FIBONACCI NUMBERS AND GENERALIZED BINOMIAL COEFFICIENTS

V. E. Hoggatt, Jr.
San Jose State College, San Jose, California

1. INTRODUCTION

The first time most students meet the binomial coefficients is in the expansion

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j , \quad n \geq 0,$$

where

$$\binom{n}{m} = 0 \quad \text{for } m > n , \qquad \binom{n}{n} = \binom{n}{0} = 1, \qquad \text{and}$$

(1) $$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1} , \quad 0 < m < n .$$

Consistent with the above definition is

(2) $$\binom{n}{m} = \frac{n(n-1)\cdots 2\cdot 1}{m(m-1)\cdots 2\cdot 1\ (n-m)(n-m-1)\cdots 2\cdot 1} = \frac{n!}{m!(n-m)!} ,$$

where

$$n! = n(n-1)(n-2)\cdots 2\cdot 1 \quad \text{and} \quad 0! = 1 .$$

Given the first lines of Pascal's arithmetic triangle one can extend the table to the next line using directly definition (2) or the recurrence relation (1).

We now can see just how the ordinary binomial coefficients $\binom{n}{m}$ are related to the sequence of integers 1, 2, 3, ..., k, ... Let us generalize this observation using the Fibonacci sequence.

2. THE FIBONOMIAL COEFFICIENTS

Let the Fibonomial coefficients (which are a special case of the generalized binomial coefficients) be defined as

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{F_n F_{n-1} \cdots F_2 F_1}{(F_m F_{m-1} \cdots F_2 F_1)(F_{n-m} F_{n-m-1} \cdots F_2 F_1)} , \quad 0 < m < n ,$$

and $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1$, where F_n is the nth Fibonacci number, defined by

$$F_n = F_{n-1} + F_{n-2}\,, \quad F_1 = F_2 = 1 \,.$$

We next seek a convenient recurrence relation, like (1) for the ordinary binomial coefficients, to get the next line from the first few lines of the Fibonomial triangle.

To find two such recurrence relations we recall the Q-matrix,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} ,$$

for which it is easily established by mathematical induction that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \; n \geq 0.$$

The laws of exponents hold for the Q-matrix so that $Q^n = Q^m Q^{n-m}$. Thus

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n-m+1} & F_{n-m} \\ F_{n-m} & F_{n-m-1} \end{pmatrix}$$

$$= \begin{pmatrix} F_{m+1}F_{n-m+1} + F_mF_{n-m} & F_{m+1}F_{n-m} + F_mF_{n-m-1} \\ F_mF_{n-m+1} + F_{m-1}F_{n-m} & F_mF_{n-m} + F_{m-1}F_{n-m-1} \end{pmatrix}$$

yielding, upon equating corresponding elements,

(A) $$F_n = F_{m+1}F_{n-m} + F_mF_{n-m-1} \qquad \text{(upper right)} \,,$$

(B) $$F_n = F_mF_{n-m+1} + F_{m-1}F_{n-m} \qquad \text{(lower left).}$$

Define C so that

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{F_nF_{n-1}\cdots F_2F_1}{(F_mF_{m-1}\cdots F_2F_1)(F_{n-m}F_{n-m-1}\cdots F_2F_1)} = F_nC \;.$$

With C defined above, then

$$\begin{bmatrix} n-1 \\ m \end{bmatrix} = F_{n-m}C \quad \text{and} \quad \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = F_mC \,.$$

Returning now to identity (A), we may write for $C \neq 0$,

$$F_n C = F_{m+1}(F_{n-m}C) + F_{n-m-1}(F_m C)$$

but by the definition of C, we have derived

$$\text{(D)} \qquad \begin{bmatrix} n \\ m \end{bmatrix} = F_{m+1}\begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m-1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} .$$

Similarly, using identity (B), one can establish

$$\text{(E)} \qquad \begin{bmatrix} n \\ m \end{bmatrix} = F_{m-1}\begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m+1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.$$

It is thus now easy to establish by mathematical induction that if the Fibonomial coefficients are integers for an integer n ($m = 0, 1, \ldots, n$), then they are integers for an integer $n + 1$ ($m = 0, 1, 2, \ldots, n + 1$).

