The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF
THE FIBONACCI ASSOCIATION

VOLUME 15 NUMBER 2

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APRIL 1977

The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

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The Quarterly is entered as third-class mail at the University of Santa Clara Post Office, California, as an official publication of the Fibonacci Association.

The Quarterly is published in February, April, October, and December, each year.

Typeset by
HIGHLANDS COMPOSITION SERVICE
P. O. Box 760
Clearlake Highlands, Calif. 95422

PROPERTIES OF SOME FUNCTIONS SIMILAR TO LUCAS FUNCTIONS

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1. INTRODUCTION

The ordinary Lucas functions are defined by

$$v_n = a_1^n + a_2^n, \qquad u_n = (a_1^n - a_2^n)/(a_1 - a_2), \tag{1.1}$$

where a_1, a_2 are the roots of

$$x^2 = Px - Q,$$

$\Delta = (a_1 - a_2)^2 = P^2 - 4Q$, and P, Q are coprime integers. These functions and their remarkable properties have been discussed by many authors. The best known works are those of Lucas [7] and Carmichael [3]. Lehmer [6] has dealt with a more general form of these functions for which $P = \sqrt{R}$ and R, Q are coprime integers.

Bell [1] attributed the existence of the many properties of the Lucas functions to the simplicity of the functions' form. He added, "this simplicity vanishes, apparently irrevocably, when we pass beyond second order series." The purpose of this paper is to define a set of third order functions W_n, V_n, U_n, and to show that these functions possess much of the "arithmetic fertility" of the Lucas functions.

Consider first the functions v_n and u_n, which are defined in the following manner. We let ρ_1, ρ_2 be the roots of

$$x^2 = rx + s$$

and

$$2a_1 = v_1 + u_1\rho_1, \qquad 2a_2 = v_1 + u_1\rho_2,$$

where s, r, v_1, u_1 are given integers. We then put

$$v_n = \frac{2}{\delta}\begin{vmatrix} a_1^n & \rho_1 \\ a_2^n & \rho_2 \end{vmatrix}, \qquad u_n = \frac{2}{\delta}\begin{vmatrix} 1 & a_1^n \\ 1 & a_2^n \end{vmatrix},$$

where

$$\delta = \begin{vmatrix} 1 & \rho_1 \\ 1 & \rho_2 \end{vmatrix}.$$

If we select values for s, r, v_1, u_1 such that v_n, u_n are both integers for all non-negative integer values of n, then $P = a_1 + a_2$ and $Q = a_1a_2$ will be integers. If we further restrict our choices of values for r, s, v_1, u_1 such that $(P, Q) = 1$, then it can be easily shown that the resulting functions v_n and u_n have many properties analogous to those of the ordinary Lucas functions. Indeed, if we select $s = \Delta$, $r = 0$, $v_1 = P$, $u_1 = 1$, the functions u_n and v_n are the functions given by (1.1).

In this paper we shall be concerned with the third order analogues of the above functions. We let ρ_1, ρ_2, ρ_3 be the roots of

$$x^3 = rx^2 + sx + t \quad \text{and} \quad 3a_i = W_1 + V_1\rho_i + U_1\rho_i^2 \quad (i = 1, 2, 3),$$

where r, s, t, W_1, V_1, U_1 are given integers. We define

$$W_n = \frac{3}{\delta}\begin{vmatrix} a_1^n & \rho_1 & \rho_1^2 \\ a_2^n & \rho_2 & \rho_2^2 \\ a_3^n & \rho_3 & \rho_3^2 \end{vmatrix}, \quad V_n = \frac{3}{\delta}\begin{vmatrix} 1 & a_1^n & \rho_1^2 \\ 1 & a_2^n & \rho_2^2 \\ 1 & a_3^n & \rho_3^2 \end{vmatrix}, \quad U_n = \frac{3}{\delta}\begin{vmatrix} 1 & \rho_1 & a_1^n \\ 1 & \rho_2 & a_2^n \\ 1 & \rho_2 & a_3^n \end{vmatrix}, \tag{1.3}$$

where

$$\delta = \begin{vmatrix} 1 & \rho_1 & \rho_1^2 \\ 1 & \rho_2 & \rho_2^2 \\ 1 & \rho_3 & \rho_3^2 \end{vmatrix} \neq 0.$$

We also put $P = a_1 + a_2 + a_3$, $Q = a_1a_2 + a_2a_3 + a_3a_1$, $R = a_1a_2a_3$, $\Delta = \delta^2$.

Let N be the set of positive integers. If we restrict the values of r, s, t, W_1, V_1, U_1 such that

(1) W_n, V_n, U_n are all integers for any $n \in N$,

(2) P, Q, R are integers and $(P,Q,R) = 1$,

(3) there exists $\mu \in N$ such that $U_i \equiv U_{i+k\mu}$ (mod 3) for all $i, k \in N$,

the functions W_n, V_n, U_n have several characteristics similar to those of the Lucas functions. Functions similar to W_n, V_n, U_n have been discussed by Williams [10] and [11, $(q = 3)$], but for these functions $r = s = 0$, $\Delta = t$.

Conditions (1) and (2) are analogous to the two restrictions placed on the functions of (1.2). These two restrictions guarantee that there exists an integer $m \in N$ such that $u_i \equiv u_{i+km}$ (mod 2) for any $i, k \in N$; however, we shall see that conditions (1) and (2) do not imply (3).

It is necessary to demonstrate what the conditions on r, s, t, W_1, V_1, U_1 are such that (1), (2), (3) are true. In order to do this, we require several identities satisfied by W_n, V_n and U_n. These identities, which are independent of (1), (2), (3), are given in Section 2.

2. IDENTITIES

It is not difficult to see from (1.3) that

$$3^{n-1}(W_n + \rho V_n + \rho^2 U_n) = (W_1 + \rho V_1 + \rho^2 U_1)^n, \tag{2.1}$$

where $\rho = \rho_1, \rho_2, \rho_3$. It follows that

$$\begin{aligned} 3W_{n+m} &= W_nW_m + tV_nU_m + tU_nV_m + trU_mU_n, \\ 3V_{n+m} &= V_nW_m + W_nV_m + sV_mU_n + sV_nU_m + (rs + t)U_nU_m. \\ 3U_{n+m} &= W_mU_n + W_nU_m + V_nV_m + rU_mV_n + rU_nV_m + (r^2 + s)U_nU_m, \end{aligned} \tag{2.2}$$

$$\begin{aligned} 3W_{2m} &= W_m^2 + 2tV_mU_m + trU_m^2, \\ 3V_{2m} &= (sr + t)U_m^2 + 2sV_mU_m + 2V_mU_m, \\ 3U_{2m} &= V_m^2 + 2W_mU_n + 2rU_mV_m + (r^2 + s)U_m^2, \end{aligned} \tag{2.3}$$

$$\begin{aligned} 9W_{3m} &= W_m^3 + tV_m^3 + t(r^2 + 2rs + t)U_m^3 + 6tW_mV_mU_m \\ &\quad + 3trW_mU_m^2 + 3trU_mV_m^2 + 3t(r^2 + s)U_m^2V_m \\ 9V_{3m} &= sV_m^3 + (sr + t)(r^2 + 2s)U_m^3 + 6sW_mV_mU_m + 3V_mW_m^2 \\ &\quad + 3(sr + t)W_mU_m^2 + 3(t + rs)U_mV_m^2 + 3(s^2 + sr^2 + t)U_m^2V_m \\ 9U_{3m} &= rV_m^3 + (r^4 + 3r^2s + s^2 + 2tr)U_m^3 + 6rW_mV_mU_m + 3U_mW_m^2 \\ &\quad + 3W_mV_m^2 + 3(r^2 + s)W_mU_m^2 + 3(r^2 + 2rs + t)U_m^2V_m, \end{aligned} \tag{2.4}$$

$$\begin{aligned} 3R^mW_{-m} &= W_m^2 + rW_mV_m + (r^2 + 2s)W_mU_m - sV_m^2 - (rs + t)U_mV_m + (s^2 - rt)U_m^2, \\ 3R^mV_{-m} &= -W_mV_m - rV_m^2 - r^2U_mV_m + (rs + t)U_m^2, \\ 3R^mU_{-m} &= -W_mU_m + V_m^2 + rU_mV_m - sU_m^2. \end{aligned} \tag{2.5}$$

By using methods similar to those of Williams [12], we can show that

$$9R^m W_{n-m} = \begin{vmatrix} W_n & tU_m & tV_m + rtU_m \\ V_n & W_m + sU_m & sV_m + (rs+t)U_m \\ U_n & V_m + rU_m & W_m + rV_m + (r^2+s)U_m \end{vmatrix},$$

(2.6)
$$9R^m V_{n-m} = \begin{vmatrix} W_m & W_n & tV_m + rtU_m \\ V_m & V_n & sV_m + (rs+t)U_m \\ U_m & U_n & W_m + rV_m + (r^2+s)U_m \end{vmatrix},$$

$$9R^m U_{n-m} = \begin{vmatrix} W_m & tU_m & W_n \\ V_m & W_m + sU_m & V_n \\ U_m & V_n + rU_m & U_n \end{vmatrix},$$

(2.7)
$$27R^m = \begin{vmatrix} W_m & tU_m & tV_m + rtU_m \\ V_m & W_m + sU_m & sV_m + (rs+t)U_m \\ U_m & V_m + rU_m & W_m + rV_m + (r^2+s)U_m \end{vmatrix},$$

(2.8)
$$\begin{vmatrix} W_n & V_n & U_n \\ W_{n+m} & V_{n+m} & U_{n+m} \\ W_{n+2m} & V_{n+2m} & U_{n+2m} \end{vmatrix} = R^n N_m,$$

(2.9)
$$27\begin{vmatrix} W_n & W_{n+m} & W_{n+2m} \\ W_{n+m} & W_{n+2m} & W_{n+3m} \\ W_{n+2m} & W_{n+3m} & W_{n+4m} \end{vmatrix} = -R^n t^2 N_m^2,$$

$$27\begin{vmatrix} V_n & V_{n+m} & V_{n+2m} \\ V_{n+m} & V_{n+2m} & V_{n+3m} \\ V_{n+2m} & V_{n+3m} & V_{n+4m} \end{vmatrix} = -R^n (rs+t) N_m^2,$$

$$27\begin{vmatrix} U_n & U_{n+m} & U_{n+2m} \\ U_{n+m} & U_{n+2m} & U_{n+3m} \\ U_{n+2m} & U_{n+3m} & U_{n+4m} \end{vmatrix} = -R^n N_m^2,$$

where

$$N_m = 3\begin{vmatrix} V_m & U_m \\ V_{2m} & U_{2m} \end{vmatrix} = (V_m + rU_m)^3 - rU_m(V_m + rU_m)^2 - sU_m^2(V_m + rU_m) - tU_m^3.$$

Let

$$P_m = a_1^m + a_2^m + a_3^m, \quad Q_m = a_1^m a_2^m + a_2^m a_3^m + a_3^m a_1^m, \quad R_m = a_1^m a_2^m a_3^m = R^m.$$

From (2.1) and (2.7), we have

(2.10)
$$3P_m = 3W_m + rV_m + (r^2 + 2s)U_m,$$

(2.11)
$$9Q_m = 3W_m^2 + 2rV_m U_m + (2r^2 + 4s)U_m W_m - sV_m^2 - (sr + 3t)U_m V_m + (s^2 - 2tr)U_m^2$$

(2.7)
$$27R_m = W_m^3 + tV_m^3 + t^2 U_m^3 - (3t + rs)W_m V_m U_m + rW_m^2 V_m - sV_m^2 W_m$$
$$+ (2s + r^2)W_m^2 U_m + (s^2 - 2rt)W_m U_m^2 + trV_m^2 U_m - tsV_m U_m^2.$$

If

$$\epsilon_m = \begin{vmatrix} 1 & a_1^m & a_1^{2m} \\ 1 & a_2^m & a_2^{2m} \\ 1 & a_3^m & a_3^{2m} \end{vmatrix}$$

and $E_m = \epsilon_m^2$, then

(2.14) $$27\epsilon_m = -27R^{2m}\epsilon_{-m} = \delta N_m$$

and

(2.15) $$3^6 E_m = \Delta N_m^2 .$$

It should be noted that

(2.16) $$E_m = P_m^2 Q_m^2 + 18P_m Q_m R_m - 4Q_m^3 - 4P_m^3 R_m - 27R_m^2$$

and

$$\Delta = r^2s^2 - 18rst + 4s^3 - 4r^3t - 27t^2 .$$

If

$$F(x,y) = x^3 - rx^2y - sxy^2 - ty^3 ,$$

we see from (2.14) and (2.5), that

$$R^m F(V_m + rU_m, U_m) = F\{(tU_m^2 - rW_mU_m - W_mV_m)/3, (-W_mU_m + V_m^2 + rU_mV_m - sU_m^2)/3\}.$$

If W_1, V_1, U_1 are selected such that $W_1 = 3a$, $V_1 = 3b$, $U_1 = 3c$, where a, b, c are integers and $a + \rho_1 b + \rho_1^2 c$ is a unit of the cubic field generated by adjoining ρ_1 to the rationals, we can obtain an infinitude of integer solutions of the Diophantine equation

$$F(x,y) = F(z,w).$$

If we define

$$Z_n = \frac{1}{\delta}\begin{vmatrix} 1 & \rho_1 & \rho_1^n \\ 1 & \rho_2 & \rho_2^n \\ 1 & \rho_3 & \rho_3^n \end{vmatrix},$$

then (Bell [1])

$$\rho^n = (Z_{n+2} - rZ_{n+1} - sZ_n) + (Z_{n+1} - rZ_n)\rho + Z_n\rho^2 ,$$

where $\rho = \rho_1, \rho_2, \rho_3$. Using this result together with (2.1), we obtain

(2.17) $$3^{m-1}W_{nm} = \sum_{i,j} \frac{m!}{i!\,j!\,(m-i-j)!}(Z_{2j+i+2} - rZ_{2j+i+1} - sZ_{2j+1})W_n^{m-i-j}V_n^iU_n^j ,$$

$$3^{m-1}V_{nm} = \sum_{i,j} \frac{m!}{i!\,j!\,(m-i-j)!}(Z_{2j+i+1} - rZ_{2j+1})W_n^{m-i-j}V_n^iU_n^j ,$$

$$3^{m-1}U_{nm} = \sum_{i,j} \frac{m!}{i!\,j!\,(m-i-j)!} Z_{2j+i}W_n^{m-i-j}V_n^iU_n^j ,$$

where the sum is taken over integers $i, j \geqslant 0$ such that $0 \leqslant i + j \leqslant m$.

Finally, it should be noted that for a fixed value of n, each of $W_{n+km}, V_{n+km}, U_{n+km}$ can be represented as a linear combination of the kth powers of the roots of the equation

$$x^3 = P_m x^2 - Q_m x + R_m ;$$

consequently, we have

(2.18)
$$W_{n+(k+3)m} = P_m W_{n+(k+2)m} - Q_m W_{n+(k+1)m} + R_m W_{n+km},$$
$$V_{n+(k+3)m} = P_m V_{n+(k+2)m} - Q_m V_{n+(k+1)m} + R_m V_{n+km},$$
$$U_{n+(k+3)m} = P_m U_{n+(k+2)m} - Q_m U_{n+(k+1)m} + R_m U_{n+km}.$$

The identities (2.1), (2.2), (2.6), (2.7), (2.9), (2.17), (2.18) are analogous to Lucas' important identities (7), (49), (51), (46), (32) and (33), (49), and (13), respectively.

3. PRELIMINARY RESULTS

We will now show how to obtain values for r, s, t, W_1, V_1, U_1 in such a way that W_n, V_n, U_n are integers for any $n \in N$. We require two lemmas.

Lemma 1. If W_n, V_n, U_n are integers for all $n \in N$, then P, Q, R are integers and one of the following is true.

(i) $3 \mid (W_1, V_1, U_1)$†
(ii) $3 \mid W_1, 3 \nmid U_1, V_1 \equiv -rU_1 \pmod 3, 3 \mid t$, and $3 \mid s$
(iii) $3 \mid W_1, 3 \nmid U_1, V_1 \equiv rU_1 \pmod 3, 3 \mid t$ and $r^2 + s \equiv 0 \pmod 3$
(iv) $3 \nmid W_1, 3 \mid V_1, 3 \nmid U_1, W_1 \equiv -U_1 \pmod 3, s \equiv 1 \pmod 3$, and $t \equiv -r \pmod 3$
(v) $3 \nmid W_1 V_1 U_1, W_1 \equiv U_1 \pmod 3, V_1 \equiv tU_1 \pmod 3, 3 \mid s, 3 \mid r$, and $3 \nmid t$

Proof. Since W_2, V_2, U_2 are integers, it follows from (2.3) that one of the cases (i), (ii), (iii), (iv) or (v) must be true. In each of these cases, we see that

$$rV_1 + (r^2 + 2s)U_1 \equiv 0 \pmod 3;$$

hence, P is an integer.

