SOLUTIONS TO PROBLEMS

Solutions to problems posed previously are given here. Where a problem solution appeared in the Fibonacci Quarterly, date and page numbers are given.

Page 5

The solutions to "Problems For Exploration" were given by Ken Siler in "Fibonacci Summations," Fibonacci Quarterly, Vol. 1, No. 3, October, 1963, pp. 67-69, as follows:

$$\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1 \qquad \sum_{k=1}^{n} F_{4k-2} = F_{2n}^2$$

$$2\sum_{k=1}^{n} F_{3k-1} = F_{3n+1} - 1 \qquad \sum_{k=1}^{n} F_{4k} = F_{2n+1}^2 - 1$$

$$2\sum_{k=1}^{n} F_{3k-2} = F_{3n} \qquad \sum_{k=1}^{n} F_{4k-3} = F_{2n-1}F_{2n}$$

$$2\sum_{k=1}^{n} F_{3k} = F_{3n+2} - 1 \qquad \sum_{k=1}^{n} F_{4k-1} = F_{2n}F_{2n+1}$$

In that paper is derived the general formula,

$$\sum_{k=1}^{n} F_{ak-b} = \frac{(-1)^a F_{an-b} - F_{a(n+1)-b} + (-1)^{a-b} F_b + F_{a-b}}{(-1)^a + 1 - L_a}$$

for the a-th Lucas number L_a and the Fibonacci numbers with subscript ak-b .

Page 17

B-1, B-2, B-3 are each proved by mathematical induction in Vol. 1, No. 3, October, 1963, pp. 76-78.

Page 26

Use the formulas for the lambda number developed in the article <u>or</u> use elementary row and column operations to simplify the resulting determinant.

Page 45

B-4 (Solution by Joseph Erbacher and J. L. Brown, Jr., FQ, 2:1, February, 1964, p. 80) Using the Binet formula,

$$F_{2n+j} = \frac{(a^2)^n a^j - (b^2)^n b^j}{a - b} = \frac{(1 + a)^n a^j - (1 + b)^n b^j}{a - b}$$

Since

$$a^2 = a + 1, \quad b^2 = b + 1 \quad \text{when} \quad a = (1 + \sqrt{5})/2, \quad b = (1 - \sqrt{5})/2,$$

we have

$$F_{2n+j} = \frac{1}{a - b}\left[\sum_{i=0}^{n} \binom{n}{i} a^{i+j} - \sum_{i=0}^{n} \binom{n}{i} b^{i+j}\right] = \sum_{i=0}^{n} \binom{n}{i} \frac{a^{i+j} - b^{i+j}}{a - b}$$

$$= \sum_{i=0}^{n} \binom{n}{i} F_{i+j} \quad .$$

If j = 0, we have the original problem. The identity also holds, with arbitrary j, for Lucas numbers $L_n = F_{n+1} + F_{n-1}$.

B-5 (Solution by J. L. Brown, Jr., FQ, 1:3, October, 1963, p. 79.)

Let a_n for $n \geq 1$ be the number of different ways of being paid n dollars in one and two dollar bills, taking order into account. Consider the case where $n \geq 2$. Since a one-dollar bill is received as the last bill if and only if n - 1 dollars have been received previously and a two-dollar bill is received as the last bill if and only if n - 2 dollars have been received previously, the two possibilities being mutually exclusive, we have $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. But $a_1 = 1$, $a_2 = 2$; therefore, $a_n = F_{n+1}$ for $n \geq 1$.

B-9 (Solution by Francis D. Parker, FQ, 1:4, Dec., 1963, p. 76)

Since

$$\frac{1}{F_{n-1}F_{n+1}} = \frac{F_n}{F_{n-1}F_nF_{n+1}} = \frac{F_{n+1} - F_{n-1}}{F_{n-1}F_nF_{n+1}} = \frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}} \ ,$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}}\right) = \left(\frac{1}{1\cdot 1} - \frac{1}{1\cdot 2}\right) + \left(\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}\right)$$

$$+ \left(\frac{1}{2\cdot 3} - \frac{1}{3\cdot 5}\right) + \dots = 1$$

Similarly,

$$\frac{F_n}{F_{n-1}F_{n+1}} = \frac{F_{n+1} - F_{n-1}}{F_{n-1}F_{n+1}} = \frac{1}{F_{n-1}} - \frac{1}{F_{n+1}} \qquad \text{and}$$

$$\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{8}\right) + \ldots = 2$$

