We would like to find a criterion involving characteristic numbers which would enable us to determine if two sequences belong to the same family or not. We conclude with conjectures in this direction:

Conjecture 1: $D_S = D_T \Rightarrow S \leftrightarrow T$

Conjecture 2: $S \leftrightarrow T \Longleftrightarrow D_S D_T$ is an $\mathcal{L}$-factor times a rational square. It would also be desirable to have an algorithm to produce the derivation given the $\mathcal{L}$-factor.

Conjecture 3: p is a Brousseau number $\Rightarrow$ each of the powers of p corresponds to a distinct family of sequences.

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AN ESTIMATE FOR THE LENGTH OF A FINITE JACOBI ALGORITHM

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There are many papers concerning the length of the continued fraction expansion of a rational number (see, e.g., M. Mendès-France [2]). Following a method given by J. D. Dixon [1] in an elementary way, an estimate can be given for the length of the Jacobi algorithm of a rational point.

The Jacobi algorithm may be described in the following way: Let

$$B = \{x = (x_1, \ldots, x\ ) \mid 0 \le x_j < 1,\ 1 \le j \le n\}.$$

If $x = (0, \ldots, 0)$, then $Tx = x$. If $x_1 = \cdots = x_t = 0$, $x_{t+1} > 0$ for $0 \le t < n$, then,

$$T(0, \ldots, 0, x_{t+1}, \ldots, x_n) = (0, \ldots, 0, x_{t+2}/x_{t+1} - [x_{t+2}/x_{t+1}], \ldots, 1/x_{t+1} - [1/x_{t+1}]).$$

We define $x^{(g)} = T^g x$. We say that the algorithm of x has length $L(x) = G$ if

$$G = \min\{g \ge 0\ x^{(g)} = (0, \ldots, 0)\}.$$

Let $x^{(s)} = (0, \ldots, 0, x_{t+1}^{(s)}, \ldots, x_n^{(s)})$, then we define

$$k_0^{(s+1)} = \cdots = k_{t-1}^{(s+1)} = 0$$

$$k_t^{(s+1)} = 1 \text{ (if } t = 0, \text{ then } k_0^{(s+1)} = 1)$$

$$k_{t+1}^{(s+1)} = [x_{t+2}^{(s)}/x_{t+1}^{(s)}],\ \ldots,\ k_n^{(s+1)} = [1/x_{t+1}^{(s)}]$$

$$A_i^{(j)} = \delta_{ij} \text{ for } 0 \le i,\ j \le n$$

$$A_i^{(n+1)} = 0 \text{ for } 1 \leq i \leq n, \; A_0^{(n+1)} = 1$$

$$A_i^{(s+n+1)} = \sum_{j=0}^{n} A_i^{(s+j)} k_j^{(s)}, \; 0 \leq i \leq n.$$

Then, an easy induction shows

$$x_i = \frac{A_i^{(s+n+1)} + \sum_{j=1}^{n} A_i^{(s+j)} x_j^{(s)}}{A_0^{(s+n+1)} + \sum_{j=1}^{n} A_0^{(s+j)} x_j^{(s)}}$$

for $1 \leq i \leq n$.

We want to prove the following

Theorem: Let $x = (a_1/b, \ldots, a_n/b) \in B$ be a rational point. Then

(1) Let $\theta > 1$ and $\theta^n + 1 = \theta^{n+1}$, then $L(x) \leq (\log \theta)^{-1} \log b$.

(2) Let $0 < \sigma < 1$. Then there is an $\eta = \eta(\sigma) > 0$ with the following property: Denote by $N(z)$ the number of rational points x satisfying $b \leq z$ such that $L(x) \leq \eta \log b$, then $N(z) = O(z^{n+\sigma})$.

Remark: Since the order of magnitude of the number of rational points satisfying $b \leq z$ is z^{n+1}, the result (2) states that in some sense almost all rational points satisfy $L(x) > \eta \log b$.

We first need a lemma, well known for the Jacobi algorithm without "Störungen" (that means $x_1^{(g)} \neq 0$ for all g; see O. Perron [3]).

Lemma: For $a \geq 0$,

$$(A_1^{(a+n)}, A_2^{(a+n)}, \ldots, A_n^{(a+n)}, A_0^{(a+n)}) = 1.$$

Proof: This is clear for $a = 0$. Therefore, we put $a = g + 1 \geq 1$. Suppose that $k_{t-1}^{(g)} = \cdots = k_0^{(g)} = 0$, $k_t^{(g)} = 1$ and $k_{s-1}^{(g-1)} = \cdots = k_0^{(g-1)} = 0$, $k_s^{(g-1)} = 1$, where $0 \leq s \leq t$. Then the following relations hold ($0 \leq i \leq n$):

$$A_i^{(g+n+1)} = A_i^{(g+n)} k_n^{(g)} + \cdots + A_i^{(g+t+1)} k_{t+1}^{(g)} + A_i^{(g+t)}$$

$$A_i^{(g+n)} = A_i^{(g-1+n)} k_n^{(g-1)} + \cdots + A_i^{(g+s)} k_{s+1}^{(g-1)} + A_i^{(g+s-1)}.$$

We introduce the matrices:

$$M_g \text{ with rows } (A_1^{(g+j)}, \ldots, A_n^{(g+j)}, A_0^{(g+j)}), \; s \leq j \leq n;$$

$$M_{g+1} \text{ with rows } (A_1^{(g+1+h)}, \ldots, A_n^{(g+1+h)}, A_0^{(g+1+h)}), \; t \leq h \leq n;$$

$$M_{g+1}^* \text{ with rows } (A_1^{(g+h)}, \ldots, A_n^{(g+h)}, A_0^{(g+h)}), \; t \leq h \leq n.$$

Then M_g has rank $n + 1 - s$, and M_{g+1} and M_{g+1}^* both have rank $n + 1 - t$.