Now, from (2.18) and the fact that $V_0 = U_0 = 0$, we have

$$V_3 = PV_2 - QV_1,$$
$$U_3 = PU_2 - QU_1;$$

thus, QV_1 and QU_1 are both integers. Since $9Q$ is an integer, we see that Q is an integer if $3 \nmid V_1$ or $3 \nmid U_1$. If $3 \mid (V_1, U_1)$, then it is clear from (2.11) that Q is an integer. Using the equations

$$V_4 = PV_3 - QV_2 + RV_1, \qquad U_4 = PU_3 - QU_2 + RU_1$$

and (2.7), we can show that R must also be an integer.

Lemma 2. If the conditions of (i) of Lemma 1 are true, Q and R are integers.
If the conditions of (ii) hold, Q and R are integers if and only if $9 \mid t$.
If the conditions of (iii) hold, Q and R are integers if and only if $t \equiv r(s - 2r^2) \pmod 9$.
If the conditions of (iv) hold, Q and R are integers if and only if $s \equiv 1 - tr - r^2 \pmod 9$.
If the conditions of (v) hold, Q and R are integers if and only if $s \equiv t^2 - 1 - tr \pmod 9$.

Proof. The proof of the first statement of the lemma is clear from Eqs. (2.11) and (2.7). We show how the other statements can be proved by demonstrating the truth of the fourth statement. (The proofs of the others are similar.)

We write

$$W_1 = -U_1 + 3L, \quad V_1 = 3K,$$

where L, K are integers. Substituting these values for W_1 and V_1 in (2.11), we get

$$9Q \equiv 2U_1^2[1 - s - tr - r^2] \pmod 9.$$

Hence, Q is an integer if and only if

$$s \equiv 1 - tr - r^2 \pmod 9.$$

---

†If $x, y, z, \cdots$ are rational integers, we write as usual $x \mid y$ for x divides y, $x \nmid y$ for x does not divide y, and $(x, y, z, \cdots)$ for the greatest common divisor of $x, y, z, \cdots$. We also write $y^n \| x$ to indicate that $y^n \mid x$ and $y^{n+1} \nmid x$.

Assuming that Q is an integer and repeating the above method using (2.7), we get

$$27R \equiv [-1 + t^2 + 2s + r^2 - s^2 + 2rt]\,U_1^3 \pmod{27}.$$

Thus,

$$3R \equiv ((t + r)/3 - (s - 1)/3)((t + r)/3 + (s - 1)/3)U_1^3 \pmod 3.$$

Since $(s - 1)/3 \equiv r(t + r)/3$ and $3 \nmid r$, we see that R is an integer if Q is.

The answer to the problem of this section is given as

Theorem 1. W_n, V_n, U_n are integers for any $n \in N$ if and only if one of the following is true.

(a) $3 \mid (W_1, V_1, U_1)$
(b) $3 \mid W_1, 3 \nmid U_1, V_1 \equiv -rU_1 \pmod 3, 3 \mid s, 9 \mid t$
(c) $3 \mid W_1, 3 \nmid U_1, V_1 \equiv rU_1 \pmod 3, 3 \nmid s, r^2 + s \equiv 0 \pmod 3, t \equiv r(s - 2r^2) \pmod 9$
(d) $3 \nmid W_1, 3 \mid V_1, 3 \nmid U_1, W_1 \equiv U_1 \pmod 3, s \equiv 1 \pmod 3, t \equiv -r \pmod 3, s \equiv 1 - tr - r^2 \pmod 9$
(e) $3 \nmid W_1 V_1 U_1, W_1 \equiv U_1 \pmod 3, V_1 \equiv tU_1 \pmod 3, 3 \mid s, 3 \mid r, 3 \nmid t, s \equiv t^2 - 1 - tr \pmod 9$.

Proof. By Lemmas 1 and 2, one of the above conditions is necessary in order for W_n, V_n, U_n to be integers for any $n \in N$. To show sufficiency of the conditions, we note that in each case W_2, V_2, U_2, P, Q, R are integers. The fact that W_n, V_n, U_n are integers for any $n \in N$ follows by induction on (2.18).

Corollary. Let $n \in N$.

If the conditions of (a) are true,

$$W_n \equiv V_n \equiv U_n \equiv 0 \pmod 3.$$

If the conditions of (b) hold,

$$W_n \equiv 0, \quad V_n \equiv -rU_n \pmod 3.$$

If the conditions of (c) hold,

$$W_n \equiv 0, \quad V_n \equiv rU_n \pmod 3.$$

If the conditions of (d) hold,

$$W_n \equiv -U_n, \quad V_n \equiv 0 \pmod 3.$$

If the conditions of (e) hold,

$$W_n \equiv U_n, \quad V_n \equiv tU_n \pmod 3.$$

Proof. These results are easily verified for $n = 2$. The results for general $n \in N$ follow by using induction on (2.18).

For the sake of brevity, we shall say that the functions W_n, V_n, U_n are given by (a), (b), (c), (d), or (e) if W_1, V_1, U_1, r, s, t obey the conditions of the cases (a), (b), (c), (d), or (e) above. From this point on, we consider only those functions W_n, V_n, U_n which are given by one of these cases.

4. CONGRUENCE PROPERTIES MODULO 3

Since $3 \mid (W_n, V_n, U_n)$ for W_n, V_n, U_n given by (a), we will confine ourselves here to an investigation of the congruence properties (mod 3) of W_n, V_n, U_n when they are given by (b), (c), (d) or (e). In each of these cases, $9 \mid \Delta$ and we let $H = \Delta/9$. From the corollary to Theorem 1, we see that it is sufficient to discuss U_n only.

We define μ to be the least positive integer such that

$$U_i \equiv U_{i+k\mu} \pmod 3$$

for all $i, k \in N$. We further define

$$B = \{X_1, X_2, \cdots, X_\mu\},$$

where $U_i \equiv U_1 X_i \pmod 3$.

Lemma 3. For W_n, V_n, U_n given by (b), (c), (d) or (e), μ and B are determined from the following results.

Case (i) $3 \nmid Pr$. The values of μ, R (mod 3), B are functions of the values of H and Q (mod 3). These values (mod 3) are given in Table 1.

Table 1

H	Q	μ	R	B
1	$0, 1$	2	0	$\{1, (Q+1)P\}$
1	-1	2	P	$\{1, 0\}$
-1	1	4	P	$\{1, 0, -1, 0\}$
-1	$0, -1$	$4, 8$	$P(1+Q)$	$\{1, (Q-1)P, -1, 0, -1, -(Q-1)P, 1, 0\}$
0	P	6	$P-1$	$\{1, 1, 0, -1, -1, 0\}$
0	$-P$	3	$P+1$	$\{1, -1, 0\}$

Case (ii). $3 \nmid P$, $3 \mid r$

In this case, $\mu = 2$, $R \equiv PQ(Q-1) \pmod 3$, and $B = \{1, P+PQ\}$.

Case (iii). $3 \mid P$

In this case, $Q \equiv -H \pmod 3$ and the value of R is independent of Q and H. The values of μ and B are given in Table 2.

Table 2

H	Q	R	μ	B
0	0	$-F$	6	$\{1, F, 0, F, -1, 0\}$
-1	1	$F \equiv 0$	4	$\{1, 0, -1, 0\}$
-1	1	$F \not\equiv 0$	8	$\{1, F, -1, 0, -1, -F, 1, 0\}$
1	-1	0	2	$\{1, F\}$

Here

$$F = (-W_1 + sU_1)/3 \quad \text{for } W_n, V_n, U_n \text{ given by (b)},$$
$$F = (-W_1 + rV_1 + (tr - 3 - r^2)U_1)/3 \quad \text{for } W_n, V_n, U_n \text{ given by (c)},$$
$$F = (-W_1 + rV_1 - sU_1)/3 \quad \text{for } W_n, V_n, U_n \text{ given by (d)},$$

and

$$F = (-W_1 - tV_1 + (s+2)U_1)/3 \quad \text{for } W_n, V_n, U_n \text{ given by (e)}.$$

Proof. For W_n, V_n, U_n given by (b), put

$$W_1 = 3L, \quad V_1 = -rU_1 + 3K, \quad a = s/3, \quad b = rt/9, \quad A_1 = L + rK + aU_1, A_2 = L, \quad A_3 = L + aU_1.$$

Then it can be shown by substitution into (2.10), (2.11), (2.7), that

$$P \equiv A_1 + A_2 + A_3, \quad Q \equiv A_1A_2 + A_2A_3 + A_3A_1 + b, \quad R \equiv A_1A_2A_3 + bA_1,$$
$$A_2A_3 \equiv (A_2 + A_3)^2 - (A_2 - A_3)^2 \equiv (A_2 + A_3)^2 - a^2 \pmod 3.$$

Also, if $3 \nmid r$, $H \equiv a^2 - b \pmod 3$ and if $3 \mid r$, $H \equiv 0 \pmod 3$. Hence, if $3 \nmid r$,

$$Q \equiv P(A_2 + A_3) - H \pmod 3$$
$$R \equiv \begin{cases} P(Q-H)(Q+H-1) \pmod 3 & \text{when } 3 \nmid P \\ (A_2+A_3)(H-1) \pmod 3 & \text{when } 3 \mid P, \end{cases}$$
$$U_2 \equiv U_1(A_2 + A_3) \equiv U_1(PQ + PH) \quad \text{when } 3 \nmid P.$$

If $3 \mid r$,

$$P \equiv 2aU_1 \pmod 3$$
$$Q \equiv P(A_2 + A_3) - a^2 \pmod 3$$
$$R \equiv \begin{cases} PQ(Q-1) \pmod 3 & \text{when } 3 \nmid P \\ -(A_2+A_3) \pmod 3 & \text{when } 3 \mid P \end{cases}$$
$$U_2 \equiv (A_2 + A_3)U_1 \equiv P(Q+1)U_1 \quad \text{when } 3 \nmid P.$$

The proof of the lemma for W_n, V_n, U_n given by (b) follows by using induction on (2.18).

For (c), put

$$W_1 = 3L,\quad V_1 = rU_1 + 3K,\quad a = rt/3 - 1,\quad b = r(t - r(s - 2r^2))/9,\quad A_1 = L + (a+1)U_1,$$
$$A_2 = L - rK,\quad A_3 = L + 2rK + aU_1.$$

Then

$$H \equiv a^2 - b,\quad P \equiv A_1 + A_2 + A_3,\quad Q \equiv A_1A_2 + A_2A_3 + A_3A_1 + b,$$
$$R \equiv A_1A_2A_3 + bA_1,\quad (A_2 - A_3)^2 \equiv a^2,\quad A_2A_3 \equiv (A_2 + A_3)^2 - a^2 \pmod 3.$$

For (d), put

$$W_1 = U_1 + 3L,\quad V_1 = 3K,\quad a = rK,\quad b = r(t + sr)/9,\quad A_1 = L + rK + U_1(r^2 - 1)/3,$$
$$A_2 = L + \sqrt{s}\,K + U_1(s - 1)/3,\quad A_3 = L - \sqrt{s}\,K + U_1(s - 1)/3.$$

Then

$$H \equiv (a - P)^2 - b,\quad P \equiv A_1 + A_2 + A_3,\quad Q \equiv A_1A_2 + A_1A_3 + A_2A_3 + b,$$
$$R \equiv A_1A_2A_3 - b(a + A_2 + A_3),\quad (A_2 - A_3)^2 \equiv a^2,\quad A_2A_3 \equiv (A_2 + A_3)^2 - a^2 \pmod 3.$$

For (e), put

$$V_1 = tU_1 + 3K,\quad W_1 = U_1 + 3L,\quad A_1 = L + tK + U_1(1 + 2t^2)/3,$$
$$A_2 = L + \beta_1 K + \beta_1 U_1 r/3,\quad C = L + \beta_2 K + \beta_2 U_1 r/3,$$

where β_1, β_2 are the zeros of $x^2 + (t - r)x + 1$. Then $H \equiv 0$, $P \equiv A_1 + A_2 + A_3$.

$$Q \equiv A_1A_2 + A_3A_1 + A_2A_3,\quad R \equiv A_1A_2A_3,\quad (A_2 - A_3)^2 \equiv 0,\quad A_2A_3 \equiv (A_2 + A_3)^2 \pmod 3.$$

The remainder of the proof of this lemma for W_n, V_n, U_n given by (c), (d), or (e) can now be obtained in the same way as that for W_n, V_n, U_n given by (b).

Corollary. If $n \in N$, $3 \mid U_n$ if and only if $\psi \mid n$, where ψ is the least positive integer value for m such that $3 \mid U_m$. From the statement of Lemma 3, it is clear that $\psi = \mu$, $\mu/2$ or no value for ψ exists.

In the statement of Lemma 3, we have neglected the case for which $3 \nmid Pr$, $3 \mid Q$ and $3 \mid H$. In this case, it can be shown that μ does not exist. By the definition of W_n, V_n, U_n, we exclude this case; hence, we may not have values of r, s, t, W_1, V_1, U_1 such that $3 \nmid Pr$, $3 \mid F$, $27 \mid \Delta$ for W_n, V_n, U_n given by (b) or (c) or values of r, s, t, W_1, V_1, U_1 such that $3 \nmid P$, $3 \mid (F + P)$, $27 \mid \Delta$ for W_n, V_n, U_n given by (d).

We have now found the conditions on r, s, t, W_1, V_1, U_1 in order that the functions W_n, V_n, U_n satisfy the requirements (1) and (3) of Section 1. We give the conditions for $(P,Q,R) = 1$ ((2) of Section 1) in Section 5.

5. FURTHER RESTRICTIONS ON r, s, t, W_1, V_1, U_1

It is not immediately clear how to select r, s, t, W_1, V_1, U_1 In order that $(P,Q,R) = 1$. We show how such selections may be made in

Theorem 2. Let $3G = (2r^2 + 6s)V_1 + (2r^3 + 7rs + 9t)U_1$.

1. If W_n, V_n, U_n are given by (a), $(P,Q,R) = 1$ if and only if $(W_1, V_1, U_1) = 3$ and $(P,G,\Delta) = 2^\alpha 3^\beta$, where $\alpha > 0$ only if $2 \nmid (s + r)(V_1 + U_1)$ and $\beta > 0$ only if none of the following is true.
   (i) $3 \mid r$ and $W_1 + tV_1 + t^2U_1 \equiv 0 \pmod 9$
   (ii) $3 \nmid r$, $s \equiv 1 \pmod 3$, and $W_1 + tU_1 \equiv 0 \pmod 9$
   (iii) $3 \nmid r$, $3 \mid s$, and $W_1(W_1 + rV_1 + U_1) \equiv 0 \pmod{27}$
   (iv) $3 \nmid r$, $s \equiv -1 \pmod 3$, and $9 \mid W_1$.
2. If W_n, V_n, U_n are given by (b), (c), (d), or (e), then $(P,Q,R) = 1$ if and only if $(W_1, V_1, U_1) = 1$ and $(P,G,H) = 2^\alpha 3^\gamma$, where $\alpha > 0$ only if $2 \nmid (s + r)(V_1 + U_1)$ and $\gamma > 0$ only if $3 \nmid F$.

Proof. We first prove the necessity of the conditions of the theorem.

If p ($\neq 3$) is a prime and $p \mid (W_1, V_1, U_1)$, then it is clear from (2.10), (2.11), and (2.7) that $p \mid (P,Q,R)$. If $9 \mid (W_1, V_1, U_1)$, then $3 \mid (P,Q,R)$. Hence, if $(P,Q,R) = 1$, $(W_1, V_1, U_1) \mid 3$.