B-10 (Solution by Charles Wall, FQ, 1:4, Dec., 1963, p. 77)
Since

$$\frac{L_n + \sqrt{5}\,F_n}{2} = \frac{a^n + b^n + a^n - b^n}{2} = a^n$$

where $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$, we have

$$\left(\frac{L_n + \sqrt{5}\,F_n}{2}\right)^p = a^{np} = \frac{a^{np} + b^{np} + a^{np} - b^{np}}{2} = \frac{L_{np} + \sqrt{5}\,F_{np}}{2} \quad .$$

Page 51

H-8 (Solution by John Allen Fuchs and Joseph Erbacher, FQ, 1:3, October, 1963, pp. 51-52.) The squares of the Fibonacci numbers satisfy the linear homogeneous recursion relationship $F^2_{n+3} = 2F^2_{n+2} + 2F^2_{n+1} - F^2_n$. We may use this recursion formula to substitute for the last row of the given determinant, D_n, and then apply standard row operations to get

$$D_n = \begin{vmatrix} F^2_n & F^2_{n+1} & F^2_{n+2} \\ F^2_{n+1} & F^2_{n+2} & F^2_{n+3} \\ 2F^2_{n+1} + 2F^2_n - F^2_{n-1} & 2F^2_{n+2} + 2F^2_{n+1} - F^2_n & 2F^2_{n+3} + 2F^2_{n+2} - F^2_{n+1} \end{vmatrix}$$

$$= \begin{vmatrix} F^2_n & F^2_{n+1} & F^2_{n+2} \\ F^2_{n+1} & F^2_{n+2} & F^2_{n+3} \\ -F^2_{n-1} & -F^2_n & -F^2_{n+1} \end{vmatrix} = -D_{n-1} \; .$$

It follows that $D_n = (-1)^{n-1}D_1$. Since $D_1 = 2$, $D_n = 2(-1)^{n-1} = 2(-1)^{n+1}$.

B-28 (Solution by Marjorie Bicknell, FQ, 2:2, April, 1964, p. 159)

By considering combinations of Fibonacci numbers which give minimum and maximum values to sums of the form abc + def + ghi, the following determinant seems to have the maximum value obtainable with the nine Fibonacci numbers given:

$$\begin{vmatrix} F_{10} & F_4 & F_7 \\ F_6 & F_9 & F_3 \\ F_2 & F_5 & F_8 \end{vmatrix} = F_{10}F_9F_8 + F_7F_6F_5 + F_4F_3F_2 - (F_{10}F_3F_5 + F_9F_2F_7 + F_8F_4F_6)$$
$$= 39796 - 1496 = 38300 .$$

B-13 Expand the determinant by its last row, obtaining $F_n = F_{n-1} + F_{n-2}$, making possible a proof by mathematical induction since $F_1 = 1$ and $F_2 = 2$.

Page 64

B-14 (Solution by Charles Wall, FQ, 1:4, Dec., 1963, pp. 79-80)

Since

$$\sum_{n=1}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$

let x = 0.1 in one case and (-0.1) in the other.

Page 69

Since Euler's famous formula gives $e^{i\pi} = -1$, the curious formula becomes just $\emptyset = 2 \cos \pi/5$, which is proved in the article referred to when the problem was posed.

B-18 (Solution by J. L. Brown, Jr., FQ, 2:1, Feb., 1964, pp. 74-75.)

It is well known (e. g., I. J. Schwatt, "An Introduction to the Operations with Series," Chelsea Pub. Co., p. 177) that $\cos \pi/5 = (1 + \sqrt{5})/4$ and $\sin \pi/10 = (\sqrt{5} - 1)/4$. Therefore, $a = (1 + \sqrt{5})/2 = 2 \cos \pi/5$ and $b = (1 - \sqrt{5})/2 = -2 \sin \pi/10$, and

$$F_n = \frac{a^n - b^n}{a - b} = 2^{n-1} \cdot \frac{\cos^n \frac{\pi}{5} - (-1)^n \sin^n \frac{\pi}{10}}{\cos \frac{\pi}{5} + \sin \frac{\pi}{10}}$$

$$= 2^{n-1} \sum_{k=0}^{n-1} (-1)^k \cos^{n-k-1} \frac{\pi}{5} \sin^k \frac{\pi}{10}$$

as stated. We have made use of the algebraic identity

$$\frac{x^n - y^n}{x - y} = \sum_{k=0}^{n-1} x^{n-k-1}y^k .$$

Page 85

H-19 (Solution by Michael Goldberg, FQ, 2:2, April, 1964, pp. 130-131)