Let $d = (A_1^{(g+n+1)}, \ldots, A_n^{(g+n+1)}, A_0^{(g+n+1)})$ denote the greatest common divisor. Then d divides all $(n + 1 - t) \times (n + 1 - t)$ determinants of M_{g+1} and therefore of M_{g+1}^* as well.

Now the Laplacian expansion for determinants shows that d is a divisor of all $(n + 1 - s) \times (n + 1 - s)$ determinants of the matrix M_g. Repeating the argument, we finally see that d divides determinants of M_0, but $|\det M_0| = 1$.

Proof of the Theorem: If $L(x) = G$, then $a_i/b = A_i^{(G+n+1)}/A_0^{(G+n+1)}$ for $1 \leq i \leq n$. Therefore $b = d_G A_0^{(G+n+1)}$. From this, we first obtain

$$b \geq A_0^{(G+n+1)} \geq \theta^G$$

and

$$\log b \geq G \log \theta.$$

The number of rational points satisfying $b \leq z$ is smaller than or equal to the number of allowed algorithms (see O. Perron [3] or F. Schweiger [4]) such that $d_G A_0^{(G+n+1)} \leq z$.

Since $A_0^{(G+n+1)} \geq k_n^{(s)} \ldots k_n^{(G)}$ and given $k_n^{(g)}$ there are at most

$$(k_n^{(g)} + 1)^{n-1} \leq 2^{n-1} (k_n^{(g)})^{n-1}$$

possible values for the digits $k_j^{(g)}$, $1 \leq j \leq n - 1$, we have the estimate (we write c_j instead of $k_n^{(j)}$):

$$N(z) \leq \sum_{G \leq \eta \log z} \left(\sum_{q_1 \cdots q_G d_G \leq z} (2^{n-1})^G (q_1 \ldots q_G)^{n-1} \left(\frac{z}{q_1 \ldots q_G d_G} \right)^s \right)$$

where $s > n$ will be chosen. This shows

$$N(z) = 0\left(z^s \sum_{G \leq \eta \log z} 2^{(n-1)G} \sum_{q_1 = 1}^{\infty} \cdots \sum_{q_G = 1}^{\infty} \sum_{q_G = 1}^{\infty} (q_1 \ldots q_G d_G)^{n-1-s} \right)$$

$$= 0\left(z^s \sum_{G \leq \eta \log z} (2^{n-1} \zeta(s + 1 - n))^{G+1} \right) = 0\left(z^s (2^{n-1} \zeta(s + 1 - n))^{\eta \log z} \right).$$

We put $s = n + \varepsilon$ and obtain $N(z) = 0(z^\alpha)$ where

$$\alpha = n + \varepsilon + \eta(\log \zeta(1 + \varepsilon) + (n - 1)\log 2).$$

Choosing $\varepsilon > 0$ and $\eta = \eta(\varepsilon)$, we may obtain

$$\varepsilon + \eta[\log \zeta(1 + \varepsilon) + (n - 1)\log 2] \leq \sigma.$$

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SOLUTION OF THE RECURRENT EQUATION $u_{n+1} = 2u_n - u_{n-1} + u_{n-3}$

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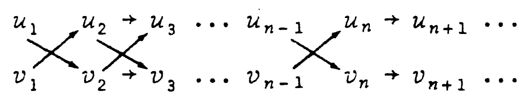
To find the general term of the sequence $\{u_n\}$, we introduce an auxiliary sequence $\{v_n\}$, intertwined with $\{u_n\}$ in the following way:



where

(1) $$\begin{cases} u_{n+1} = v_{n-1} + u_n, \\ v_{n+1} = u_{n-1} + v_n. \end{cases}$$

It is clear that both sequences are determined as soon as u_1, v_1 $(= u_3 - u_2)$, and u_2, v_2 $(= u_4 - u_3)$ are given. $\{u_n\}$ solves our problem since

$$u_{n+1} = v_{n-1} + u_n = u_{n-3} + v_{n-2} + u_n = u_{n-3} + (u_n - u_{n-1}) + u_n.$$

1. Adding the equations in (1) memberwise, we obtain:

$$u_{n+1} + v_{n+1} = (u_{n-1} + v_{n-1}) + (u_n + v_n),$$

which implies that $\{u_n + v_n\}$ *is a Fibonacci sequence* $\{F_n\}$ whose first two terms are

$$u_1 + v_1 \ (= u_1 - u_2 + u_3) \quad \text{and} \quad u_2 + v_2 \ (= u_2 - u_3 + u_4).$$

2. Our problem would be completely solved if we would have an expression for $u_n - v_n = \varepsilon_n$. Subtracting the equations in (1) memberwise, we obtain:

$$\begin{aligned} \varepsilon_{n+1} &= \varepsilon_n - \varepsilon_{n-1}, \\ &= (\varepsilon_{n-1} - \varepsilon_{n-2}) - \varepsilon_{n-1} \quad \text{(replacing } n \text{ by } n - 1 \text{ above)}, \\ &= -\varepsilon_{n-2}, \\ &= -(-\varepsilon_{n-5}) \quad \text{(replacing } n \text{ by } n - 3 \text{ above)}, \\ &= \varepsilon_{n-5}. \end{aligned}$$