Recalling

$$L_m = F_{m+1} + F_{m-1}$$

and adding (D) and (E) yields

$$\text{(3)} \qquad 2\begin{bmatrix} n \\ m \end{bmatrix} = L_m\begin{bmatrix} n-1 \\ m \end{bmatrix} + L_{n-m}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} ,$$

where L_m is the mth Lucas number. From (3) it is harder to show that the Fibonomial coefficients are integers.

3. THE FIBONOMIAL TRIANGLE

Pascal's arithmetic triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$$\binom{n}{0} \quad \binom{n}{1} \quad \cdots \quad \binom{n}{m} \quad \cdots \quad \binom{n}{n-1} \quad \binom{n}{n}$$

has been the subject of many studies and has always generated interest. We note here to get the next line we merely use the recurrence relation (1). Here we point out two interpretations, one of which shows a direction for generalization. The usual first meeting with Pascal's triangle lies in the binomial theorem expansion of $(x + y)^n$. However, of much interest to us is the difference equation interpretation. The difference equation satisfied by n^0 is

$$(n + 1)^0 - n^0 = 0 \quad ,$$

while the difference equation satisfied by n is

$$(n + 2) - 2(n + 1) + n = 0 \quad .$$

For n^2 the difference equation is

$$(n + 3)^2 - 3(n + 2)^2 + 3(n + 1)^2 - n^2 = 0 \ .$$

Certainly one notices the binomial coefficients with alternating signs appearing here. In fact,

$$\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (n + m + 1 - j)^m = 0 \ .$$

It is this connection with the difference equations for the powers of the integers that leads us naturally to the Fibonomial triangle.

Similar to the difference equation coefficients array for the powers of the positive integers which results in Pascal's arithmetic triangle with alternating signs, there is the Fibonomial triangle made up of the Fibonomial coefficients, with doubly alternated signs. We first write down the Fibonomial triangle for the first six levels.

						1						
					1		1					
				1		1		1				
			1		2		2		1			
		1		3		6		3		1		
	1		5		15		15		5		1	
1		8		40		60		40		8		1

The top line is the zeroth row and the coefficients in the difference equation satisfied by F_n^k are the numbers in the $(k + 1)$st row. Of course, we can get the next line of Fibonomial coefficients by using our recurrence relation (D),

$$\begin{bmatrix} n \\ m \end{bmatrix} = F_{m+1} \begin{bmatrix} n-1 \\ m \end{bmatrix} + F_{n-m-1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}, \quad 0 < m < n \quad .$$

We now rewrite the Fibonomial triangle with appropriate signs so that the rows are properly signed to be the coefficients in the difference equation satisfied by F_n^k .

	1						
F_n^0 :	1	-1					
F_n^1 :	1	-1	-1				
F_n^2 :	1	-2	-2	+1			
F_n^3 :	1	-3	-6	+3	+1		
F_n^4 :	1	-5	-15	+15	+5	-1	
F_n^5 :	1	-8	-40	+60	+40	-8	-1

Thus, from the above we may write

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0 \qquad \text{and}$$

$$F_{n+5}^4 - 5F_{n+4}^4 - 15F_{n+3}^4 + 15F_{n+2}^4 + 5F_{n+1}^4 - F_n^4 = 0 \quad .$$

The auxiliary polynomial for the difference equation satisfied by F_n^m is

$$\sum_{h=0}^{m+1} \begin{bmatrix} m+1 \\ h \end{bmatrix} (-1)^{h(h+1)/2} x^{m+1-h}$$

which shows that the sign pattern of doubly alternating signs persists. (See [1], [2].) (Further generalizations given in the original paper are here omitted.)

REFERENCES

1. Dov Jarden, Recurring Sequences, Riveon Lematematika, Jerusalem, Israel, 1958, pp. 42-45.
2. A. P. Hillman and V. E. Hoggatt, "The Characteristic Polynomial of the Generalized Shift Matrix," Fibonacci Quarterly, Vol. 3, No. 2, April, 1965, pp. 91-94.
3. Terrence A. Brennan, "Fibonacci Powers and Pascal's Triangle in a Matrix," Fibonacci Quarterly, Vol. 2, No. 2, April, 1964, pp. 93-103.