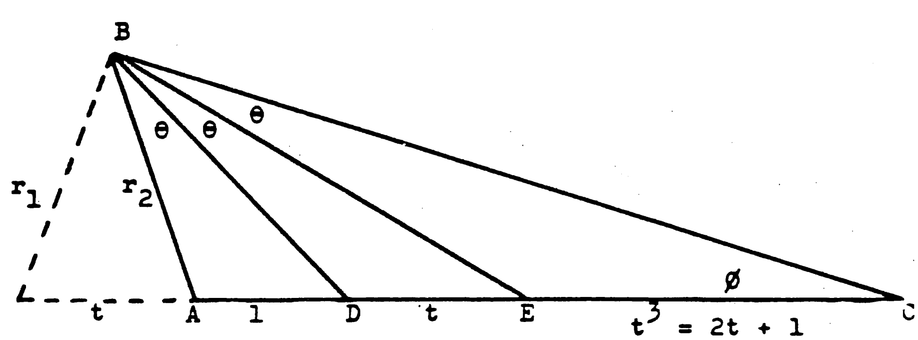
As $n \to \infty$, the ratio F_{n+1}/F_n approaches $t = (\sqrt{5} + 1)/2$, and F_{n+3}/F_n approaches $t^3 = 2t + 1$. Hence, the limiting triangle ABC can be drawn by taking points D and E on AC so that AD = 1, DE = t, and EC = 2t + 1. Since BD is a bisector of $\angle$ABE, the point B must lie on the circle which is the locus of points whose distances to A and E are in the ratio AD/DE = 1/t. The circle passes through D. If the diameter of the circle is $2r_1 = x + 1$, then $x/(x + 1 + t) = 1/t$ from which

$$r_1 = t/(t - 1) = t^2 = t + 1 .$$

Simarly, BE is a bisector of the angle DBC. The point B must lie on a circle which is the locus of points whose distances from D and C are in the ratio $DE/EC = t/t^3 = 1/t^2$. If the diameter of the circle is $2r_2 = y + t$, then $y/(y + t + t^2) = 1/t^2$ from which

$$r_2 = t^2 = t + 1 = r_1 .$$

Hence, $\cos \angle BAE = -t/2(t + 1) = -(\sqrt{5} - 1)/4$ and $\angle BAE = 108°$. From which $2\theta = 90° - 108°/2 = 36°$, $\theta = 18°$; $\emptyset = 180° - 108° - 3\theta = 18°$.



Page 145

B-39 (Solution by Brian Scott, FQ, 2:4, Dec., 1964, p. 327)

The solution is by induction on n. $F_{3+2} = F_5 = 5 < 8 = 2^3$ and $F_{4+2} = F_6 = 8 < 16 = 2^4$. Assume as the induction hypothesis that $F_{(n-2)+2} < 2^{n-2}$ and $F_{(n-1)+2} < 2^{n-1}$. Then

$$F_{n+2} = F_{(n-1)+2} + F_{(n-2)+2} < 2^{n-1} + 2^{n-2} = 2^{n-2}(2 + 1) < 2^{n-2}\cdot 2^2 = 2^n .$$

Therefore, $F_{n+2} < 2^n$ for all $n \geq 3$.

B-41(Solution by John L. Brown, Jr., FQ, 2:4, Dec., 1964, pp. 328-329.)

No. For, assume $F_i < F_j < F_h < F_k$ are in arithmetic progression, so that $F_j - F_i = d = F_k - F_h$. Then

$$d = F_j - F_i < F_j$$

while

$$d = F_k - F_h \geq F_k - F_{k-1} = F_{k-2} \geq F_j ,$$

since $k \geq j + 2$. This is a contradiction, so that four distinct positive Fibonacci numbers cannot be in arithmetic progression.

B-42 (Solution by H. H. Ferns)

The following three identities are readily proved by applying Binet's formula.

