Theorem: The probability that n tosses of a fair coin will contain a run of at least r consecutive heads, $r \leq n$, is given by $1 - F_{n+2}^{(r)}/2^n$.

Proof: Apply Lemma 3 to (4) with $m = n - r + 1$.

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COMBINATORIAL IDENTITIES DERIVED FROM UNITS

SUSAN C. SEEDER

Grinnell College, Grinnell, IA 50112

ABSTRACT

We shall derive two combinatorial identities by considering units in infinite classes of cubic fields. This is a comparatively new application of units.

0. INTRODUCTION

We shall begin by stating a result of Bernstein and Hasse [2] concerning systems of units in infinitely many fields.

Theorem: Let $P(x)$ be a polynomial of degree $n \geq 2$ with the form

$$P(x) = (x - D_0)(x - D_1) \ldots (x - D_{n-1}) - d,\ d \geq 1,\ D_i, d \in Z,\ D_0 \equiv D_i \pmod{d},$$
$$D_0 - D_i \geq 2d(n - 1),\ (i = 1, \ldots, n - 1),\ D_0 > D_1 > \ldots > D_{n-1}.$$

Then $P(x)$ has exactly n distinct real roots; $P(x)$ is irreducible over Q; and if w is the largest root of $P(x)$, then

$$e_i = \frac{(w - D_i)^n}{d} \qquad (i = 0, \ldots, n - 1)$$

are different units of $Q(w)$. Furthermore, any $n - 1$ of these units form a system of independent units.

1. COMBINATORIAL IDENTITIES FROM UNITS

Consider the cubic polynomials $P(x) = (x - D_0)(x - D_1)(x - D_2) - 1$; D_i as above. First we work with the case $D_2 = 0$; later we will eliminate this condition. Now it is clear that itself is a unit in $Q(w)$ with $N(w) = 1$. We proceed by expressing the integral powers of w. For any integer $n \geq 0$, let

$$w^n = x_n + y_n w + z_n w^2 \qquad (x_n, y_n, z_n \in Z). \tag{1.1}$$

Calculating directly and taking into account that $w^3 = 1 - Bw + Aw^2$ where $A = D_0 + D_1$ and $B = D_0 D_1$, we have

$$\begin{aligned} w^{n+1} &= z_n + (x_n - Bz_n)w + (y_n + Az_n)w^2 \\ w^{n+2} &= (y_n + Ax_n) + (z_n - By_n - ABz_n)w + (x_n - Bz_n + Ay_n + A^2 z_n)w^2; \end{aligned} \tag{1.2}$$

so that

$$x_{n+1} = z_n;\ y_{n+1} = x_n - Bx_{n+1};\ z_{n+1} = x_{n-1} - Bx_n + Ax_{n+1}. \tag{1.3}$$

From (1.1) and (1.3), we obtain

$$w^n = x_n + (x_{n-1} - Bx_n)w + (x_{n-2} - Bx_{n-1} + Ax_n)w^2 \tag{1.4}$$

along with the recursion formula

(1.5) $$x_{n+3} = x_n - Bx_{n+1} + Ax_{n+2};\ n \geq 0.$$

Now in order to write x_n explicitly, we shall make use of generating functions together with (1.5) to obtain

(1.6) $$\sum_{n=0}^{\infty} x_n u^n = (1 - Au + Bu^2) \sum_{n=0}^{\infty} (A - Bu + u^2)^n u^n.$$

It should be noted that for the sake of convergence, u can be chosen such that $|Au - Bu^2 + u^3| < 1$. Equating coefficients in (1.6),

(1.7) $$x_n = \sum_{i=2}^{n-2} \sum_{\ell=1} (-1)^i \begin{pmatrix} & n - i - 1 & \\ n - 2i + \ell & i - 2\ell & \ell - 1 \end{pmatrix} A^{n-2i+\ell} B^{i-2\ell}.$$

In the same manner we calculate the negative powers of w.

(1.8) $$w^{-n} = r_n + s_n w + t_n w^2,\quad (n \in N;\ r_n, s_n, t_n \in Z)$$

(1.9) $$w^{-n} = r_n + (r_{n-2} - Ar_{n-1})w + r_{n-1}w^2 \text{ where } r_{n+3} = r_n - Ar_{n+1} + Br_{n+2}.$$

As before, we apply generating functions to obtain

$$\sum_{n=0}^{\infty} r_n u^n = \sum_{n=0}^{\infty} (B - Au + u^2)^n u^n.$$

Comparing coefficients of equal powers of u gives us the relation

(1.10) $$r_n = \sum_{i=0}^{n-1} \sum_{\ell=0} (-1)^i \begin{pmatrix} & n - i & \\ n - 2i + \ell & i - 2\ell & \ell \end{pmatrix} A^{i-2\ell} B^{n-2i+\ell}.$$

We return to formulas (1.1) and (1.8) and multiply right and left sides together to obtain, after some rearrangements, the following three equations with r_n, s_n, t_n as unknowns:

$$1 = x_n r_n + z_n s_n + (y_n + Az_n)t_n;$$
$$0 = y_n r_n + (x_n - Bz_n)s_n + (-By_n + z_n - ABz_n)t_n;$$
$$0 = z_n r_n + (y_n + Az_n)s_n + (x_n + Ay_n - Bz_n + A_n^2 z_n)t_n.$$