Now, suppose that p ($\neq 3$) is a prime divisor of (P,G,Δ). Since

$$3W_1 \equiv -rV_1 - (r^2+2s)U_1 \pmod p,$$

we have

$$-27Q \equiv (r^2+3s)V_1^2 + (2r^3+7rs+9t)U_1V_1 + (r^4+4sr^2+6tr+s^2)U_1^2 \pmod p.$$

Since

$$-3\Delta = (2r^3+7rs+9t)^2 - 4(r^2+3s)(r^4+4sr^2+6tr+s^2), \tag{5.1}$$

we see that

$$27\cdot 4\cdot(r^2+3s)Q \equiv 9G^2 \pmod p.$$

If $p\nmid 2(r^2+3s)$, then $p\mid Q$. If $p\mid(r^2+3s)$, then, from (5.1), $p\mid(2r^3+7rs+9t)$. As a consequence of these two facts, we deduce that $p\mid(rs+9t)$ and $p\mid(3tr-s^2)$; thus, $p\mid(r^4+4sr^2+6tr+s^2)$ and $p\mid Q$. Combining (2.15) and (2.16), we get

$$27^2(P^2Q^2+18PQR-4Q^3-4P^3R-27R^2) = \Delta N_1^2;$$

consequently, if $p\mid(Q,P,\Delta)$ and $p\neq 3$, then $p\mid R$. Thus, if $(P,Q,R)=1$, then $(P,G,\Delta)=2^\alpha 3^\beta$, $((P,G,H)=2^\alpha 3^\gamma)$. If $2\mid(P,G,\Delta)$ and $(P,Q,R)=1$, then $2\nmid Q$. Q is odd if and only if $(s+q)(V_1+U_1)$ is. If $3\mid(P,G,\Delta)$ (or $3\mid(P,G,H)$) and $(P,Q,R)=1$, then $3\nmid(Q,R)$. We will show the conditions under which $3\nmid(Q,R)$ for part 1 of the theorem only. The conditions for part 2 are quite easy to obtain from results used in the proof of Lemma 3.

Since $3\mid P$ and $3\mid\Delta$, we have

$$rV_1/3 \equiv -(r^2+2s)U_1/3 \pmod 3 \quad\text{and}\quad r^2s^2+s \equiv rt \pmod 3.$$

We now deal with four cases.

(i) $\underline{3\mid r}$. If $3\mid r$, then $3\mid s$ and $3\mid Q$. Hence $3\nmid(Q,R)$ if and only if $3\nmid(W_1/3+tV_1/3+t^2U_1/3\,.)$

(ii) $\underline{3\nmid r,\ s\equiv 1 \pmod 3}$. Here we have $9\mid V_1$ and $tr\equiv -1 \pmod 3$; thus, $s^2-2tr\equiv 0 \pmod 3$ and $3\mid Q$. Hence, $3\nmid(Q,R)$ if and only if $3\nmid(W_1/3+tU_1/3)$.

(iii) $\underline{3\nmid r,\ 3\mid s}$. We must have $3\mid t$ and $3\mid Q$. (R,Q) is not divisible by 3 if and only if $9\nmid W_1(W_1/3+rV_1/3+U_1/3)$.

(iv) $\underline{3\nmid r,\ s\equiv -1 \pmod 3}$. Once more, we get $3\mid t$. Also $U_1\equiv -V_1 \pmod 9$; hence $3\mid Q$. $3\nmid(R,Q)$ if and only if $\underline{3\nmid W_1/3}$.

We now show the sufficiency of the conditions. Let p $(\neq 3)$ be a prime such $p\mid(P,\ Q,\ R)$ and $p\nmid\Delta$. Put $T=V_1+rU_1$. Since $p\mid E_1$ and $p\nmid\Delta$, we must have $p\mid N_1$ and

$$T^3 - rT^2U_1 - sTU_1^2 - tU_1^3 \equiv 0 \pmod p. \tag{5.2}$$

Also

$$3W_1 \equiv -rT - 2sU_1 \pmod p \quad\text{and}\quad p\mid 27Q;$$

hence,

$$T^2(-r^2-3s) + U_1T(-sr-9t) + U_1^2(-s^2+3tr) \equiv 0 \pmod p. \tag{5.3}$$

If $p\mid U_1$, then $p\mid V_1$ and $p\mid W_1$. Suppose $p\nmid U_1$; then

$$\begin{vmatrix} -9t-rs & -3s-r^2 & 0 \\ -s & -r & 1 \\ 3rt-s^2 & -9t-rs & -3s-r^2 \end{vmatrix} TU_1^{-1} + \begin{vmatrix} -3rt-s^2 & -3s-r^2 & 0 \\ -t & -r & 1 \\ 0 & -9t-rs & -3s-r^2 \end{vmatrix} \equiv 0 \pmod p.$$

Evaluating the determinants, we have

$$-3\Delta TU_1^{-1} + r\Delta \equiv 0 \pmod p$$

and, consequently, $T\equiv 3^{-1}rU_1 \pmod p$. Putting this result into (5.2) and (5.3), we get $r^2+3s\equiv 0 \pmod p$ and $2r^3+9sr+27t\equiv 0 \pmod p$. By (5.1) $p\mid\Delta$, this is a contradiction; thus $p\mid(W_1,\ V_1,\ U_1)$.

If $3\mid(P,Q,R)$ and $3\nmid\Delta$, then W_n, V_n, U_n are given by (a) and we discuss two cases. If $3\mid r$, then $3\nmid s$ and from (2.10), we must have $9\mid U_1$. Using these results in (2.11) and (2.7), we see that $9\mid V_1$ and $9\mid W_1$. If $3\nmid r$, we obtain from (2.10) the fact that

$$V_1/3 \equiv -r(1+2s)U_1/3 \pmod 3.$$

Putting this result into (2.11), we deduce

$$(-s-s^2+tr)(U_1/3)^2 \equiv 0 \pmod 3.$$

Since $3 \nmid \Delta$, $3 \mid U_1/3$ and $3 \mid V_1/3$, from (2.7), we have $3 \mid W_1/3$.

If p ($\neq 3$) is a prime and $p \mid (P, Q, R, \Delta)$, then

$$4 \cdot 27(r^2 + 2s)Q \equiv 9G^2 \pmod{p}$$

and $p \mid G$. If $p = 2$, then $2 \mid (P, G, \Delta)$ and we have $2 \mid (s + r)(U_1 + V_1)$.

If $3 \mid (P, Q, R, \Delta)$ and W_n, V_n, U_n are given by (a), it follows from (2.10) that

$$rV_1/3 + (r^2 + 2s)U_1/3 \equiv 0 \pmod{3}.$$

Hence,

$$G \equiv 2r(rV_1/3 + (r^2 + 2s)U_1/3) + 3rsU_1/3 + 9tU_1/3 + 6sV_1/3 \equiv 0 \pmod{3}$$

and $3 \mid (P, G, \Delta)$. By the reasoning given above, one of (i), (ii), (iii), or (iv) must be true. If W_n, V_n, U_n are given by (b), (c), (d), or (e), then by Lemma 3, $3 \mid H$, and we have

$$-4 \cdot 27 \cdot (r^2 + 2s)Q \equiv 9G^2 \pmod{27};$$

hence, $3 \mid (P, G, H)$ and $3 \mid F$.

The values of α, β, γ in Theorem 2 can be bounded. We give these bounds in

Lemma 4. If $(P, Q, R) = 1$, then $\alpha < 3$, $\beta < 4$, and $\gamma < 6$.

Proof. If $8 \mid (P, G, \Delta)$, then

$$-12(r^2 + 3s)Q \equiv 9G^2 \pmod{8}.$$

Since $2 \nmid (r^2 + 3s)$, we have $2 \mid Q$ and it follows that $2 \mid R$.

If $\beta \geqslant 4$,

$$3W_1/3 + rV_1/3 + (r^2 + 2s)U_1/3 \equiv 0 \pmod{81}$$

and

$$3Q \equiv -[(r^2 + 3s)(V_1/3)^2 + (2r^3 + 7rs + 9t)(U_1/3)(V_1/3) + 3(s^2 - 2rt)(U_1/3)^2] \pmod{243}.$$

If $27 \nmid (r^2 + 3s)$, then $9 \mid Q$. If $27 \mid (r^2 + 3s)$, we have $3 \mid r$, $3 \mid s$ and $(r/3)^2 + (s/3) \equiv 0 \pmod{3}$. Since $81 \mid \Delta$, we also have $r/3 \equiv t \pmod{3}$. Since

$$-3Q \equiv (7rs + 9t)(U_1/3)(V_1/3) + (6rt + s^2)(U_1/3)^2 \pmod{27}$$

and $7rs + 9t \equiv 6tr + s^2 \equiv 0 \pmod{27}$, it follows that $9 \mid Q$. From the facts that $9 \mid Q$, $81 \mid \Delta$, $27 \mid N_1$, and

$$E_1 = \Delta(N_1/27)^2,$$

we see that $3 \mid R$.

If $\gamma \geqslant 6$, then $3^8 \mid -3\Delta$ and

$$-4 \cdot 27(r^2 + 3s)Q \equiv 9G^2 \pmod{3^8};$$

hence, $3^5 \mid (r^2 + 3s)Q$. It is not difficult to show that $9 \mid Q$. Since $3 \mid N_1$ and $3^8 \mid \Delta$, we have $3^{10} \mid \Delta N_1^2$, and consequently, $3 \mid R$.

6. PROPERTIES OF W_n, V_n, U_n

In the following sections, we will be demonstrating several divisibility properties of the W_n, V_n, U_n functions. Most of these results depend upon

Theorem 3. If $n \in N$, $(W_n, V_n, U_n) \mid 3$.

Proof. Suppose p ($\neq 3$) is a prime such that $p \mid (W_2, V_2, U_2)$. From (2.10), (2.11), (2.7), it is clear that $p \mid P_2$, $p \mid Q_2$, $p \mid R$. Since $P_2 = P^2 - 2Q$ and $Q_2 = Q^2 - 2RP$, we have $p \mid (P, Q, R)$, which is impossible by definition of W_n, V_n, U_n. If $9 \mid (W_2, V_2, U_2)$, then $3 \mid R$, $3 \mid P_2$, $9 \mid Q_2$; hence, $3 \mid (P, Q, R)$. The theorem is true for $n = 1, 2$.

Suppose $n > 2$ is the least positive integer such that $p \mid (W_n, V_n, U_n)$, where p ($\neq 3$) is a prime. Since $P \mid R$, by (2.18), it follows that

$$PW_{n-1} \equiv QW_{n-2}, \quad PV_{n-1} \equiv QV_{n-2}, PU_{n-1} \equiv QU_{n-2} \pmod{p}.$$

If $p \mid P$, then $p \nmid Q$; hence, $p \mid (W_{n-2}, V_{n-2}, U_{n-2})$, which is impossible by the definition of n. If $p \mid Q$, then

$p|(W_{n-1}, V_{n-1}, U_{n-1})$, which is also impossible. This enables us to write

$$W_{n-1} \equiv P^{-1}QW_{n-2}, \quad V_{n-1} \equiv P^{-1}QV_{n-2}, \quad U_{n-1} \equiv P^{-1}QU_{n-2} \pmod p,$$

where $P^{-1}Q \not\equiv 0 \pmod p$. From (2.2), we see that

$$W_n \equiv P^{-1}QW_{n-1}, \quad V_n \equiv P^{-1}QV_{n-1}, \quad U_n \equiv P^{-1}QU_{n-1} \pmod p$$

and consequently $p|(W_{n-1}, V_{n-1}, U_{n-1})$, which is impossible.

Suppose $n > 2$ is the least positive integer such that $9|(W_n, V_n, U_n)$. From (2.2), it is evident that

$$3|(W_{n+1}, V_{n+1}, U_{n+1}).$$

If ψ has the same meaning as that assigned to it in the corollary of Lemma 3, we have $\psi|n$ and $\psi|n+1$; that is, $\psi = 1$. Since $3|W_{n-3}$ and $3|R$, we have

$$P(W_{n-1}/3) \equiv Q(W_{n-2}/3) \pmod 3$$

and similar results for V_{n-1} and U_{n-1}. By reasoning similar to that above, we obtain the result that

$$3|(W_{n-1}/3, V_{n-1}/3, U_{n-1}/3),$$

which cannot be.

Corollary. If $n \in N$, $(U_n, V_n, R)|3$.

Proof. If p ($\neq 3$) is a prime and $p|(U_n, V_n, R)$, then $p|W_n$, which contradicts the theorem. If $9|(U_n, V_n, R)$, then by (2.7), $81|W_n^3$ and $9|W_n$, which is also a contradiction.

We have, with the aid of Theorem 3 and Lemma 3, completely characterized all the divisors of (W_n, V_n, U_n). We will now begin to develop some results concerning $D_n = (V_n, U_n)$. It will be seen that the divisibility properties of D_n are similar to those of Lucas' u_n (Carmichael's D_n). In fact, we have analogues of Carmichael's theorems I, II, III, IV, VI, X, XII, XIII, XVII (corollary), in Theorem 3 (corollary), Theorem 3, Lemma 3, Theorem 4, Theorem 5 (corollary), Theorem 7, Theorem 8, Theorem 8 (corollary), Theorem 7 (corollary), respectively. We also have the analogues of Corollaries I and II of Carmichael's Theorem VIII as a consequence of Theorem 5 and a result of Ward [9].

Theorem 4. If $n, k \in N$ and $m|D_n$, then $m|D_{kn}$.

Proof. This theorem is true for $k = 1$. Suppose it is true for $k = j$.

Since

$$3V_{(j+1)n} = V_nW_{jn} + W_nV_{jn} + sV_{jn}U_n + sV_nU_{jn} + (rs+t)U_nU_{jn}$$

and

$$3U_{(j+1)n} = W_{jn}U_n + U_{jn}W_n + V_nV_{jn} + rU_{jn}V_n + rU_nV_{jn} + (r^2+s)U_nU_{jn},$$

we have $m|D_{(j+1)m}$, when $3\nmid m$. If $3|m$, then $3|W_n$ and $3|W_{jn}$; hence, $3m|3V_{(j+1)n}$, $3m|3U_{(j+1)n}$ and $m|D_{(j+1)m}$. The theorem is true by induction.

Let D_ω be the first term of the sequence

$$D_1, D_2, D_3, \cdots, D_k, \cdots$$

in which m occurs as a factor. We call $\omega = \omega(m)$ the *rank of apparition of* n.

Theorem 5. If $n \in N$ and m is a divisor of D_n, then $\omega(m)|n$.

Proof. Suppose $\omega\nmid n$; then $n = k\omega + j$ $(0 < j < \omega)$. From (2.2)

$$3V_n = V_jW_{k\omega} + W_jV_{k\omega} + sV_jU_{k\omega} + sV_{k\omega}U_j + (rs+t)U_{k\omega}U_j,$$

$$3U_n = U_jW_{k\omega} + W_jU_{k\omega} + V_jV_{k\omega} + rU_{k\omega}V_j + rU_jU_{k\omega} + (r^2+s)U_{k\omega}U_j.$$

If $3\nmid m$, $m|(V_jW_{k\omega}, U_jW_{k\omega})$. Since $m|D_{k\omega}$, $(m, W_{k\omega}) = 1$ and $m|D_j$.

If $3|m$, then $3|W_{k\omega}$ and $3|W_n$. If ψ is the rank of apparition of 3, we know that $\psi|n$ and $\psi|k\omega$; hence, $\psi|j$ and $3|(W_j, V_j, U_j)$. We now have $3m|(V_jW_{k\omega}, U_jW_{k\omega})$. If $3\|m$, then $(m/3, W_{k\omega}) = 1$, $m/3|(V_j, U_j)$, $3|(V_j, U_j)$ and consequently $m|D_j$. If $3^\alpha\|m$ and $\alpha > 1$, then $3\|W_{k\omega}$ and $m|D_j$.

If $\omega \nmid n$, we can find $j < \omega$ such that $m \mid D_j$. This contradicts the definition of ω.

Corollary. If $n,m \in N$, then $D_{(m,n)} = (D_m, D_n)$.

Proof. This result follows from the theorem and a result of Ward [9].

Corollary. If m,n are integers and $(m,n) = 1$, $\omega(mn)$ is the least common multiple of $\omega(m)$ and $\omega(n)$.

7. THE LAWS OF REPETITION AND APPARITION

We have defined the rank of apparition of an integer m without having shown whether it exists or, if it does exist, what its value is. We give in this section those values of m for which ω exists and we partially answer the question of the value of ω for these m values. *The Law of Repetition* describes how $\omega(p^n)$ (p a prime) may be determined once $\omega(p)$ is known. In order to prove the Law of Repetition, we must first give a few preliminary results.

Lemma 5. Suppose $3 \nmid R$ and $3 \mid D_m$; then $3 \mid (P_m, Q_m)$ if and only if $9 \mid D_{3m}$. If $3 \nmid \Delta$, then $3 \mid (P_m, Q_m)$ if and only if $9 \mid D_m$.