(1) $$2F_{n+1} = F_n + L_n$$

(2) $$L_n^2 - 5F_n^2 = 4(-1)^n$$

(3) $$2L_{n+1} = 5F_n + L_n$$

Eliminating L_n from (1) and (2) gives F_{n+1}, while eliminating F_n from (2) and (3) gives L_{n+1}:

$$F_{n+1} = \frac{F_n + \sqrt{5F_n^2 + 4(-1)^n}}{2} , \qquad L_{n+1} = \frac{L_n + \sqrt{5}\sqrt{L_n^2 - 4(-1)^n}}{2} .$$

B-44 (Solution by Douglas Lind, FQ, 3:1, February, 1965, p. 75)
Assume the maximum,

(1) $$n^k < F_{r+1}, F_{r+2}, \ldots, F_{r+n} < n^{k+1} .$$

Now

$$\sum_{j=1}^{n-1} F_{r+j} = \sum_{j=1}^{r+n-1} F_j - \sum_{j=1}^{r} F_j = F_{r+n+1} - F_{r+2} .$$

But by (1),

$$\sum_{j=1}^{n-1} F_{r+j} + F_{r+2} > n \cdot n^k$$

and hence

$$F_{r+n+1} > n^{k+1}$$

thus proving the proposition.

B-47 (Solution by Sidney Kravitz, FQ, 3:1, February, 1965, p. 77)
Let F_n be the n-th Fibonacci number. We note that $F_n > 1$ for $n > 2$, that F_j divides F_{mj}, and that j is a divisor of $(k + 2)! + j$ for $3 \leq j \leq k + 2$. Thus the k consecutive Fibonacci numbers $F_{(k+2)!+3}$, $F_{(k+2)!+4}$, $\ldots$, $F_{(k+2)!+k+2}$ are divisible by $F_3, F_4, \ldots, F_{k+2}$ respectively.

B-58 (Solution by Douglas Lind, FQ, 3:3, October, 1965, pp. 236-237)
Since $L_k = F_{k-1} + F_{k+1}$, the assertion is equivalent to

(1) $$F_n = F_{k-1} + F_{k+1} .$$

If $k \geq 3$, then $n > k + 1$, and (1) is clearly impossible since

$$F_{k-1} + F_{k+1} < F_k + F_{k+1} = F_{k+2} \leq F_n .$$

Impossibility for $k \geq 3$ implies impossibility for $k \leq -3$ since only signs are different. For $-3 < k < 3$ we find $F_{-2} = L_{-1} = 1$, $F_3 = L_0 = 2$, $F_1 = L_1 = 1$, and $F_4 = L_2 = 3$, corresponding to $k = -1, 0, 1$, and 2 respectively. Hence these are the only solutions.

B-62 (Solution by J. L. Brown, Jr., FQ, 3:3, October, 1965, p. 239)

From the identity $F_{2n+1} = F_n^2 + F_{n+1}^2$, $(n \geq 1)$ it follows that $F_{2n+1} < (F_n + F_{n+1})^2 = F_{n+2}^2$. Therefore, any representation $F_{2n+1} = F_k^2 + F_m^2$ $(k \leq m)$ must have both k and $m \leq n + 1$. Then $k \geq n$, for otherwise $F_k^2 + F_m^2 < F_n^2 + F_{n+1}^2 = F_{2n+1}$ for $k > 2$.

B-95 (Solution by Charles W. Trigg, FQ, 5:2, April, 1967, p. 204)

For $n \geq 3$, F_k is divisible by 2^n if k is of the form $2^{n-2} \cdot 3(1 + 2m)$. F_k is divisible by 2^n but by no higher power of 2. Hence, the highest power of 2 that exactly divides $F_1 F_2 F_3 \cdots F_{100}$ is

$$[103/6] + 3[106/12] + 4[112/24] + 5[124/48] + 6[148/96] + 7[196/192]$$

or 80. As usual, [x] indicates the largest integer in x.

(Editorial note: The results in the above solution indicate that the answer may also be expressed as

$$[100/3] + 2[100/6] + [100/12] + [100/24] + [100/48] + [100/96]$$
$$= 33 + 32 + 8 + 4 + 2 + 1 = 80.)$$

H-2 This was a world famous problem. J. H. E. Cohn proved the truth of the conjecture in "Square Fibonacci Numbers, Etc.," Fibonacci Quarterly, Vol. 2, No. 2, April, 1964, pp. 109-113. Also, it was established that $L_1 = 1$ and $L_3 = 4$ are the only Lucas numbers which are perfect squares.