m+4n+2} + H_{m-2}$ $\quad m = 2, 3, \ldots;$

(c) $11\sum_{0}^{n}(-1)^r H_{m+5r} = (-1)^n(5H_{m+5n+1} + 2H_{m+5n}) + 4H_m - 5H_{m-1}$ $\quad m = 1, 2, \ldots;$

(d) $4\sum_{0}^{n} H_k H_{2k+1} + 2H_0^2 = H_{2n+3}H_n + H_{2n}H_{n+3};$

(e) $3\sum_{0}^{n}(-1)^r H_{m+2r}^2 = (-1)^n H_{m+2n}H_{m+2n+2} + H_m H_{m-2}$ $\quad m = 2, 3, \ldots;$

(f) $7\sum_{0}^{n}(-1)^r H_{m+4r}^2 = (-1)^n H_{m+4n}H_{m+4n+4} + H_m H_{m-4}$ $\quad m = 4, 5, \ldots;$

(g) $2\sum_{1}^{n} H_{k+2}H_{k+1}^2 = H_{n+3}H_{n+2}H_{n+1} - H_0H_1H_2;$

(h) $2\sum_{1}^{n}(-1)^r H_r H_{r+1}^2 = (-1)^n H_n H_{n+1} H_{n+2} - H_0H_1H_2;$

(i) $2\sum_{1}^{n}(-1)^r H_{r+1}^3 = (-1)\ (H_{n+1}^2 H_{n+4} - H_n H_{n+2} H_{n+3}) - E,$

where $E = p^3 - 3pq^2 - q^3$.

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DIVISIBILITY PROPERTIES OF A GENERALIZED FIBONACCI SEQUENCE

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This note gives some divisitility properties of the generalized Fibonacci numbers viz $H_0 = q$, $H_1 = p$, $H_{n+1} = bH_n + cH_{n-1}$ $(n \geq 1)$, denoted henceforth by (b, c, p, q) GF sequence. The results have similarity to those of Dov Jarden [1].

For the Horadam generalized Fibonacci sequence: $H_0 = q$, $H_1 = p$, $H_{n+1} = H_n + H_{n-1}$ $(n \geq 1)$, we have

Theorem 1: $H_{n+k} + (-1)\ H_{n-k}$ is divisible by H for all $n \geq k$.

Proof: The proof easily follows from the identity

(1) $$H_{n+k} + (-1)^k H_{n-k} = L_k H_n.$$

Corollary a: $H_{n+k}^2 + (-1)^{2k+1}H_{n-k}^2$ is divisible by H_n; and

Corollary b: $H_{n+k}^3 + (-1)^{3k+2}H_{n-k}^3$ is divisible by H_n.

Divisibility properties of (b, c, p, q) GF sequence.

Theorem 2: If $(m,n) = 1$ and $q = 0$, $H_m H_n / H_{mn}$.

Proof: $H_n = (gr^n - hs^n)/(r - s)$ and $H_{mn} = (gr^{mn} - hs^{mn})/(r - s)$, where r and s are the roots of $x^2 - bx - c = 0$ and $g = p - sq$ and $h = p - rq$.

It is easily seen that H_m or H_n divides H_{mn} if $g = h$. Since $r = s$ leads to the degenerate case, we must have $q = 0$. Also, it is necessary that $(m,n) = 1$.

Theorem 3: If $p^2 - bpq - cq^2 = 0$, then $H_m H_n / H_{mn}$.

Proof: By the identity

$$H_n^2 - H_{n+1}H_{n-1} = (-c)^{n-1}e, \tag{2}$$

where $e = p^2 - bpq - cq^2$, the desired result follows.

Theorem 4: For $p = cq(1 - b)/(b^2 + c + 1 - b)$, if $c^2 = (-1 - b)(1 + 2c)$, then $H_m H_n / H_{mn}$.

It is known from [2] that $H_n = pU_n + cqU_{n-1}$, where the nth member of the U sequence is defined by $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = bU_{n+1} + cU_n$ $(n > 0)$.

On suitably combining this relation with

$$2(pU_n + cqU_{n-1}) = (pU_{n+1} + cqU_n) + (pU_{n-1} + cqU_{n-2}), \tag{3}$$

it is easy to see that (b, c, p, q) GF sequence results in an A.P. Therefore, if $H_m H_n$ were to divide H_{mn}, we would get

$$c^2 = (1 - b)(1 + 2c).$$

Further equating the initial term of the A.P. with the common difference, we get either $c = 0$ or $p(b^2 + c + 1 - b) = cq(1 - n)$.

The case $c = 0$ is already discussed in Theorem 3; hence, the other condition gives the desired result of divisibility.

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PYTHAGOREAN PENTIDS

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1. INTRODUCTION

Let $T_n = n(n + 1)/2$ denote the nth triangular number. Then we have

$$\begin{aligned}(T_{2r})^2 + (T_{2r} + 1)^2 + (T_{2r} + 2)^2 + \cdots + (T_{2r} + r) \\ = (T_{2r} + r + 1)^2 + (T_{2r} + r + 2)^2 + \cdots + (T_{2r} + 2r)^2\end{aligned} \tag{1.1}$$

and

$$\begin{aligned}(T_{2r} + 9k)^2 + (T_{2r} + 1 + 12k)^2 + \cdots + (T_{2r} + r + 12k)^2 \\ = (T_{2r} + r + 1 + 12k)^2 + (T_{2r} + r + 2 + 12k)^2 + \cdots + (T_{2r} + 2r + 15k)^2, \\ r = 1, 2, 3, \ldots; k = 1, 2, 3, \ldots .\end{aligned} \tag{1.2}$$

This gives a generalized identity of squares of numbers with $r + 1$ terms on the left-hand side and r terms on the right-hand side. But the triangular numbers are a particular case of the generalized Tribonacci sequence having a recurrence relation

$$X_{n+3} = 3X_{n+2} - 3X_{n+1} + X_n,\ n \geq 0, \text{ with } X_0 = 0,\ X_1 = 1, \text{ and } X_2 = 3. \tag{1.3}$$

Therefore, the properties of the generalized Tribonacci sequence are also properties of the triangular numbers.

The case $r = 1$ in equation (1.1) gives the well-known Pythagorean triad (3, 4, 5). For $r = 2$, we have the Pythagorean pentid (10, 11, 12, 13, 14). Pythagorean triads have been studied by various authors, particularly by Teigen and Hadwin [6] and by Shannon and Horadam [5]. The object of this note is to extend the results of the above-mentioned authors to the Pythagorean pentids. Similar extensions are also possible for the general Pythagorean n-tids of (1.1).