Proof. If $9 \mid D_k$, then $3 \mid W_k$ and $3 \mid (P_k, Q_k)$. If $\Delta \equiv r^2s^2 + s - tr \not\equiv 0 \pmod 3$ and $3 \mid (P_m, Q_m)$, then

$$r(V_m/3) + (r^2 + 2s)(U_m/3) \equiv 0 \pmod 3$$

and

$$-s(V_m/3)^2 - sr(U_m/3)(V_m/3) + (s^2 - 2tr)(U_m/3)^2 \equiv 0 \pmod 3 .$$

If $3 \mid r$, then $3 \nmid s$; hence, if $3 \mid U_m/3$, $9 \mid D_m$. If $3 \nmid r$, then $(V_m/3) \equiv -r(r^2 + 2s)U_m/3 \pmod 3$; thus,

$$-\Delta(U_m/3)^2 \equiv 0 \pmod 3$$

and $9 \mid D_m$.

If $9 \mid D_{3m}$, we have $3 \mid P_{3m}$ and $3 \mid Q_{3m}$. Now

$$P_{3m} = P_m^3 - 3Q_mP_m + 3R_m,$$
$$Q_{3m} = Q_m^3 - 3R_mP_mQ_m + 3R_m^2;$$

consequently, $3 \mid (P_m, Q_m)$. If $3 \mid (P_m, Q_m)$, then since

$$V_{3m}/3 = P_mV_{2m}/3 - Q_mV_m/3 \equiv 0 \pmod 3$$

and

$$U_{3m}/3 = P_mU_{2m}/3 - Q_mU_m/3 \equiv 0 \pmod 3,$$

we have $9 \mid D_{3m}$.

Lemma 6. Suppose $3 \nmid R$, $3 \mid D_m$, and $3 \mid \Delta$. If $3 \nmid P_m$, $9 \mid D_{2m}$ if and only if one of the following is true.

(i) $3 \mid s$, $3 \mid t$, $3 \nmid r$, $W_m \equiv U_m \not\equiv 0 \pmod 9$, and $9 \mid V_m$.
(ii) $s \equiv 1 \pmod 3$, $t \equiv -r \not\equiv 0 \pmod 3$, $W_m \equiv -U_m \not\equiv 0 \pmod 9$ and $V_m \equiv rU_m \pmod 9$.
(iii) $s \equiv -1 \pmod 3$, $3 \mid t$, $3 \nmid r$, and $W_m \equiv -rV_m + U_m \not\equiv 0 \pmod 9$.

Proof. Since $3 \nmid P_m$ and $3 \mid \Delta$, it is clear that $3 \nmid r$.

We show the necessity of one of (i), (ii), or (iii). If $9 \mid D_{2m}$, then

$$(sr + t)(U_m/3)^2 + 2s(V_m/3)(U_m/3) + 2(V_m/3)(W_m/3) \equiv 0 \pmod 3$$

and

$$(V_m/3)^2 + 2(W_m/3)(U_m/3) + 2r(U_m/3)(V_m/3) + (r^2 + s)(U_m/3) \equiv 0 \pmod 3.$$

If $9 \mid U_m$, then $9 \mid V_m$ and $3 \mid P_m$, which is impossible. If $9 \mid V_m$, then $3 \mid (rs + t)$ and $(r^2 + s)U_m \equiv W_m \pmod 9$. Now since $3 \nmid (s + 1)$, we have $3 \mid s - 1$ or $3 \mid s$. If $3 \mid (s - 1)$, then $3 \mid (r^2 + 2s)$ and $3 \mid P_m$. If $3 \mid s$, then $3 \mid t$ and $W_m \equiv U_m \not\equiv 0 \pmod 9$.

If $9 \nmid U_m$ and $9 \nmid V_m$, then

$$W_m - (sr + t)V_m + sU_m \equiv 0 \pmod 9$$
$$W_m + rV_m - (1 + r^2 + s)U_m \equiv 0 \pmod 9$$

and

$$r(s+1)^2V_m \equiv -(s+1)U_m \pmod 9.$$

If $3|s$, $rV_m \equiv -U_m \pmod 9$ and $3|P_m$. If $s \equiv 1 \pmod 3$, then $t \equiv -r \pmod 3$, $rV_m \equiv U_m \not\equiv 0 \pmod 9$ and $W_m \equiv -U_m \pmod 9$. If $s \equiv -1 \pmod 3$, then $3|t$, and $W_m \equiv -rV_m + U_m \pmod 9$.

It is clear that any one of the conditions (i), (ii), or (iii) is sufficient for $9|D_{2m}$.

Theorem 6. If $3\nmid R$, ψ is the rank of apparition of 3, and $9\nmid D_\psi$, then the rank of apparition of 9 is $\sigma\psi$, where the value of σ is given below.

I. $3\nmid\Delta$.

In this case, W_n, V_n, U_n are given by (a) and the value of σ is a function of the values (modulo 3) of $N_1/27$, Δ, P, Q. The values of σ are given in Table 3.

Table 3

$N_1/27$	Δ	P	Q	σ
0	±1	P	Q	2
±1	-1	±1	±1	4
±1	-1	±1	0	8
±1	-1	0	Q	8
±1	1	P	Q	13

II. $3|\Delta$.

Here $\sigma = 2$ if $3\nmid P_\psi$ and one of the following is true.

(i) $3|s$, $3|t$, $3\nmid r$, $W_\psi \equiv U_\psi \not\equiv 0 \pmod 9$ and $9|V_\psi$;

(ii) $s \equiv 1 \pmod 3$, $t \equiv -r \not\equiv 0 \pmod 3$, $W_\psi \equiv -U_\psi \not\equiv 0 \pmod 9$, and $V_\psi \equiv rU_\psi \pmod 9$;

(iii) $s \equiv -1 \pmod 3$, $3|t$, $3\nmid r$, and $W_\psi \equiv -rV_\psi + U_\psi \not\equiv 0 \pmod 9$.

$\sigma = 3$ if $3|P_\psi$.

$\sigma = 6$ if $3\nmid P_\psi$ and none of (i), (ii), (iii) is true.

Proof. Since $3|D_\psi$, we have $27|N_\psi$; hence

$$E_\psi = \Delta(N_\psi/27)^2.$$

If $3|\Delta$,

$$P_\psi R_\psi \equiv Q_\psi(Q_\psi P_\psi^2 - 1) \pmod 3.$$

If $3|P_\psi$, then $3|Q_\psi$ and $9|(V_{3\psi}, U_{3\psi})$. If $3\nmid P_\psi$, then

$$R_\psi \equiv P_\psi Q_\psi(Q_\psi - 1) \pmod 3;$$

thus, $Q_\psi \equiv -1 \pmod 3$ and $R_\psi \equiv -P_\psi \pmod 3$. Since

$$P_{2\psi} = P_\psi^2 - 2Q_\psi \equiv 0 \quad \text{and} \quad Q_{2\psi} = Q_\psi^2 - 2R_\psi P_\psi \equiv 0 \pmod 3,$$

it follows from Lemma 5 that $9|D_{6\psi}$ and $9\nmid D_{3\psi}$. From Lemma 6, we see that $9|D_{2\psi}$ if and only if one of (i), (ii) or (iii) is true.

If $3\nmid\Delta$ and $81|N_1$, then $3|E_1$, $3|(P_2, Q_2)$ and $\sigma = 2$.

If $\Delta \equiv -1 \pmod 3$ and $81|N_1$, then

$$PR \equiv P^2Q^2 - Q + 1 \pmod 3.$$

Using the formulas

$$P_{2k} = P_k^2 - 2Q_k \quad \text{and} \quad Q_{2k} = Q_k^2 - 2P_kR_k,$$

we see that if $3|P$, then $Q \equiv 1 \pmod 3$ and $P_2 \equiv Q_2 \equiv 1 \pmod 3$, $Q_4 \equiv P_4 \equiv -1 \pmod 3$, $Q_8 \equiv P_8 \equiv 0 \pmod 3$; consequently, $\sigma = 8$. The remaining results for this case are proved in the same way.

If $\Delta \equiv 1 \pmod 3$ and $81\nmid N_1$, then

$$PR \equiv P^2Q^2 - Q - 1 \pmod 3.$$

Using the formulas

$$P_{n+3} = PP_{n+2} - QP_{n+1} + RP_n$$
$$Q_{n+3} = QQ_{n+2} - PRQ_{n+1} + R^2Q_n,$$

we see that if $3|P$, then $Q \equiv -1 \pmod 3$ and $P_{13} \equiv Q_{13} \equiv 0 \pmod 3$. If $3\nmid P$, then $R \equiv P(Q^2 - Q - 1)$ and $P_{13} \equiv Q_{13} \equiv 0 \pmod 3$.

Theorem 7. (Law of Repetition). Let p be a prime. If, for $\lambda > 0$, $p^\lambda \neq 3, 2$ and $p^\lambda \| D_m$, then

$$p^{\alpha+\lambda} \| D_{m\nu p^\alpha}, \quad \text{where} \quad (\nu, p) = 1.$$

If $p^\lambda = 2$ and ν is odd, $p^{\alpha+1} | D_{m\nu p^\alpha}$ and $4 | D_{m\nu}$. If $p^\lambda = 3$ and $3\nmid R$, then

$$3^{\alpha+1} | D_{m\tau 3^{\alpha-1}} \quad \text{and} \quad 9\nmid D_{m\nu}, \quad \text{if} \quad \tau\nmid \nu.$$

Here

$$\tau = \sigma/(m/\psi, \sigma),$$

where ψ, σ have the meanings assigned to them in Theorem 6. If $3|R$, then $3\|D_n$ for any $n \in N$.

Proof. Since p is a divisor of $p!/[i!j!(p-i-j)!]$ when $i,j \neq 0,p$, we have (from (2.17))

$$3^{p-1}V_{mp} \equiv pW_m^{p-1}V_m \pmod{p^{\lambda+2}}$$
$$3^{p-1}U_{mp} \equiv pW_m^{p-1}U_m \pmod{p^{\lambda+2}}$$

if $p \neq 2$ or if $p = 2$ and $\lambda > 1$. If $p \neq 3$, then $p\nmid W_n$; hence $p^{\lambda+1} \| D_{mp}$. By induction $p^{\lambda+\alpha} \| D_{mp^\alpha}$. If

$$p^{\lambda+\alpha+1} | D_{m\mu p^\alpha}, \quad \text{then} \quad p^{\lambda+\alpha+1} | (D_{mp^\alpha\mu}, D_{mp^{\alpha+1}}) = D_{mp^\alpha},$$

which is impossible. If $p = 2$ and $\lambda = 1$, $3V_{2m} \equiv 3U_{2m} \equiv 0 \pmod 4$; hence, $2^{\alpha+1} | D_{2^\alpha m}$ and $4\nmid D_{m\mu}$.

If $3^\lambda \| D_m$ and $\lambda > 1$, then $3|W_m$ and $3\lambda \geqslant \lambda + 4$, $2\lambda + 2 \geqslant \lambda + 4$. Using the triplication formulas (2.4), we have

$$3^{\lambda+3} | 9V_{3m} \quad \text{and} \quad 3^{\lambda+3} | 9U_{3m}$$

or $3^{\lambda+1} | D_{3m}$. Also

$$9V_{3m} \equiv 3V_m W_m^2 \pmod{3^{\lambda+4}}$$
$$9U_{3m} \equiv 3U_m W_m^2 \pmod{3^{\lambda+4}}.$$

Since $9\nmid W_m$,

$$3^{\lambda+2} \nmid D_{3m} \quad \text{and} \quad 3^{\lambda+1} \| D_{3m}.$$

If $3\|D_m$, then $\psi | m$ and $9|D_n$ if and only if $\sigma\psi | n$. Since $\sigma\psi | m\tau$, we have $9|D_{m\tau}$ and $3^{\alpha+1} | D_{m\tau 3^{\alpha-1}}$. If $\tau\nmid\nu$, then $\sigma\psi \nmid \nu m$ and $9\nmid D_{\nu m}$.

If $3|R$ and $9|D_n$, then $81|W_n^3$ or $9|W_n$, which is impossible.

The *Law of Apparition* gives those primes for which the rank of apparition exists and also gives us some information concerning the value of the rank of apparition. We first define an auxiliary function y_n.

If p is a prime such that $p\nmid 3N_1R$, we define the function y_n to be the Lucas function u_n of (1.1), where $a_1 + a_2 \equiv g \pmod p$, $a_1a_2 \equiv h^3 \pmod p$, and

$$h = r^2 + 3s, \qquad g = 2r^3 + 9rs + 27t.$$

Theorem 8. (Law of Apparition). If p is a prime such that $p\nmid R$, then ω, the rank of apparition of p, exists. If $p = 3$, then $\omega = \psi$. Suppose $p\nmid 3R$; then $\omega(p) | \Phi(p)$, where the value of Φ is given below.

We let $p \equiv q \pmod 3$, where $|q| = 1$.

If $p\nmid \Delta N_1$ and $(\Delta|p) = -1$, then $(p-1)\nmid\omega$ and $\Phi(p) = p^2 - 1$.

If $p\nmid \Delta N_1 h$ and $(\Delta|p) = +1$, then $\Phi(p) = p - 1$, when $y_{(p-q)/3} \equiv 0 \pmod p$; $\Phi(p) = p^2 + p + 1$, when $y_{(p-q)/3} \not\equiv 0 \pmod p$.

If $p\nmid \Delta N_1$, $(\Delta|p) = +1$, and $p|h$, then $p \equiv 1 \pmod 3$ and $\Phi(p) = p - 1$, when $(g|p)_3 = 1$; $\Phi(p) = p^2 + p + 1$, when $(g|p)_3 \neq 1$.

If $p \nmid \Delta$ and $p \mid N_1$, then $\Phi(p) = p - 1$.
If $p = 2$ and $p \mid \Delta$, then $\Phi(p) = 4$.
If $p \neq 2$, $p \mid \Delta$ and $p \nmid N_1$, then $p \mid \omega$ and $\Phi(p) = p(p-1)$.
If $p \neq 2$, $p \mid \Delta$ and $p \mid N_1$, then $\Phi(p) = p$, when $p \mid G$; $\Phi(p) = p - 1$, when $p \nmid G$.

Proof. These results may be deduced without much difficulty from (2.15) and results of Engstrom [5], Ward [8], and Cailler [2]. (See also Duparc [4].)

Corollary. If we define $\Phi(p^n) = p^{n-1}\Phi(p)$ for $p \neq 3$, $\Phi(3^2) = \sigma\psi$, $\Phi(3^n) = 3^{n-2}\Phi(3^2)$, and $\Phi(mn)$ to be the least common multiple of $\Phi(m)$ and $\Phi(n)$, when $(m,n) = 1$, then $\omega(m) \mid \Phi(m)$.

If p is of the form $3k + 1$ and $p \nmid \Delta N_1 R$, we can sharpen some of the results in the Law of Apparition.

Theorem 9. Let p ($\equiv 1 \pmod 3$) be a prime such that $p \nmid \Delta N_1 R$. If $(\Delta \mid p) = -1$, $\omega \mid (p^2 - 1)/3$ if and only if $(R \mid p)_3 = 1$. If $(\Delta \mid p) = +1$ and $y_{(p-q)/3} \not\equiv 0 \pmod p$, then $\omega \mid (p^2 + p + 1)/3$ if and only if $(R \mid p)_3 = 1$. If $(\Delta \mid p) = +1$ and $y_{(p-q)/3} \equiv 0 \pmod p$, $\omega \mid (p - 1)/3$ only if $(R \mid p)_3 = 1$.

Proof. If $(\Delta \mid p) = -1$, then $(E_1 \mid p) = -1$ and the polynomial $x^3 - Px^2 + Qx - R$ factors modulo p into the product of a linear and irreducible quadratic factor. Let $K = GF(p^2)$ be the splitting field for this polynomial modulo p and let the roots of

$$x^3 - Px^2 - Qx - R = 0 \tag{7.1}$$

be θ, ϕ, ψ in K. Then in K

$$\theta^p = \theta, \quad \chi = \phi^p, \quad \chi^p = \phi, \quad R = \theta\phi\chi = \theta\phi^{p+1}.$$

If $R^{(p-1)/3} \equiv 1 \pmod p$, we have

$$\theta^{(p-1)/3}\phi^{(p^2-1)/3} = 1 \quad \text{and} \quad \theta^{(p^2-1)/3} = \phi^{(p^2-1)/3} = \phi^{p(p^2-1)/3}. \tag{7.2}$$

Since $p \nmid \Delta$, it follows that $p \mid D_{(p^2-1)/3}$. If $R^{(p-1)/3} \not\equiv 1 \pmod 3$, we cannot have (7.2). Since $p \nmid \Delta N_1$, it is clear that $p \nmid D_{(p^2-1)/3}$.

