

A PRIMER FOR THE FIBONACCI NUMBERS: PART XI

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MULTISECTION GENERATING FUNCTIONS FOR THE COLUMNS OF PASCAL'S TRIANGLE

1. INTRODUCTION

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

be the generating function for the sequence $\{a_n\}$. Often one desires generating functions which multisection the sequence $\{a_n\}$,

$$G_i(x) = \sum_{j=0}^{\infty} a_{i+mj} x^j, \quad (i = 0, 1, 2, \dots, m-1).$$

For the bisection generating functions the task is easy. Let

$$H_1(x^2) = \frac{f(x) + f(-x)}{2},$$

$$H_2(x^2) = \frac{f(x) - f(-x)}{2x};$$

then clearly $H_1(x^2)$ and $H_2(x^2)$ contain only even powers of x so that

$$H_1(x) = \sum_{n=0}^{\infty} a_{2n} x^n \quad \text{and} \quad H_2(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n$$

are what we are looking for.

Let us illustrate this for the Fibonacci sequence. Here

$$f(x) = \frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n ;$$

then

$$H_1(x) = \frac{x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n} x^n$$

and

$$H_2(x) = \frac{1-x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n+1} x^n .$$

Exercise: Find the bisection generating functions for the Lucas sequence.

Let us find the general multisection generating functions for the Fibonacci sequence, using the method of H. W. Gould (See [1]). The Fibonacci sequence enjoys the Binet Form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} , \quad \alpha = \frac{1 + \sqrt{5}}{2} , \quad \beta = \frac{1 - \sqrt{5}}{2} .$$

Let $f(x) = 1/(1-x)$; then

$$\begin{aligned} \sum_{n=0}^{\infty} F_{mn+j} x^n &= \frac{\alpha^j f(\alpha^m x) - \beta^j f(\beta^m x)}{\alpha - \beta} \\ &= \frac{1}{\alpha - \beta} \left(\frac{\alpha^j}{1 - \alpha^m x} - \frac{\beta^j}{1 - \beta^m x} \right) \\ &= \frac{\frac{\alpha^j - \beta^j}{\alpha - \beta} + (\alpha\beta)^j \frac{\alpha^{m-j} - \beta^{m-j}}{\alpha - \beta} x}{1 - (\alpha^m + \beta^m)x + (\alpha\beta)^m x^2} \\ &= \frac{F_j + (-1)^j F_{m-j} x}{1 - L_m x + (-1)^m x^2} , \quad (j = 0, 1, 2, \dots, m-1), \end{aligned}$$

since $\alpha\beta = -1$, $\alpha^m + \beta^m = L_m$, and $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$.

Exercise: Find the general multisection generating function for the Lucas sequence.

The same technique can be used on any sequence having a Binet Form. The general problem of multisectioning a general sequence rapidly becomes very complicated according to Riordan [2], even in the classical case.

2. COLUMN GENERATORS OF PASCAL'S TRIANGLE

The column generators of Pascal's left-justified triangle [3], [4], [5], are

$$G_k(x) = \frac{x^k}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} x^n, \quad k = 0, 1, 2, \dots$$

We now seek generating functions which will m -sect these,

$$G_i(m, k; x) = \sum_{n=0}^{\infty} \binom{i+k+mn}{k} x^{n+k+1}, \quad (i = 0, 1, \dots, m-1).$$

We first cite an obvious little lemma.

Lemma 1:
$$\binom{n}{k} = \sum_{j=1}^m \binom{n-j}{k-1} + \binom{n-m}{k}.$$

Definition: Let $G_{i,k}(x)$, $i = 0, 1, 2, \dots, m-1$, be the m generating functions

$$G_{i,k}(x) = \sum_{n=0}^{\infty} \binom{i+k+mn}{k} x^{i+mn+k}.$$

Lemma 2:

$$G_{i,k+1}(x) = \frac{xG_{i,k}(x) + x^2G_{i-1,k}(x) + \dots + x^mG_{i-m+1,k}(x)}{1 - x^m}.$$

The proof follows easily from Lemma 1.

