TRIANGULAR DISPLAYS OF INTEGERS

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The purpose of this article is to exhibit some properties of certain binomial coefficients that are generated in Theorem 1 below. We display the integers in a triangular form and show that their occurrence within that structure follows a regular pattern.

We make use of the kth difference $\Delta_h^k f(x)$ of a function, this difference being defined by

$$\Delta_h^k f(x) = f(x + kh) - \binom{k}{1} f[x + (k - 1)h] + \cdots + (-1)^k f(x),$$

where h is a positive real number.

Although the following theorem is a special case of [1, Theorem 2], we present an independent proof that is more appropriate to the present context.

Theorem 1: Let $x_0, x_1, \ldots, x_k$ and $y_0, y_1, \ldots, y_{mk}$ be two sets of real numbers such that $x_0 < x_1 < \ldots < x_k$, $y_0 < y_1 < \ldots < y_{mk}$, $x_s = y_{ms}$, $s = 0, 1, \ldots, k$, and $y_i - y_{i-1} = h$, $i = 1, 2, \ldots, mk$. Then

$$\Delta_{mh}^k f(x_0) = \sum_{i=0}^{(m-1)k} \alpha_i \Delta_h^k f(y_i), \tag{1}$$

where the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{(m-1)k}$ are positive, symmetrical [that is, $\alpha_i = \alpha_{(m-1)k-i}$, $i = 0, 1, \ldots (m - 1)k$], and have sum equal to m^k. More specifically,

$$\alpha_i = \begin{cases} \binom{i+k-1}{k-1}, & 0 \le i < m \\ \binom{i+k-1}{k-1} - \binom{k}{1}\binom{i-m+k-1}{k-1}, & m \le i < 2m \\ \vdots & \\ \binom{i+k-1}{k-1} - \binom{k}{1}\binom{i-m+k-1}{k-1} + \cdots + (-1)^q \binom{k}{q}\binom{i-mq+k-1}{k-1}, & qm \le i < (q+1)m, \end{cases}$$

where $(m - 1)k = mq + r$, $0 \le r < m$.

Proof: In [2, Theorem 6, p. 150] it is proved that for any positive integer n,

$$\Delta_h^k f(x) = \sum_{i_1=0}^{n-1} \sum_{i_2=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} \Delta_{\frac{h}{n}}^k f\left[x + (i_1 + \cdots + i_k)\frac{h}{n}\right],$$

from which we readily deduce that

$$\Delta_{mh}^k f(x_0) = \sum_{i_1=0}^{m-1} \sum_{i_2=0}^{m-1} \cdots \sum_{i_k=0}^{m-1} \Delta_h^k f[x + (i_1 + \cdots + i_k)h].$$

We now observe that α_p is equal to the number of ways in which p can be expressed as a sum $i_1 + \cdots + i_k$, where $0 \le i_t \le m - 1$, $t = 1, 2, \ldots, k$. Consequently, α_p is equal to the coefficient of x^p in the expansion

$$\sum_{r=0}^{(m-1)k} \alpha_r x^r \equiv (1 + x + x^2 + \cdots + x^{m-1})^k = (1 - x^m)^k (1 - x)^{-k}. \tag{2}$$

It is now clear from (2) that the α_i are positive, symmetrical and have the form specified. That their sum is m^k follows by putting $x = 1$ in the left-hand side of (2).

When $k = 3$, for example, we display the coefficients α_i in the following triangular array:

(3)

m																			
1										1									
2						1	3						3	1					
3					1	3	6			7			6	3	1				
4				1	3	6	10	12				12	10	6	3	1			
5			1	3	6	10	15	18		19		18	15	10	6	3	1		
6		1	3	6	10	15	21	25	27		27	25	21	15	10	6	3	1	
7	1	3	6	10	15	21	28	33	36	37	36	33	28	21	15	10	6	3	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

We now make some observations in relation to the coefficients in (3). First, as predicted by Theorem 1, the sum of the integers in row r is equal to r^3. Second, each integer in the above table is either a multiple of 3 or leaves a remainder of +1 when divided by 3. Furthermore, for any particular row, the first entry, namely 1, and every third successive entry, are exactly those integers which leave remainder +1 when divided by 3. We summarize this discussion in the following theorem.

Theorem 2: Each integer in arrangement (3) is either a multiple of 3 or leaves a remainder of +1 on division by 3. If we label the integers in any one row as $\alpha_0, \alpha_1, \ldots,$ then $\alpha_i \equiv 1 \pmod 3$ when $i \equiv 0 \pmod 3$, and $\alpha_i \equiv 0 \pmod 3$ when $i \not\equiv 0 \pmod 3$. Consequently, in row m, there are m coefficients which leave remainder +1 on division by 3, and $2(m - 1)$ which are a multiple of 3.

Proof: The form of the coefficients α_i is specified in Theorem 1. Since 3 is a prime number, the remainders after division by 3 are completely determined by the term

$$\binom{i+k-1}{k-1} = \binom{i+2}{2} = \frac{(i+1)(i+2)}{2}.$$

If $i \not\equiv 0 \pmod 3$, then i is of the form $3m - 1$ or $3m - 2$, where m is a positive integer. In either case, it is easy to see that $\frac{(i+1)(i+2)}{2}$ is divisible by 3. If, on the other hand, $i \equiv 0 \pmod 3$, then we can write $i = 3m$, and

$$\frac{(i+1)(i+2)}{2} = \frac{(3m+1)(3m+2)}{2}.$$

Consequently,

$$\frac{(3m+1)(3m+2)}{2} - 1 = \frac{9m(m+1)}{2},$$

and this is easily seen to be divisible by 3.

We can generalize the results of Theorem 2 as follows:

Theorem 3: Let k be a prime number. Then each coefficient α_i of Theorem 1 is either a multiple of k, or leaves a remainder of +1 on division by k. In any one row, $\alpha_i \equiv 1 \pmod k$ when $i \equiv 0 \pmod k$, and $\alpha_i \equiv 0 \pmod k$ when $i \not\equiv 0 \pmod k$. Consequently, in row m there are m coefficients which leave remainder +1 on division by k, and $(m - 1)(k - 1)$ which are a multiple of k.

The proof is similar to that of Theorem 2, and will not be included.

REFERENCES

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2. D. V. Widder. *Laplace Transform.* Princeton: Princeton University Press, 1946.

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PYTHAGOREAN TRIANGLES AND MULTIPLE ANGLES

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In a paper dealing with Pythagorean triangles, Gruhn [1] asked how many pairs of primitive Pythagorean triangles exist in which the sine of one of the acute angles of the second triangle equals the sine of twice either of the acute angles of the first triangle. This question may be generalized to determining pairs of primitive Pythagorean triangles where an acute angle of the second is N times an acute angle of the first (here N can take on any positive integer value). In addition, it may be asked whether any relationship exists among the generators of such primitive Pythagorean triangles.

It is necessary to review first some known results from number theory and trigonometry. A Pythagorean triangle is a right triangle whose sides are positive integers. Such triangles will be designated by the triple (x,y,z) which satisfies the equation $x^2 + y^2 = z^2$. In the case where x and y are relatively prime, the triangle is said to be primitive. Formulas for the sides of primitive Pythagorean triangles in terms of generators m and n are (see [2]):