If $(\Delta \mid p) = +1$ and $p \nmid y_{(p-q)/3}$, the polynomial $x^3 - rx^2 - sx - t$ is irreducible modulo p; hence, the polynomial $x^3 - Px^2 + Qx - R$ is irreducible modulo p. If $K = GF(p^3)$ is the splitting field of this polynomial (modulo p) and θ, ϕ, χ are the roots of (7.1) in K, then

$$\theta^p = \phi, \quad \theta^{p^2} = \chi, \quad \theta^{p^3} = \theta, \quad R = \theta^{1+p+p^2}.$$

If $R^{(p-1)/3} \equiv 1 \pmod p$,

$$\theta^{(p^3-1)/3} = 1 \quad \text{and} \quad \theta^{p(p^2+p+1)/3} = \theta^{p^2(p^2+p+1)/3} = \theta^{(p^2+p+1)/3};$$

hence $p \mid D_{(p^2+p+1)/3}$. If $R^{(p-1)/3} \not\equiv 1 \pmod p$, then $p \nmid D_{(p^2+p+1)/3}$.

If $(\Delta \mid p) = +1$ and $p \mid y_{(p-q)/3}$, the polynomial $x^3 - Px^2 + Qx - R$ splits modulo p into the product of three linear factors. It is not difficult to show that if $p \mid D_{(p-1)/3}$, then $R^{(p-1)/3} \equiv 1 \pmod p$.

We have not discussed the functions

$$B_n = (W_n, V_n) \quad \text{and} \quad C_n = (W_n, U_n)$$

which are somewhat analogous in their divisibility properties to Lucas' V_n or Carmichael's S_n. The functions B_n and C_n behave in a rather complicated fashion and in a further paper results concerning these functions will be presented together with other results on the W_n, V_n, U_n functions.

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PHI AGAIN: A RELATIONSHIP BETWEEN THE GOLDEN RATIO AND THE LIMIT OF A RATIO OF MODIFIED BESSEL FUNCTIONS

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In his study of infinite continued fractions whose partial quotients form a general arithmetic progression, D. H. Lehmer derived a formula for their evaluation in terms of modified Bessel Functions [1]. We have

$$F(a,b) = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots = [a_0, a_1, a_2, \cdots], \tag{1}$$

where $a_n = an + b$. It was shown that

$$F(a,b) = \frac{I_{\alpha-1}(2/a)}{I_\alpha(2/a)}, \tag{2}$$

where $\alpha = b/a$ and I_α is the modified Bessel function

$$I_\alpha(z) = i^{-\alpha} J_\alpha(iz) \sum_{m=0}^{\infty} \frac{(z/2)^{\alpha+2m}}{\Gamma(m+1)\Gamma(\alpha+m+1)}. \tag{3}$$

Using (1) and (2) with $c\alpha = 2/a$ and $b = c/2$, we have

$$F(a,b) = [b, a+b, 2a+b, \cdots] = \frac{I_{\alpha-1}(c\alpha)}{I_\alpha(c\alpha)}. \tag{4}$$

As $\alpha \to \infty$ $(a \to 0)$, in the limit (Theorem 5 of [1]),

$$\lim_{\alpha \to \infty} \frac{I_{\alpha-1}(c\alpha)}{I_\alpha(c\alpha)} = F(0,b) = [b, b, b, \cdots]. \tag{5}$$

But, for $b = 1$, $(c = 2)$, $F(0,1)$ is the positive root of the quadratic equation

$$1 + \frac{1}{x} = x \tag{6}$$

which is represented by the infinite continued fraction expansion $[1, 1, 1, \cdots]$.

[Continued on p. 152.]

LIMITS OF QUOTIENTS FOR THE CONVOLVED FIBONACCI SEQUENCE AND RELATED SEQUENCES

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If $\{F_n\}_{n=1}^{\infty}$ is the sequence of Fibonacci numbers defined recursively by

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geqslant 3$$

then $C_1(x)$, the generating function for the sequence $\{F_n\}_{n=1}^{\infty}$, is given by

$$C_1(x) = (1 - x - x^2)^{-1} = \sum_{i=0}^{\infty} F_{i+1} x^i . \tag{1}$$

Letting $C_n(x)$ be the generating function for the Cauchy convolution product of $C_1(x)$ with itself n times and $F_{i+1}^{(n)}$ be the coefficient of x^i in the nth convolution, we have

$$C_n^{(x)} = (1 - x - x^2)^{-n} = \sum_{i=0}^{\infty} F_{i+1}^{(n)} x^i, \quad n \geqslant 1. \tag{2}$$

In a personal communique, V.E. Hoggatt, Jr., pointed out that he and Marjorie Bicknell have shown that

$$\lim_{n \to \infty} \frac{F_{n+1}^{(r)}}{F_n^{(r)}} = a \tag{3}$$

and

$$\lim_{n \to \infty} \frac{F_n^{(r)}}{F_n^{(r+1)}} = 0, \tag{4}$$

where $a = (1 + \sqrt{5})/2$.

An immediate consequence of (3) is

$$\lim_{n \to \infty} \frac{F_{n+k}^{(r)}}{F_{n+m}^{(r)}} = a^{k-m} \tag{5}$$

while by using (4), we obtain

$$\lim_{n \to \infty} \frac{F_{n+k}^{(r)}}{F_{n+m}^{(r+1)}} = 0. \tag{6}$$

The purpose of this note is to extend the results of (3) and (4) to the columns of the convolution array formed by a sequence of generalized Fibonacci numbers as well as to the array generated by the numerator polynomials of the generating functions for the row sequences associated with the convolution array formed by the given sequence of generalized Fibonacci numbers.

The sequence $\{H_n\}_{n=1}^{\infty}$ of generalized Fibonacci numbers defined recursively by

$$H_1 = 1, \quad H_2 = P, \quad H_n = H_{n-1} + H_{n-2}, \quad n \geqslant 3$$

has generating function $C_1^*(x)$ given by

(7) $$C_1^*(x) = \sum_{i=0}^{\infty} H_{i+1}x^i = \frac{1+(P-1)x}{1-x-x^2} = \sum_{i=0}^{\infty} (F_{i+1}+(P-1)F_i)x^i .$$

Using $C_n^*(x)$ for the Cauchy convolution product of $C_1^*(x)$ with itself n times and $H_{i+1}^{(n)}$ for the coefficient of x^i in the nth convolution, we have

(8) $$C_n^*(x) = \sum_{i=0}^{\infty} H_{i+1}^{(n)}x^i = \left(\frac{1+(P-1)x}{1-x-x^2}\right)^n = \sum_{i=0}^{\infty} F_{i+1}^{(n)}x^i \sum_{j=0}^{n} \binom{n}{j}(P-1)^j x^j$$

$$= \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\binom{n}{j}(P-1)^j F_{i-j+1}^{(n)}\right)x^i .$$

Hence,

(9) $$H_{i+1}^{(n)} = \sum_{j=0}^{i}\binom{n}{j}(P-1)^j F_{i-j+1}^{(n)} .$$

Using (5) together with the fact that $\binom{n}{j} = 0$ for $j > n$, we have

$$\lim_{i\to\infty}\frac{H_{i+1}^{(n)}}{F_{i-n}^{(n)}} = \lim_{i\to\infty}\sum_{j=0}^{i}\binom{n}{j}(P-1)^j F_{i-j+1}^{(n)}/F_{i-n}^{(n)} = \sum_{j=0}^{n}\binom{n}{j}(P-1)^j a^{n-j+1}$$

$$= a\lim_{i\to\infty}\sum_{j=0}^{i}\binom{n}{j}(P-1)^j F_{i-j}^{(n)}/F_{i-n}^{(n)} = a\lim_{i\to\infty}\frac{H_i^{(n)}}{F_{i-n}^{(n)}}$$

so that

(10) $$\lim_{i\to\infty}\frac{H_{i+1}^{(n)}}{H_i^{(n)}} = a$$

and

(11) $$\lim_{i\to\infty}\frac{H_{i+k}^{(n)}}{H_{i+m}^{(n)}} = a^{k-m} .$$

By (6) and an argument similar to that used in the derivation of (10), we have

$$\lim_{i\to\infty}\frac{H_i^{(n)}}{F_{i-n}^{(n+1)}} = 0$$

while

$$\lim_{i\to\infty}\frac{H_i^{(n+1)}}{F_{i-n}^{(n+1)}} = \sum_{j=0}^{n+1}\binom{n+1}{j}(P-1)^j a^{n-j} \neq 0$$

so that

(12) $$\lim_{i\to\infty}\frac{H_i^{(n)}}{H_i^{(n+1)}} = 0$$

and

(13) $$\lim_{i\to\infty}\frac{H_{i+k}^{(n)}}{H_{i+m}^{(n+1)}} = 0 .$$

Let $R_{(n)}^*(x)$ be the generating function for the sequence of elements in the nth row of the convolution array formed by the powers of $C_1^*(x)$. Then

(14) $$R_n^*(x) = \sum_{i=0}^{\infty} H_n^{(i+1)} x^i .$$

In [1], it is shown that

(15) $$R_1^*(x) = (1-x)^{-1}$$

(16) $$R_2^*(x) = P(1-x)^{-2}$$

and

(17) $$R_n^*(x) = \frac{(1+(P-1)x)N_{n-1}^*(x) + (1-x)N_{n-2}^*(x)}{(1-x)^n} = \frac{N_n^*(x)}{(1-x)^n}, \quad n \geqslant 3 ,$$

where $N_n^*(x)$ is a polynomial of degree $n-2$ for $n \geqslant 2$.

Let $G_n^*(x)$ be the generating function for the nth column of the left-adjusted triangular array formed by the coefficients of the $N_n^*(x)$ polynomials. In [1], it is shown that

(18) $$G_1^*(x) = C_1^*(x)$$

(19) $$G_2^*(x) = DC_2(x)$$

and

(20) $$G_n^*(x) = \frac{(P-1-x)}{(1-x-x^2)} G_{n-1}^*(x), \quad n \geqslant 3 ,$$

where $D = P^2 - P - 1$. By induction, it can be shown that

(21) $$G_n^*(x) = \frac{(P-1-x)^{n-2}}{(1-x-x^2)^n}, \quad n \geqslant 3$$

which by an argument similar to that of (8) yields

(22) $$G_n^*(x) = D \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} (-1)^j \binom{n-2}{j} (P-1)^{n-j-2} F_{i-j+1}^{(n)} \right) x^i .$$

If we let $g_{i+1}^{(n)}$ be the coefficient of x^i in $G_n^*(x)$ then we see that

(23) $$g_{i+1}^{(1)} = F_{i+1} + (P-1)F_i$$

(24) $$G_{i+1}^{(2)} = DF_{i+1}^{(2)}$$

and

(25) $$g_{i+1}^{(n)} = D \sum_{j=0}^{i} (-1)^j \binom{n-2}{j} (P-1)^{n-j-2} F_{i-j+1}^{(n)}, \quad n \geqslant 3.$$

Following arguments similar to those given in obtaining (10) through (13), we have

(26) $$\lim_{i \to \infty} \frac{g_{i+1}^{(n)}}{g_i^{(n)}} = a$$

(27) $$\lim_{i \to \infty} \frac{g_{i+k}^{(n)}}{g_{i+m}^{(n)}} = a^{k-m}$$

(28) $$\lim_{i \to \infty} \frac{g_i^{(n)}}{g_i^{(n+1)}} = 0$$

and

(29) $$\lim_{i \to \infty} \frac{g_{i+k}^{(n)}}{g_{i+m}^{(n+1)}} = 0.$$

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SUMMATION OF MULTIPARAMETER HARMONIC SERIES

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1. INTRODUCTION

Consider the multiparameter alternating harmonic series denoted and defined by

(1) $$\omega(j; k_1, \cdots, k_n) = \sum_{i=0}^{\infty} (-1)^i/(j+s_i),$$

where j and the k_i are positive integers, $s_0 = 0$, $s_n = S$, and

$$s_i = [i/n]S + \sum_{t=1}^{i, \bmod n} k_t.$$

Note that the parameters $k_1, \cdots, k_n$ are successive cyclic denominator increments. In the ensuing treatment summation formulas for such series, to be called ω-series, are developed which admit evaluation in terms of elementary functions. An example is included to illustrate the formulas.

2. SUMMATION FORMULAS

The expression of the summation formulas for the ω-series (1) is based upon the following two lemmas.

Lemma 1.

(2) $$\omega(j; k) = (\tfrac{1}{2}k)G(j/k) = \int_0^1 x^{j-1}dx/(1+x^k)$$
$$= (-1)^{j-1}(r/k)\ln(1+x)$$
$$-(2/k)\sum_{i=0}^{q-1} [P_i(x)\cos((2i+1)j\pi/k) - Q_i(x)\sin((2i+1)j\pi/k)]\Big|_0^1,$$

[Continued on page 144.]

FIBONACCI CONVOLUTION SEQUENCES

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The Fibonacci convolution sequences $\left\{F_n^{(r)}\right\}$ which arise from convolutions of the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, \cdots, F_n, \cdots\}$ lead to some new Fibonacci identities, limit theorems, and determinant identities.

1. THE FIBONACCI CONVOLUTION SEQUENCES

Let the rth Fibonacci convolution sequence be denoted $\left\{F_n^{(r)}\right\}$; note that $F_n^{(0)} = F_n$, the nth Fibonacci number. Then

$$F_n^{(1)} = \sum_{i=0}^{n} F_{n-i} F_i \tag{1.1}$$

$$F_n^{(r)} = \sum_{i=0}^{n} F_{n-i}^{(r-1)} F_i \ . \tag{1.2}$$

However, there are some easier methods of calculation.

Let the Fibonacci polynomials $F_n(x)$ be defined by

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \qquad F_0(x) = 0, \quad F_1(x) = 1 \ . \tag{1.3}$$

Then, since $F_n(1) = F_n$, the recursion relation for the Fibonacci numbers, $F_{n+2} = F_{n+1} + F_n$, follows immediately by taking $x = 1$. In a similar manner we may write recursion relations for $\left\{F_n^{(r)}\right\}$.

From (1.3), taking the first derivative we have

$$F'_{n+2}(x) = xF'_{n+1}(x) + F'_n(x) + F_{n+1}(x).$$

Since $F'_n(1) = F_n^{(1)}$, taking $x = 1$ gives us the recursion relation for $\left\{F_n^{(1)}\right\}$,

$$F_{n+2}^{(1)} = F_{n+1}^{(1)} + F_n^{(1)} + F_{n+1} \ . \tag{1.4}$$

Since the generating function for the Fibonacci polynomials is

$$\frac{Y}{1 - xY - Y^2} = \sum_{n=1}^{\infty} F_n(x) Y^n , \tag{1.5}$$

while the generating function for the Fibonacci convolution sequences is

$$\left(\frac{x}{1 - x - x^2}\right)^{r+1} = \sum_{n=1}^{\infty} F_n^{(r)} x^n , \tag{1.6}$$

it is easy to see that

$$F_n^{(r)} = F_n^{(r)}(1)/r! \ , \tag{1.7}$$

where $F_n^{(r)}(x)$ is the rth derivative of the Fibonacci polynomial $F_n(x)$. Thus we can write

$$F_{n+2}^{(r+1)} = F_{n+1}^{(r+1)} + F_n^{(r+1)} + F_{n+1}^{(r)} \ , \tag{1.8}$$

which enables us to make the following table with a minimum of effort.

We can extend our sequences for negative subscripts to write

$$F_{-n}^{(r)} = (-1)^{n+1} F_n^{(r)} , \tag{1.9}$$

n	F_n	$F_n^{(1)}$	$F_n^{(2)}$	$F_n^{(3)}$	$F_n^{(4)}$	...
0	0	0	0	0	0	...
1	1	0	0	0	0	...
2	1	1	0	0	0	...
3	2	2	1	0	0	...
4	3	5	3	1	0	...
5	5	10	9	4	1	...
6	8	20	22	14	5	...
7	13	38	51	40	20	...
8	21	71	111	105	65	...
9	34	130	233	256	190	...
10	55	235	474	594	511	...
...	...	...	...	...	...	...

where we note that $\left\{F_n^{(r)}\right\}$ has $2r+1$ zeros, and $F_{r+1}^{(r)} = 1$, $F_{r+2}^{(r)} = r$.

Equation (1.9) can be established for $r = 1$ quite easily by induction. Assume that (1.9) holds for 1, 2, 3, $\cdots$, r, and for $r+1$ for $n = 1, 2, \cdots, k$. Then by (1.8)

$$F_{k+1}^{(r+1)} = F_k^{(r+1)} + F_{k-1}^{(r+1)} + F_k^{(r)} = (-1)^{k+1}F_{-k}^{(r+1)} + (-1)^k F_{-k+1}^{(r+1)} + (-1)^{k+1}F_{-k}^{(r)}$$

$$= (-1)^{k+2}[F_{-k+1}^{(r+1)} - F_{-k}^{(r+1)} - F_{-k}^{(r)}] = (-1)^{k+2}F_{-k-1}\ ,$$

which is equivalent to (1.9) for $n = k+1$, finishing a proof by induction.