The determinant of the system is equal to the norm of w^n as can be seen from (1.1) and (1.2). But $N(w) = 1$. Therefore, we have

(1.11) $$r_n = \begin{vmatrix} x_n - Bz_n & -By_n + z_n - ABz_n \\ y_n + Az_n & x_n + Ay_n - Bz_n + A_n^2 z_n \end{vmatrix}.$$

From (1.2), (1.5), and (1.11),

$$r_n = \begin{vmatrix} x_{n+3} - Ax_{n+2} & x_{n+4} - Ax_{n+3} \\ x_{n+2} & x_{n+3} \end{vmatrix}.$$

(1.12) $$r_n = x_{n+3}^2 - x_{n+2}x_{n+4}.$$

We have at last reached our first combinatorial identity by considering (1.12) in conjunction with (1.7) and (1.10), which express the x_n and the r_n as combinatorial functions.

If we now consider x_n as an unknown and solve the original system of equations, the determinant of the system becomes $-N(w^{-n}) = -1$; so that

$$x_n = -\begin{vmatrix} r_n - Bt_n & -Bs_n + t_n - ABt_n \\ s_n + At_n & r_n + As_n - Bt_n + A_n^2 t_n \end{vmatrix}.$$

Substituting for s_n and t_n in terms of r_n from (1.9) and recalling that $r_{n+1} = r_{n-2} - Ar_{n-1} + Br_n$, we obtain our second combinatorial identity

(1.13) $$x_n = r_{n-3}^2 - r_{n-2}r_{n-4} \qquad (n \geq 5).$$

Note that no generality was lost by assuming $D_2 = 0$, since by setting $w - D_2 = \overline{w}$ and working with the equation

$$\overline{w}^3 + (2D_2 - D_0 - D_1)\overline{w}^2 + (D_2 - D_0)(D_2 - D_1)\overline{w} - 1 = 0,$$

we would obtain the same identities with A and B replaced by $\overline{A} = -2D_2 + D_0 + D_1$ and $\overline{B} = (D_2 - D_0)(D_2 - D_1)$, respectively.

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A STOLARSKY ARRAY OF WYTHOFF PAIRS

DAVID R. MORRISON

Harvard University, Cambridge, MA 02138

A *positive Fibonacci sequence* is a sequence $\{s_k\}$ such that $s_{k+1} = s_k + s_{k-1}$ and $s_k > 0$ for k sufficiently large. A *Stolarsky array* is an array $A = \{A_{m,n} : m, n \quad 0\}$ of natural numbers such that:

(a) the rows $\{A_{m,1}, A_{m,2}, \ldots\}$ are positive Fibonacci sequences;
(b) every natural number occurs exactly once in the array;
(c) every positive Fibonacci sequence is a row of the array, after a shift of indices. That is, given a positive Fibonacci sequence $\{s_j\}$, there exist m and k such that

$$A_{m,n} = s_{n+k}.$$

The first such array[1] was constructed by Stolarsky [8]. In this note, we will construct a new Stolarsky array using Wythoff pairs. By inspecting the tables of these two arrays, it is easy to obtain more Stolarsky arrays. (For example, in either table, the 4 may be shifted from the second to the third row.) It would be interesting to have a classification of the Stolarsky arrays.

Let $\alpha = \frac{1}{2}(1 + \sqrt{5})$ be the golden ratio, and let $[\ ]$ denote the greatest integer function. The *Wythoff pairs* are the pairs of numbers $([n\alpha], [n\alpha^2])$ which give the winning positions in Wythoff's game (see [5], for example). These pairs have two remarkable properties:

1. *Beatty complementarity* [2]—Every natural number m is either of the form $[n\alpha]$ or of the form $[n\alpha^2]$, but not both.

2. *Connell's formula* [4]—

$$[n\alpha] + [n\alpha^2] = [[n\alpha^2]\alpha].$$

<u>*Lemma 1*</u>: Let $s_1 = [k\alpha]$, $s_2 = [k\alpha^2]$ generate a positive Fibonacci sequence. Then (s_{2j-1}, s_{2j}) is a Wythoff pair for every $j > 0$.

<u>*Proof*</u>: Since $\alpha^2 = \alpha + 1$, we have

$$(*) \qquad n + [n\alpha] = [n\alpha^2].$$

Suppose $(s_{2j-3}, s_{2j-2}) = ([m\alpha], [m\alpha^2])$ is a Wythoff pair. Then by Connell's formula,

$$s_{2j-1} = [m\alpha] + [m\alpha^2] = [[m\alpha^2]\alpha],$$

while by formula (*),

$$s_{2j} = [m\alpha^2] + [[m\alpha^2]\alpha] = [[m\alpha^2]\alpha^2].$$

Thus, (s_{2j-1}, s_{2j}) is a Wythoff pair, and the lemma follows by induction.

We define the *Wythoff array* to be an array $W = \{W_{m,n}\}$ which is Fibonacci in its rows, and is generated by:

$$W_{m,1} = [[m\alpha]\alpha],\ W_{m,2} = [[m\alpha]\alpha^2].$$

The first 100 terms of the Wythoff array are listed in Table 1.

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[1]That (c) holds for Stolarsky's array does not seem to have been noticed. We will verify it as Corollary 2, below.