Let

$$(1 + x + x^2 + \dots + x^{m-1})^n = \sum_{j=0}^{n(m-1)} \binom{n}{j}_m x^j$$

define the row elements of the m -nomial triangle. Further, let

$$f_i(m, k; x) = \sum_{j=0}^k \binom{k}{i + jm}_m x^j, \quad i = 0, 1, \dots, m-1,$$

where j is such that $i + jm \leq k(m-1)$. These are multisectioning polynomials for the rows of the m -nomial triangle. Now, we can state an interesting theorem:

Theorem: For $i = 0, 1, 2, \dots, m-1$,

$$G_i(m, k; x) = \frac{x^{k+i} f_i(m, k; x)}{(1-x)^{k+1}}.$$

Proof: Recall first that the m -nomial coefficients obey

$$\binom{n}{r}_m = \binom{n-1}{r}_m + \binom{n-1}{r-1}_m + \dots + \binom{n-1}{r-m+1}_m$$

where the lower arguments are non-negative and less than or equal to $n(m-1)$.

Clearly, for $k = 0$, from the definition just before Lemma 2,

$$G_{i,0}(x) = \frac{x^i}{1-x^m}, \quad i = 0, 1, 2, \dots, m-1.$$

Assume now that

$$G_{i,k}(x) = \frac{x^{k+i} f_i(m, k; x^m)}{(1-x^m)^{k+1}}$$

for $i = 0, 1, 2, 3, \dots, (m-1)$. From Lemma 2,

$$G_{i,k+1}(x) = \frac{xG_{i-1,k}(x) + \dots + x^m G_{i-m+1,k}(x)}{1-x^m}.$$

Thus,

$$\begin{aligned}
G_{i,k+1}(x) &= \frac{\sum_{s=0}^{m-1} \left(\sum_{j=0}^k \binom{k}{i-s+jm}_m \right) x^{k+(i-s)+s+jm+1}}{(1-x^m)^{k+2}} \\
&= \frac{\sum_{j=0}^{k+1} \left(\sum_{s=0}^{m-1} \binom{k}{i-s+jm}_m \right) x^{k+1+i+jm}}{(1-x^m)^{k+2}} \\
&= \frac{x^{k+1+i} \sum_{j=0}^{k+1} \binom{k+1}{i+jm}_m x^{jm}}{(1-x^m)^{k+2}} \\
&= \frac{x^{k+1+i} f_i(m,k; x^m)}{(1-x^m)^{k+2}}.
\end{aligned}$$

This completes the induction.

The x^{k+1+i} merely position the column generators. Here the non-zero entries are separated by $m-1$ zeros. To get rid of the zeros, let

$$G_i(m,k; x) = \frac{x^{k+i} f_i(m,k; x)}{(1-x)^{k+1}}$$

for $i = 0, 1, 2, \dots, m-1$. This concludes the proof of the theorem.

If we write this in the form

$$G_i(m,k; x) = \sum_{j=0}^{\infty} \binom{i+jm+k}{k} x^{j+k+1} = \frac{\sum_{j=0}^{\infty} \binom{k}{i+jm}_m x^{k+i+j}}{(1-x)^{k+1}}$$

it emphasizes the relation of the multisection of the k th column of Pascal's triangle and the multisection of the k th row of the m -nomial triangle.

3. A NEAT GENERATING FUNCTION

Lemma 3:
$$\binom{n}{k} = \sum_{j=0}^r \binom{r}{j} \binom{n-r}{k-j}$$

This is easy to prove by starting with

$$\begin{aligned} \text{(A)} \quad \binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \\ &= \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k-2} \\ &= 1 \cdot \binom{n-2}{k} + 2 \cdot \binom{n-2}{k-1} + 1 \cdot \binom{n-2}{k-2}. \end{aligned}$$

Apply (A) to each term on the right repeatedly.

Now let $H_i(m, k; x)$ m -sect the k th column of Pascal's triangle ($i = 0, 1, 2, \dots, m-1$); then, using Lemma 3, it follows that

Lemma 4:
$$H_i(m, k; x) = \frac{x}{1-x} \sum_{j=1}^m \binom{m}{j} H_i(m, k-j; x).$$

The results using the method of Polya for small m and i seem to indicate the following (See [3]).

Theorem: The generating functions for the rising diagonal sums of the rows of Pascal's triangle $i + jm$ (all other rows are deleted) are given by

$$H_i(x) = \frac{(1+x)^i}{1-x(1+x)^m}, \quad i = 0, 1, \dots, m-1.$$

Exercise: Show that

$$\sum_{i=0}^{m-1} x^i H_i(x^m) = \frac{1}{1-x(1+x^m)}.$$