Returning to (1.6), recall that the recurrence relation for $\left\{F_n^{(1)}\right\}$ has auxiliary polynomial $(x^2 - x - 1)^2$, whose roots are, of course, $\alpha, \alpha, \beta, \beta$, where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then,

$$F_n^{(1)} = (A + Bn)\alpha^n + (C + Dn)\beta^n \tag{1.10}$$

for some constants A, B, C and D due to the repeated roots. Since the Fibonacci numbers are a linear combination of the same roots,

$$F_n^{(1)} = (A^* + B^*n)F_{n+1} + (C^* + D^*n)F_{n-1} \tag{1.11}$$

for some constants A^*, B^*, C^*, and D^*. By letting $n = 0, 1, 2, 3$ and solving the resulting system of equations, one finds $A^* = -1/5$, $B^* = C^* = D^* = 1/5$, resulting in

$$5F_n^{(1)} = (n-1)F_{n+1} + (n+1)F_{n-1}\ , \tag{1.12}$$

which leads easily to

$$F_n^{(1)} = (nL_n - F_n)/5 \tag{1.13}$$

where L_n is the nth Lucas number.

Returning again to the auxiliary polynomial for $\left\{F_n^{(1)}\right\}$, since $(x^2 - x - 1)^2 = x^4 - 2x^3 - x^2 + 2x + 1$, we can write

$$F_{n+4}^{(1)} = 2F_{n+3}^{(1)} + F_{n+2}^{(1)} - 2F_{n+1}^{(1)} - F_n^{(1)} \tag{1.14}$$

2. SPECIAL LIMITING RATIOS

It is well known that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha = \frac{1+\sqrt{5}}{2}\ . \tag{2.1}$$

We extend this property of the Fibonacci numbers to the Fibonacci convolution sequences. First, (1.10) gives us

$$F_n^{(1)} = (A + Bn)\alpha^n + (C + Dn)\beta^n$$

for some constants A, B, C and D. Thus one concludes

$$\lim_{n\to\infty} \frac{F_{n+1}^{(1)}}{F_n^{(1)}} = \lim_{n\to\infty} \frac{[A + B(n+1)]\,\alpha + [C + D(n+1)]\,\beta](\beta/\alpha)^n}{A + Bn + (C + Dn)(\beta/\alpha)^n} = \alpha\,.$$

Clearly, this holds for any $\{F_n^{(r)}\}$ since, by examining the auxiliary polynomial,

$$F_n^{(r)} = p_r(n)\alpha^n + q_r(n)\beta^n, \tag{2.2}$$

where $p_r(n)$ and $q_r(n)$ are polynomials in n of degree r. Then, we have

$$\lim_{n\to\infty} \frac{F_{n+1}^{(r)}}{F_n^{(r)}} = \lim_{n\to\infty} \frac{p_r(n+1)\alpha^{n+1} + q_r(n+1)\beta^{n+1}}{p_r(n)\alpha^n + q_r(n)\beta^n} = \lim_{n\to\infty} \frac{p_r(n+1)}{p_r(n)}\,\alpha = \alpha\,. \tag{2.3}$$

While it is not necessary to be able to write $p_r(n)$ and $q_r(n)$ to establish (2.3), it would be interesting to find a recurrence for these polynomials.

It is not difficult to show that

$$\lim_{n\to\infty} \frac{F_n}{F_n^{(1)}} = 0 \tag{2.4}$$

and that

$$\lim_{n\to\infty} \frac{F_n^{(r^*)}}{F_n^{(r)}} = 0, \quad r^* < r\,. \tag{2.5}$$

We also find α^2 as a value for a special limiting ratio. We define

$$W_n^{(r)} = F_{n+1}^{(r)}F_{n-1}^{(r)} - [F_n^{(r)}]^2\,. \tag{2.6}$$

For $r = 0$, the Fibonacci numbers themselves, $W_n^{(0)} = (-1)^n$, but when $r \geqslant 1$, $W_n^{(r)}$ is not a constant. However, we have the surprising limiting ratio,

$$\lim_{n\to\infty} \frac{W_{n+1}^{(r)}}{W_n^{(r)}} = \alpha^2, \quad r \geqslant 1. \tag{2.7}$$

To establish (2.7), we use (2.2) to calculate $W_n^{(r)}$ as

$$\begin{aligned} W_n^{(r)} &= [p_r(n+1)\alpha^{n+1} + q_r(n+1)\beta^{n+1}]\,[p_r(n-1)\alpha^{n-1} + q_r(n-1)\beta^{n-1}] - [p_r(n)\alpha^n + q_r(n)\beta^n]^2 \\ &= [p_r(n+1)p_r(n-1)\alpha^{2n} + q_r(n+1)q_r(n-1)\beta^{2n} + p_r(n+1)q_r(n-1)\alpha^{n+1}\beta^{n-1} \\ &\quad + p_r(n-1)q_r(n+1)\alpha^{n-1}\beta^{n+1}] - [p_r^2(n)\alpha^{2n} + 2p_r(n)q_r(n)\alpha^n\beta^n + q_r^2(n)\beta^{2n}] \\ &= [p_r(n+1)p_r(n-1) - p_r^2(n)]\,\alpha^{2n} + [q_r(n+1)q_r(n-1) - q_r^2(n)]\,\beta^{2n} + R_r(n), \end{aligned}$$

where $R_r(n)$ is a polynomial in n of degree $2r$, but each term contains a factor of α^s or β^t, where s, t are at most two, since $\alpha\beta = -1$. Then, if $p_r(n+1)p_r(n-1) - p_r^2(n) \neq 0$, we find that

$$\lim_{n\to\infty} \frac{W_{n+1}^{(r)}}{W_n^{(r)}} = \frac{F_{n+2}^{(r)}F_n^{(r)} - [F_{n+1}^{(r)}]^2}{F_{n+1}^{(r)}F_{n-1}^{(r)} - [F_n^{(r)}]^2} = \alpha^2\,.$$

Please note that for the Fibonacci numbers themselves, it is indeed true that $p = -q = 1/(\alpha - \beta)$ and

$$p(n+1)p(n-1) - p^2(n) \equiv 0.$$

That there are no other polynomials such that $p(n+1)p(n-1) - p^2(n) \equiv 0$ is proved by considering

$$F_n^{(r)} = p_r(n)\alpha^n + q_r(n)\beta^n\,,$$

where $p_r(n)$ is a polynomial of degree at most r. Consider

$$P(n) = p_r(n+1)p_r(n-1) - p_r^2(n)$$

which is a polynomial of degree at most $2r$. Thus, $P(n) \neq 0$ for more than $2r$ values of n. Clearly, then, for all large enough n, $P(n) \neq 0$.

3. DETERMINANT IDENTITIES FOR THE FIBONACCI CONVOLUTION SEQUENCES

Several interesting determinant identities can be found for the Fibonacci convolution sequences. First, we examine a class of unit determinants. Let

$$D_n = \begin{vmatrix} F_{n+3}^{(1)} & F_{n+2}^{(1)} & F_{n+1}^{(1)} & F_n^{(1)} \\ F_{n+2}^{(1)} & F_{n+1}^{(1)} & F_n^{(1)} & F_{n-1}^{(1)} \\ F_{n+1}^{(1)} & F_n^{(1)} & F_{n-1}^{(1)} & F_{n-2}^{(1)} \\ F_n^{(1)} & F_{n-1}^{(1)} & F_{n-2}^{(1)} & F_{n-3}^{(1)} \end{vmatrix} \tag{3.1}$$

Then it is easily proved that $D_n = 1$ by using (1.14), since replacing the fourth column with a linear combination of the present columns gives us the negative of the first column of D_{n+1}. That is, since

$$-F_{n+4}^{(1)} = -2F_{n+3}^{(1)} - F_{n+2}^{(1)} + 2F_{n+1}^{(1)} + F_n^{(1)},$$

$$D_n = \begin{vmatrix} F_{n+3}^{(1)} & F_{n+2}^{(1)} & F_{n+1}^{(1)} & -F_{n+4}^{(1)} \\ F_{n+2}^{(1)} & F_{n+1}^{(1)} & F_n^{(1)} & -F_{n+3}^{(1)} \\ F_{n+1}^{(1)} & F_n^{(1)} & F_{n-1}^{(1)} & -F_{n+2}^{(1)} \\ F_n^{(1)} & F_{n-1}^{(1)} & F_{n-2}^{(1)} & -F_{n+1}^{(1)} \end{vmatrix}$$

so that $D_n = D_{n+1}$ after making appropriate column exchanges. Lastly, since $D_1 = 1$, $D_n = 1$ for all n.

Now, let $D_n^{(r)}$ be the determinant of order $(2r+2)$ with successive members of the sequence $\{F_n^{(r)}\}$ written along its rows and columns in decreasing order such that $F_n^{(r)}$ appears everywhere along the minor diagonal. Since $\{F_n^{(r)}\}$ has an auxiliary polynomial of degree $(2r+2)$, $F_{n+2r+2}^{(r)}$ is a linear combination of

$$F_{n+2r+1}^{(r)},\ F_{n+2r}^{(r)},\ F_{n+2r-1}^{(r)},\ \cdots,\ F_{n+1}^{(r)},\ F_n^{(r)},$$

so that $D_n^{(r)} = \pm D_{n+1}^{(r)}$ after $(2r+1)$ appropriate column exchanges. The auxiliary polynomial $(x^2 - x - 1)^{r+1}$ has a positive constant term when r is odd, making the last column the negative of the first column of $D_{n+1}^{(r)}$, so that

$$D_n^{(r)} = (-1)^{2r+1}(-1)D_{n+1}^{(r)} = D_{n+1}^{(r)},\quad r \text{ odd};$$

but, for r even, a negative constant term makes the last column equal the first column of $D_{n+1}^{(r)}$, and

$$D_n^{(r)} = (-1)^{2r+1}D_{n+1}^{(r)} = -D_{n+1}^{(r)},\quad r \text{ even}.$$

We need only to evaluate $D_n^{(r)}$ for one value of n, then. Now, $F_n^{(r)} = 0$ for $n = 0, \pm 1, \pm 2, \cdots, \pm r$, and $F_{r+1}^{(r)} = 1$. Thus, $D_{r+1}^{(r)} = (-1)^{r+1}$ since ones appear on the minor diagonal there with zeroes everywhere below. Then, $D_n^{(r)} = 1$ when r is odd, and $D_n^{(r)} = (-1)^n$ when r is even, which can be combined to

$$D_n^{(r)} = (-1)^{n(r+1)}. \tag{3.2}$$

The special case $r = 0$ is the well known formula, $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

A second proof of (3.2) is instructive. Returning to (3.1), apply (1.8) as

$$F_{n+1}^{(r)} = F_{n+2}^{(r+1)} - F_{n+1}^{(r+1)} - F_n^{(r+1)}, \tag{3.3}$$

taking $r = 0$. Subtracting pairs of columns and then pairs of rows gives

$$D_n = \begin{vmatrix} F_{n+2} & F_{n+1} & F_{n+1}^{(1)} & F_n^{(1)} \\ F_{n+1} & F_n & F_n^{(1)} & F_{n-1}^{(1)} \\ F_n & F_{n-1} & F_{n-1}^{(1)} & F_{n-2}^{(1)} \\ F_{n-1} & F_{n-2} & F_{n-2}^{(1)} & F_{n-3}^{(1)} \end{vmatrix} = \begin{vmatrix} 0 & 0 & F_n & F_{n-1} \\ 0 & 0 & F_{n-1} & F_{n-2} \\ F_n & F_{n-1} & F_{n-1}^{(1)} & F_{n-2}^{(1)} \\ F_{n-1} & F_{n-2} & F_{n-2}^{(1)} & F_{n-3}^{(1)} \end{vmatrix} .$$

Thus,

$$D_n = (F_n F_{n-2} - F_{n-1}^2)^2 = 1 .$$

Notice that this proof can be generalized, and after sufficient subtractions, one always makes a block of zeroes in the upper left, with two smaller determinants of the same form in the lower left and upper right, so that $D_n^{(r)}$ is always a product of smaller known determinants $D_n^{(r^*)}$, $r^* < r$, making a proof by induction possible. Each higher order determinant requires more subtractions of pairs of rows and columns, but careful counting of subscripts leads one to

$$D_n^{(r)} = \begin{cases} [D_n^{(r/2)}] \cdot [D_n^{((r-2)/2)}], & r \text{ even}; \\ [D_n^{((r-1)/2)}]^2, & r \text{ odd}; \end{cases} \tag{3.4}$$

which again gives us (3.2).

The process of subtraction of pairs of columns and rows can also be applied to determinants of odd order. For example,

$$D_n^* = \begin{vmatrix} F_{n+2}^{(1)} & F_{n+1}^{(1)} & F_n^{(1)} \\ F_{n+1}^{(1)} & F_n^{(1)} & F_{n-1}^{(1)} \\ F_n^{(1)} & F_{n-1}^{(1)} & F_{n-2}^{(1)} \end{vmatrix} = \begin{vmatrix} 0 & F_n & F_{n-1} \\ F_n & F_n^{(1)} & F_{n-1}^{(1)} \\ F_{n-1} & F_{n-1}^{(1)} & F_{n-2}^{(1)} \end{vmatrix} .$$

Then, by applying (1.13) and known Fibonacci and Lucas identities, one can evaluate D_n^*. The algebra, however, is long and inelegant. One obtains, after patience,

$$D_n^* = (-1)^{n+1} F_n^{(1)} . \tag{3.5}$$

However, D_n^* can also be written out from the form given above on the right, so that

$$D_n^* = (-1)^{n+1} F_n^{(1)} = 2F_n F_{n-1} F_{n-1}^{(1)} - F_{n-1}^{(2)} F_n^{(1)} - F_n^2 F_{n-1}^{(1)}$$

$$[(-1)^{n-1} + F_{n-1}^2] F_n^{(1)} = 2F_n F_{n-1} [F_n^{(1)} - F_{n-2}^{(1)} - F_{n-1}] - F_n^2 F_{n-2}^{(1)}$$

$$[(F_{n-1}F_n - F_{n-1}^2) + F_{n-1}^2 - 2F_n F_{n-1}] F_n^{(1)} = (-2F_n F_{n-1} - F_n^2) F_{n-2}^{(1)} - 2F_n F_{n-1}^2$$

$$-F_n L_{n-2} F_n^{(1)} = -F_n L_n F_{n-2}^{(1)} - 2F_n F_{n-1}^2$$

by applying known Fibonacci identities. Finally, dividing by $-F_n$, $n \neq 0$ and rearranging, we have

$$L_{n-2} F_n^{(1)} - L_n F_{n-2}^{(1)} = 2F_{n-1}^2 , \tag{3.6}$$

which we compare with the known

$$L_{n-2} F_n - L_n F_{n-2} = 2(-1)^n .$$

If we let $D_n^{*(r)}$ denote the determinant of order $(2r+1)$ which has successive members of the sequence $\{F_n^{(r)}\}$ written along its rows and columns in decreasing order such that $\{F_n^{(r)}\}$ appears everywhere along the minor diagonal, we conjecture that

$$D_n^{*(r)} = (-1)^{r(n+1)} F_n^{(r)} . \tag{3.7}$$

Equation (3.7) has been proved for $r = 1$ above, and $r = 0$ is trivial. When $r = 2$, it is possible to prove (3.7) by using the identity

(3.8) $$F_n^{(2)} = [(5n^2 - 2)F_n - 3nL_n]/50$$

as well as (1.13). The algebra, however, is horrendous. The identity (3.8) can be derived by solving for the constants $A, B, C, D, E,$ and F in

$$F_n^{(2)} = (A + Bn + Cn^2)F_n + (D + En + Fn^2)L_n$$

which arises since $\{F_n^{(2)}\}$ has auxiliary polynomial $(x^2 - x - 1)^3$, whose roots are α, α, α and β, β, β.

Two other determinant identities follow without proof.

$$\begin{vmatrix} F_{n+2}^{(1)} & F_{n+1}^{(1)} & F_{n-1}^{(1)} \\ F_{n+1}^{(1)} & F_n^{(1)} & F_{n-2}^{(1)} \\ F_n^{(1)} & F_{n-1}^{(1)} & F_{n-3}^{(1)} \end{vmatrix} = (-1)^n[F_{n-5}^{(1)} + 2F_{n-4}]$$

$$\begin{vmatrix} F_{n+2}^{(1)} & F_n^{(1)} & F_{n-1}^{(1)} \\ F_{n+1}^{(1)} & F_{n-1}^{(1)} & F_{n-2}^{(1)} \\ F_n^{(1)} & F_{n-2}^{(1)} & F_{n-3}^{(1)} \end{vmatrix} = (-1)^n[F_{n-2}^{(1)} - F_{n-2}]$$

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TWO RECURSION RELATIONS FOR $F(F(n))$

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Some time ago, in [1], the question of the existence of a recursion relation for the sequence of Fibonacci numbers with Fibonacci numbers for subscripts was raised. In the present article we give a 6th order non-linear recursion for $f(n) = F(F(n))$.

Proposition. Let $f(n) = F(F(n))$, where $F(n)$ is the nth Fibonacci number, then

$$f(n) = (5f(n-2)^2 + (-1)^{F(n+1)})f(n-3) + (-1)^{F(n)}(f(n-3) - (-1)^{F(n+1)}f(n-6))f(n-2)/f(n-5).$$

Remark. Identity (1) below is given in [2], and identity (2) is proved similarly. Note also that $a \equiv b \pmod 3$ implies that

$$(-1)^{F(a)} = (-1)^{F(b)} = (-1)^{L(a)} = (-1)^{L(b)},$$

which is used frequently.

(1) $$F(a+b) = F(a)L(b) - (-1)^b F(a-b)$$

(2) $$5F(a)F(b) = L(a+b) - (-1)^a L(b-a).$$

Proof of Proposition. In (1), let $a = F(n-2)$, $b = F(n-1)$ to obtain

$$\begin{aligned} f(n) &= f(n-2)L(F(n-1)) - (-1)^{F(n-1)}F(-F(n-3)) \\ &= f(n-2)L(F(n-1)) - (-1)^{F(n-1)}(-1)^{F(n-3)+1}f(n-3) \\ &= f(n-2)L(F(n-1)) + (-1)^{F(n+1)}f(n-3). \end{aligned}$$

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A MATRIX SEQUENCE ASSOCIATED WITH A CONTINUED FRACTION EXPANSION OF A NUMBER

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INTRODUCTION

In Section 1, we introduce a matrix sequence each of whose terms is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, denoted by L, or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, denoted by R. We call such sequences LR-sequences. A one-to-one correspondence is established between the set of LR-sequences and the continued fraction expansions of numbers in the unit interval. In Section 2, a partial ordering of the numbers in the unit interval is given in terms of the LR-sequences and the resulting partially ordered set is a tree, called the Q-tree. A continued fraction expansion of a number is interpreted geometrically as an infinite path in the Q-tree and conversely. In Section 3, we consider a special function, g, defined on the Q-tree. We show that g is continuous and strictly increasing, but that g is not absolutely continuous. The proof that g is not absolutely continuous is a measure theoretic argument that utilizes Khinchin's constant and the Fibonacci sequence.

1. THE LR-SEQUENCE

We denote the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ by L and the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by R.

Definition. An LR-sequence is a sequence of 2×2 matrices, $M_1, M_2, \cdots, M_i, \cdots$ such that for each i, $M_i = L$ or $M_i = R$.

We shall represent points in the plane by column vectors with two components. The set $C = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \middle| \text{ both } \alpha \text{ and } \beta \text{ are non-negative and at least one of } \alpha \text{ and } \beta \text{ is positive} \right\}$ will be called the positive cone. Our present objective is to associate with each vector in the positive cone an LR-sequence.

Definition. A vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in C$ is said to accept the LR-sequence $M_1, M_2, \cdots, M_i, \cdots$ if and only if there is a sequence

$$\begin{pmatrix} \gamma_0 \\ \delta_0 \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \delta_1 \end{pmatrix}, \cdots, \begin{pmatrix} \gamma_i \\ \delta_i \end{pmatrix}, \cdots$$

whose terms are vectors in C, such that

$$\begin{pmatrix} \gamma_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

and for each $i \geqslant 1$, $\begin{pmatrix} \gamma_{i-1} \\ \delta_{i-1} \end{pmatrix} = M_i \begin{pmatrix} \gamma_i \\ \delta_i \end{pmatrix}$.

If $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in C$ and $\alpha \leqslant \beta$, then $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta - \alpha \end{pmatrix}$ and $\begin{pmatrix} \alpha \\ \beta - \alpha \end{pmatrix} \in C$.

If $\beta \leqslant \alpha$, then

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = R \begin{pmatrix} \alpha - \beta \\ \beta \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \alpha - \beta \\ \beta \end{pmatrix} \in C.$$

By induction it can be shown that every vector in C accepts at least one LR-sequence. If α is a positive irrational number, then $\begin{pmatrix} \alpha \\ 1 \end{pmatrix}$ accepts exactly one LR-sequence; if α is a positive rational number, then $\begin{pmatrix} \alpha \\ 1 \end{pmatrix}$ accepts two LR-sequences.

The expression $R^{a_0} L^{a_1} R^{a_2} \cdots$ will be used to designate the LR-sequence which consists of a_0 R's, followed by a_1 L's, followed by a_2 R's, etc.

We shall follow Khinchin's notation for continued fractions and express the continued fraction expansion of

$$\alpha, \quad \alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \qquad \text{as } \alpha = [a_0; a_1, a_2, \cdots].$$

The remainder after n elements in the expansion of α is denoted by $r_n = [a_n; a_{n+1}, a_{n+2}, \cdots]$. All the well known terms and results of continued fractions used in this paper may be found in [1].

Theorem 1. Let $\alpha = [a_0; a_1, a_2, \cdots]$ and let $\binom{\alpha}{1}$ accept the LR-sequence $R^{b_0}L^{b_1}R^{b_2}\cdots$. Then $b_i = a_i$ for all $i \geqslant 0$ and for

$$k_n = \sum_{i=0}^{n} b_i, \quad \frac{\gamma_{k_n}}{\delta_{k_n}} = r_{n+1}(\alpha)$$

if n is odd and

$$\frac{\gamma_{k_n}}{\delta_{k_n}} = \frac{1}{r_{n+1}(\alpha)}$$

if n is even.

Proof. Since $\binom{\alpha}{1}$ accepts $R^{b_0}L^{b_1}R^{b_2}\cdots$, there exists a sequence $\binom{\gamma_0}{\delta_0}, \binom{\gamma_1}{\delta_1}, \binom{\gamma_2}{\delta_2}, \cdots$, whose terms are vectors in C, such that $\binom{\gamma_0}{\delta_0} = \binom{\alpha}{1}$ and such that if n is even and $k_n \leqslant k \leqslant k_{n+1}$, then

$$\binom{\alpha}{1} = R^{b_0}L^{b_1}R^{b_2}\cdots R^{b_n}L^{k-k_n}\binom{\gamma_k}{\delta_k}$$

and if n is odd, then

$$\binom{\alpha}{1} = R^{b_0}L^{b_1}R^{b_2}\cdots L^{b_n}R^{k-k_n}\binom{\gamma_k}{\delta_k}.$$

Since

$$r_n = [a_n; r_{n+1}], \quad r_{n+1} = \frac{1}{r_n - a_n} \quad \text{and} \quad a_n = [r_n].$$

Therefore a_n is the least integer j such that $r_n - j < 1$.

We now use induction on n. For $n = 0$, $r_0 = \alpha$. Since a_0 is the least integer j such that

$$\alpha - j < 1, \quad \binom{\alpha}{1} = R^{a_0}\binom{\gamma_{a_0}}{\delta_{a_0}},$$

where $\gamma_{a_0} = \alpha - a_0$ and $\delta_{a_0} = 1$. Thus

$$b_0 = a_0 \quad \text{and} \quad \frac{\gamma_{k_0}}{\delta_{k_0}} = \frac{\alpha - a_0}{1} = \frac{1}{r_1}.$$

We assume the result for $0 \leqslant t < n$ and then consider two cases.

CASE 1. Let n be odd. Then

$$\frac{\gamma_{k_n - b_n}}{\delta_{k_n - b_n}} = \frac{\gamma_{k_{n-1}}}{\delta_{k_{n-1}}} = \frac{1}{r_n} < 1$$

and since a_n is the least integer j such that $r_n - j < 1$,

$$\binom{\gamma_{k_n - b_n}}{\delta_{k_n - b_n}} = L^{a_n}\binom{\gamma_{k_n}}{\delta_{k_n}}, \quad \text{where} \quad \gamma_{k_n} = \gamma_{k_n - b_n} \quad \text{and} \quad \delta_{k_n} = \delta_{k_n - b_n} - a_n\gamma_{k_n - b_n}.$$

Thus

$$b_n = a_n \quad \text{and} \quad \frac{\gamma_{k_n}}{\delta_{k_n}} = \frac{\gamma_{k_n - b_n}}{\delta_{k_n - b_n} - a_n\gamma_{k_n - b_n}} = \frac{1}{r_n - a_n} = r_{n+1}.$$

CASE 2. Let n be even. Then

$$\frac{\gamma_{k_n-b_n}}{\delta_{k_n-b_n}} = \frac{\gamma_{k_{n-1}}}{\delta_{k_{n-1}}} = r_n$$

and since a_n is the least integer j such that $r_n - j < 1$,

$$\begin{pmatrix}\gamma_{k_n-b_n}\\ \delta_{k_n-b_n}\end{pmatrix} = R^{a_n}\begin{pmatrix}\gamma_{k_n}\\ \delta_{k_n}\end{pmatrix}, \quad \text{where} \quad \gamma_{k_n} = \gamma_{k_n-b_n} - a_n\delta_{k_n-b_n} \quad \text{and} \quad \delta_{k_n} = \delta_{k_n-b_n}.$$

Thus

$$b_n = a_n \quad \text{and} \quad \frac{\gamma_{k_n}}{\delta_{k_n}} = \frac{\gamma_{k_n-b_n} - a_n\delta_{k_n-b_n}}{\delta_{k_n-b_n}} = r_n - a_n = \frac{1}{r_{n+1}}.$$

The preceding theorem can be extended to hold for rational α by modifying the notation as follows:

(i) If $a_n = 1$, express $[0; a_1, a_2, \cdots, a_n]$ as either

$$[0; a_1, a_2, \cdots, a_n, \infty] \quad \text{or} \quad [0; a_1, a_2, \cdots, a_{n-1}+1, \infty] \quad \text{or}$$

(ii) If $a_n \neq 1$, express $[0; a_1, a_2, \cdots, a_n]$ as either

$$[0; a_1, a_2, \cdots, a_n - 1, \infty] \quad \text{or} \quad [0; a_1, a_2, \cdots, a_n, \infty].$$

When we permit the use of these expressions we shall speak of continued fractions *in the wider sense.* One sees that the method of LR-sequences provides a common form for the continued fraction expansions for both rational and irrational numbers. (The non-uniqueness, however, of the expansion of a rational number still persists.)

Definition. Let $\alpha = [a_0; a_1, a_2, \cdots]$. The kth order convergent of α is

$$\frac{p_k(\alpha)}{q_k(\alpha)} = [a_0; a_1, a_2, \cdots, a_k],$$

where

$$p_{-1}(\alpha) = 1, \quad p_0(\alpha) = 0, \quad q_{-1}(\alpha) = 0, \quad q_0(\alpha) = 1,$$

and for $k \geqslant 1$,

$$p_k(\alpha) = a_k p_{k-1}(\alpha) + p_{k-2}(\alpha) \quad \text{and} \quad q_k(\alpha) = a_k q_{k-1}(\alpha) + q_{k-2}(\alpha).$$

When no confusion will result, we shall omit the reference to α and write p_k, q_k for $p_k(\alpha)$, $q_k(\alpha)$.

An important proposition in the theory of continued fractions is: If

$$\alpha = [a_0; a_1, a_2, \cdots, a_n, r_{n+1}], \quad \text{then} \quad \alpha = \frac{p_{n+1}}{q_{n+1}} = \frac{r_{n+1}p_n + p_{n-1}}{r_{n+1}q_n + q_{n-1}}.$$

We give an analogue of this result in the following theorem and its corrolary.

Theorem 2. If $\alpha = [0; a_1, a_2, \cdots]$, $\binom{\alpha}{1}$ accepts the LR-sequence $M_1, M_2, \cdots$, and

$$k_n = \sum_{i=1}^{n} a_i,$$

then

$$\prod_{i=1}^{k_n} M_i = \begin{cases} \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} & \text{if } n \text{ is even} \\ \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We use induction on n. For $n = 1$,

$$\prod_{i=1}^{k_1} M_i = L^{a_1} = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} = \begin{pmatrix} p_1 & p_0 \\ q_1 & q_0 \end{pmatrix}.$$

We assume the result for $1 \leqslant t < n$ and then consider two cases.

CASE 1. Let n be even.

$$\prod_{i=1}^{k_n} M_i = \left(\prod_{i=1}^{k_{n-1}} M_i\right) R^{a_n} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix}\begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_{n-1} & a_n p_{n-1} + p_{n-2} \\ q_{n-1} & a_n q_{n-1} + q_{n-2} \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}.$$

CASE 2. Let n be odd.

$$\prod_{i=1}^{k_n} M_i = \left(\prod_{i=1}^{k_{n-1}} M_i\right) L^{a_n} = \begin{pmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_n & 1 \end{pmatrix} = \begin{pmatrix} p_{n-2} + a_n p_{n-1} & p_{n-1} \\ q_{n-2} + a_n q_{n-1} & q_{n-1} \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.$$

Corollary. If $\alpha = [0; a_1, a_2, \cdots]$, $\binom{\alpha}{1}$ accepts the LR-sequence $M_1, M_2, \cdots$, and

$$k_n = \sum_{i=1}^{n} a_i, \quad \text{then} \quad \binom{\alpha}{1} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}\begin{pmatrix} \gamma_{k_n} \\ \delta_{k_n} \end{pmatrix}, \quad \text{where} \quad \frac{\gamma_{k_n}}{\delta_{k_n}} = r_{n+1}(\alpha).$$

The well known result,

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n,$$

is an immediate consequence of the above theorem and the fact that det (L) = det (R) = 1.

2. THE Q-TREE

Although $\binom{\alpha}{1}$ accepts two LR-sequences when α is rational, these two sequences coincide up through a certain initial segment.

Definition. Let α be a positive rational number and let $\binom{\alpha}{1}$ accept the LR-sequence $M_1, M_2, \cdots$. We call the initial segment $M_1, M_2, \cdots, M_n$ a head of α if and only if

$$\binom{\alpha}{1} = M_1, M_2, \cdots, M_n \binom{1}{1}.$$

If α is a positive rational number, the head of α exists and is unique. Thus if $M_1, M_2, \cdots, M_n$ is the head of α, then the two LR-sequences accepted by $\binom{\alpha}{\beta}$ are $M_1, M_2, \cdots, M_n, R, L, L, L, \cdots$ and $M_1, M_2, \cdots, M_n, L, R, R, R, \cdots$.

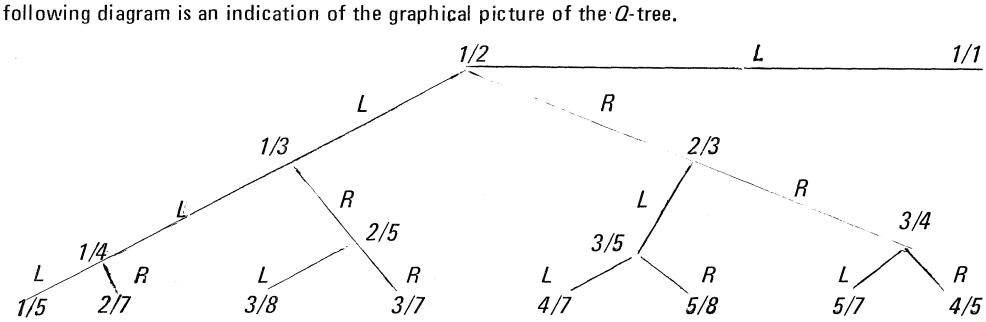
Definition. Let α_1 and α_2 be rational numbers in $(0,1]$. We say that α_1 is Q-related to α_2 if and only if the head of α_1 is an initial segment of the head of α_2.

The Q relation is a partial ordering of the rational numbers in $(0,1]$, and the resulting partially ordered set is a tree.

Definition. The set of rational numbers in $(0,1]$ partially ordered by Q is called the Q-tree.

We may now interpret the continued fraction expansion of a number (in the wider sense) geometrically as an infinite path in the Q-tree. Conversely, any infinite path in the Q-tree determines an LR-sequence and thus the continued fraction expansion (in the wider sense) for some number.

The following diagram is an indication of the graphical picture of the Q-tree.



(etc.)

Figure 1

3. THE FUNCTION g

Definition. Let $\alpha \in [0,1]$ and let $\binom{\alpha}{1}$ accept the LR-sequence $M_1, M_2, \cdots$. We then define g on the unit interval by

$$g(\alpha) = 2\sum_{j=1}^{\infty} c_j 2^{-j}, \quad \text{where} \quad c_j = \begin{cases} 0 & \text{if } M_j = L \\ 1 & \text{if } M_j = R \end{cases}.$$

It is clear that g is a one-to-one function.

Theorem 3. For $0 \leqslant \alpha \leqslant 1$, g is a strictly increasing function.

Proof. Let $0 \leqslant \alpha < \beta \leqslant 1$, $\alpha = [0; a_1, a_2, \cdots]$, $\beta = [0; b_1, b_2, \cdots]$ and let t be the least integer n such that $a_n \neq b_n$. Thus $p_k(\alpha) = p_k(\beta)$ and $q_k(\alpha) = q_k(\beta)$ for $0 \leqslant k < t$.

Now

$$\alpha < \beta \text{ iff } \frac{r_t(\beta)p_{t-1} + p_{t-2}}{r_t(\beta)q_{t-1} + q_{t-2}} - \frac{r_t(\alpha)p_{t-1} + p_{t-2}}{r_t(\alpha)q_{t-1} + q_{t-2}} > 0$$

if and only if

$$r_t(\alpha)(p_{t-2}q_{t-1} - p_{t-1}q_{t-2}) + r_t(\beta)(p_{t-1}q_{t-2} - p_{t-2}q_{t-1}) > 0$$

if and only if

$$(r_t(\alpha) - r_t(\beta))(-1)^{t-1} > 0.$$

Therefore, $r_t(\alpha) > r_t(\beta)$ if and only if t is odd. Since

$$r_t(\alpha) = [a_t; r_{t+1}(\alpha)] \quad \text{and} \quad r_t(\beta) = [b_t; r_{t+1}(\beta)], \quad a_t > b_t$$

if and only if t is odd. We consider two cases determined by the parity of t.

CASE 1. Let t be odd. In this case $a_t > b_t$. If

$$r = \sum_{i=1}^{t} a_i, \quad \text{then} \quad g(\alpha) \leqslant g\left(\frac{p_{t-1}(\alpha)}{q_{t-1}(\alpha)}\right) + \frac{2}{2^{r-1}}.$$

If

$$s = \sum_{i=1}^{t} b_i, \quad \text{then} \quad g(\beta) \geqslant g\left(\frac{p_{t-1}(\beta)}{q_{t-1}(\beta)}\right) + \frac{2}{2^{s}}.$$

Since g is a one-to-one function, $s < r$ and

$$g\left(\frac{p_{t-1}(\alpha)}{q_{t-1}(\alpha)}\right) = g\left(\frac{p_{t-1}(\beta)}{q_{t-1}(\beta)}\right) \quad \text{implies that} \quad g(\alpha) - g(\beta) \leqslant \frac{2}{2^{r-1}} - \frac{2}{2^{s}} \leqslant 0$$

with equality holding if and only if $\alpha = \beta$. Thus $g(\alpha) < g(\beta)$.

CASE 2. Let t be even. In this case $a_t < b_t$ and so $s > r$. Now

$$g(\alpha) \leqslant g\left(\frac{p_{t-1}(\alpha)}{q_{t-1}(\alpha)}\right) + \frac{2}{2^{r-a_t}} \sum_{i=1}^{a_t} \frac{1}{2^i} + \frac{2}{2^{r+1}} \quad \text{and} \quad g(\beta) \geqslant g\left(\frac{p_{t-1}(\beta)}{q_{t-1}(\beta)}\right) + \frac{2}{2^{s-b_t}} \sum_{i=1}^{b_t} \frac{1}{2^i}.$$

Since $r - a_t = s - b_t$,

$$g(\alpha) - g(\beta) = \frac{2}{2^{r-a_t}}\left[\sum_{i=1}^{a_t} \frac{1}{2^i} - \sum_{i=1}^{b_t} \frac{1}{2^i}\right] + \frac{2}{2^{r+1}} = -\sum_{i=r+1}^{s} \frac{2}{2^i} + \frac{2}{2^{r+1}} \leqslant 0$$

with equality holding if and only if $\alpha = \beta$. Thus $g(\alpha) < g(\beta)$.

Corollary. For $\alpha \in [0,1]$, $g'(\alpha)$ exists and is finite almost everywhere.

Theorem 4. For $0 \leqslant \alpha \leqslant 1$, g is a continuous function.

Proof. Let $\alpha \in [0,1]$, $\alpha = [0; a_1, a_2, \cdots]$. For any $\epsilon > 0$, choose an n such that

$$\frac{1}{2^{2n}} < \epsilon .$$

Since the even ordered convergents form an increasing sequence converging to α and the odd ordered convergents form a decreasing sequence converging to α, (see [1], p. 6 and p. 9),

$$\frac{p_{2n}}{q_{2n}} < \alpha < \frac{p_{2n+1}}{q_{2n+1}}. \quad \text{Let} \quad \delta = \left|\alpha - \frac{p_{2n+1}}{q_{2n+1}}\right|. \quad \text{Since} \quad \left|\alpha - \frac{p_{2n+1}}{q_{2n+1}}\right| < \left|\alpha - \frac{p_{2n}}{q_{2n}}\right| .$$

If $\beta \in [0,1]$ and $|\alpha - \beta| < \delta$, then either $\frac{p_{2n}}{q_{2n}} < \alpha \leqslant \beta < \frac{p_{2n+1}}{q_{2n+1}}$ or $\frac{p_{2n}}{q_{2n}} < \beta \leqslant \alpha < \frac{p_{2n+1}}{q_{2n+1}}$.

Since g is an increasing function,

$$|g(\alpha) - g(\beta)| < \left| g\left(\frac{p_{2n+1}}{q_{2n+1}}\right) - g\left(\frac{p_{2n}}{q_{2n}}\right)\right| = 2 \cdot 2^{-\sum_{i=1}^{2n+1} a_i} \leqslant \frac{2}{2^{n+1}} < \epsilon .$$

In the next theorem, we make use of the Fibonacci sequence $\langle f_n \rangle$, where $f_0 = 1$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$.

Theorem 5. The derivative of g at $u = (-1 + \sqrt{5})/2$ is infinite.

Proof. The continued fraction expansion of u is $[0; a_1, a_2, \cdots]$, where $a_i = 1$ for all $i \geqslant 1$. Therefore,

$$p_n = p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = q_{n-1} + q_{n-2} .$$

Since $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$, $q_0 = 1$, $p_n = f_n$ and $q_n = f_{n+1}$.

If

$$\frac{p_{2n}}{q_{2n}} < x \leqslant \frac{p_{2n+2}}{q_{2n+2}} < u,$$

then

$$\lim_{x \to u^-} \frac{g(u) - g(x)}{u - x} \geqslant \lim_{n \to \infty} \frac{g(u) - g\left(\frac{p_{2n+2}}{q_{2n+2}}\right)}{u - \frac{p_{2n}}{q_{2n}}} = \lim_{n \to \infty} \frac{\sum_{i=1}^{\infty} \frac{2}{2^{2i}} - \sum_{i=1}^{n+1} \frac{2}{2^{2i}}}{u - \frac{f_{2n}}{f_{2n+1}}} ,$$

which can be shown equal to (see [2], p. 15)

$$\lim_{n \to \infty} \frac{\sum_{i=n+2}^{\infty} \frac{2}{2^{2i}}}{u\left[1 - \frac{u^{-2n} - u^{2n}}{u^{-2n} + u^{2n+2}}\right]} = \lim_{n \to \infty} \frac{2(1 + u^{4n+2})}{3u(u^2 - 1 + 2u^{-4n})} \left(\frac{1}{4u^4}\right)^n .$$

Since

$$\frac{1}{4u^4} = \frac{7 + 3\sqrt{5}}{8} > 1, \quad \lim_{x \to u^-} \frac{g(u) - g(x)}{u - x} = \infty .$$

Similarly,

$$\lim_{x \to u^+} \frac{g(u) - g(x)}{u - x} = \infty .$$

We omit the details

Definition. The numbers $\alpha = [a_0; a_1, a_2, \cdots]$ and $\beta = [b_0; b_1, b_2, \cdots]$ are said to be equivalent provided there exists an N such that $a_n = b_n$ for $n \geqslant N$.

Corollary 1. If $\alpha = [a_0; a_1, a_2, \cdots]$ is equivalent to u, then $g'(\alpha) = \infty$.

Proof. Since α is equivalent to u, there exists an N such that $a_n = 1$ for $n \geqslant N$. If

$$\frac{p_{2n}}{q_{2n}} < x \leqslant \frac{p_{2n+2}}{q_{2n+2}} < \alpha < \frac{p_{2n+1}}{q_{2n+1}} ,$$

where $2n \geqslant N$, then

$$\lim_{n\to\alpha^-}\frac{g(a)-g(x)}{a-x} \geqslant \lim_{n\to\infty}\frac{g(a)-g\left(\frac{p_{2n+2}}{q_{2n+2}}\right)}{\frac{p_{2n+1}}{q_{2n+1}}-\frac{p_{2n}}{q_{2n}}} = \lim_{n\to\infty}\sum_{i=n+2}^{\infty}\frac{2}{2^{2i}}(q_{2n}q_{2n+1}).$$

Since $a_n = 1$ for $n \geqslant N$,

$$q_n \geqslant f_n = \frac{u^n-(-u)^{-n}}{\sqrt{5}} .$$

Thus

$$\lim_{n\to\alpha^-}\frac{g(a)-g(x)}{a-x} \geqslant \lim_{n\to\infty}\frac{2}{15\cdot 4^n}(u^{2n}-u^{-2n})(u^{2n+1}+u^{-2n-1}) = \lim_{n\to\infty}\frac{2}{15}(u^{8n+1}+u^{4n-1}-u^{-1})\left(\frac{1}{4u^4}\right)^n$$

Since $1/4u^4 = (7+3\sqrt{5})/8 > 1$,

$$\lim_{x\to\alpha^-}\frac{g(a)-g(x)}{a-x} = \infty .$$

Similarly

$$\lim_{x\to\alpha^+}\frac{g(a)-g(x)}{a-x} = \infty .$$

Corollary 2. In every subinterval of $[0,1]$ there exists a γ such that $g'(\gamma) = \infty$.

Proof. Let

$$(\alpha, \beta] \subset (0,1], \quad \alpha = [0; a_1, a_2, \cdots] \quad \text{and} \quad \beta = [0; b_1, b_2, \cdots] .$$

We may assume that β is not equivalent to u for if it is, there is nothing to prove.

Let t be the least integer n such that $a_n \neq b_n$. Choosing n such that $2n > t$ and $b_{2n+2} > 1$, we define

$$x = [0; b_1, b_2, \cdots, b_{2n}, \infty], \quad \gamma = [0; b_1, b_2, \cdots, b_{2n+1}, 1, 1, 1, \cdots], \quad \text{and} \quad y = [0; b_1, b_2, \cdots, b_{2n+2}, \infty].$$

Then $\alpha < x < \gamma < y < \beta$ and γ is equivalent to u. Thus the derivative of g at γ is infinite.

The measure used in this next theorem is Lebesgue measure. The measure of a set A is denoted by $m(A)$.

Theorem 6. For almost all $\alpha = [0; a_1, a_2, \cdots] \in (0, 1]$, $g'(\alpha) = 0$.

Proof. Let

$$A = \left\{\alpha \in (0,1]: \lim_{n\to\infty}\left(\prod_{i=1}^{n} a_i\right)^{1/n} = \text{Khinchin's constant}\right\} ,$$

$$B = \left\{\alpha \in (0,1]: g'(\alpha) \text{ exists and is finite}\right\}, \text{ and}$$

$$C = \left\{\alpha \in (0,1]: a_n > n\log n \text{ for infinitely many values of } n\right\}.$$

Since (see [1], pp. 93, 94),

$$m(A) = m(B) = m(C) = 1,$$
$$m(A \cap B \cap C) = 1.$$

Let

$$\alpha \in A \cap B \cap C$$

and let $\{x_n\}$ be any sequence converging to α. We define a second sequence $\{y_n\}$ in terms of the partial quotients, p_m/q_m, of α. Let

$$y_n = \left\{\frac{p_m}{q_m} : m \text{ is the greatest integer such that (i) } |\alpha - x_n| \leqslant \left|\alpha - \frac{p_m}{q_m}\right| \text{ and}\right.$$

$$\left.\text{(ii) } (\alpha - x_n) \text{ and } \left(\alpha - \frac{p_m}{q_m}\right) \text{ have the same sign}\right\}$$

We note that m is an unbounded, non-decreasing function of n and thus m goes to infinity as n does and conversely. Since g is a strictly increasing function and noting that

$$\left|a-\frac{p_{m+2}}{q_{m+2}}\right| < |a-x_n| \quad \text{and that} \quad \left(a-\frac{p_{m+2}}{q_{m+2}}\right)$$

has the same sign as

$$\left(a-\frac{p_m}{q_m}\right),$$

we have

$$\left|\frac{g(a)-g(x_n)}{a-x_n}\right| \leqslant \left|\frac{g(a)-g\left(\frac{p_m}{q_m}\right)}{a-x_n}\right|$$

$$< \left|\frac{g(a)-g\left(\frac{p_m}{q_m}\right)}{a-\frac{p_{m+2}}{q_{m+2}}}\right|$$

$$= \left|g(a)-g\left(\frac{p_m}{q_m}\right)\right| [q_{m+2}(q_{m+2}+q_{m+3}] \quad \text{[See [1], p. 20.]}$$

$$< \left|g(a)-g\left(\frac{p_m}{q_m}\right)\right| 2q_{m+3}^2$$

$$\leqslant (2\cdot 2^{-k_m})2q_{m+3}^2, \quad \text{where} \quad k_m = \sum_{i=1}^{m} a_i .$$

Since Khinchin's constant is < 3,

$$q_m = a_m q_{m-1} + q_{m-2} < 2^m \prod_{i=1}^{m} a_i$$

and $a \in A$, we have that

$$q_{m+3}^2 < \left(2^{m+3} \prod_{i=1}^{m+3} a_i\right)^2 < 2^{2m+6} 3^{2m+6}$$

for sufficiently large values of m. Now $a \in C$ implies that $k_m > m \log m$ for infinitely many values of m and thus

$$\left|\frac{g(a)-g(x_n)}{a-x_n}\right| < 2^8 \cdot 3^6 \left(\frac{36}{2^{\log m}}\right)^m$$

for infinitely many values of m and n. As n goes to infinity, m goes to infinity and hence given any positive ϵ, the inequality

$$\left|\frac{g(a)-g(x_n)}{a-x_n}\right| < \epsilon$$

will be satisfied for infinitely many values of n. Since $a \in B$, $g'(a)$ exists and therefore $g'(a) = 0$.

Corollary. The function g is not absolutely continuous.

Proof. Since g is not a constant function and for almost all $a \in (0,1]$ $g'(a) = 0$, it follows from a well known theorem that g is not absolutely continuous. (See [3], p. 90.)

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GENERALIZED LUCAS SEQUENCES

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1. INTRODUCTION

In working with linear recurrence sequences, the generating functions are of the form

$$\frac{q(x)}{p(x)} = \sum_{n=0}^{\infty} a_n x^n , \tag{1.1}$$

where $p(x)$ is a polynomial and $q(x)$ is a polynomial of degree smaller than $p(x)$. In multisecting the sequence $\{a_n\}$ it is necessary to find polynomials $P(x)$ whose roots are the kth power of the roots of $p(x)$. Thus, we are led to the elementary symmetric functions.

Let

$$p(x) = \prod_{i=1}^{n} (x - a_i) = x^n - p_1 x^{n-1} + p_2 x^{n-2} - p_3 x^{n-3} + \cdots + (-1)^k p_k x^{n-k} + \cdots + (-1)^n p_n , \tag{1.2}$$

where p_k is the sum of products of the roots taken k at a time. The usual problem is, given the polynomial $p(x)$, to find the polynomial $P(x)$ whose roots are the kth powers of the roots of $p(x)$,

$$P(x) = x^n - P_1 x^{n-1} + P_2 x^{n-2} - P_3 x^{n-3} + \cdots + (-1)^n P_n . \tag{1.3}$$

There are two basic problems here. Let

$$S_k = a_1^k + a_2^k + a_3^k + \cdots + a_n^k , \tag{1.4}$$

where

$$p(x) = (x - a_1)(x - a_2) \cdots (x - a_n) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n$$

and $c_k = (-1)^k p_k$. then Newton's Identities (see Conkwright [1])

$$\begin{aligned} S_1 + c_1 &= 0 \\ S_2 + S_1 c_1 + 2c_2 &= 0 \\ &\cdots \\ S_n + S_{n-1} c_1 + \cdots + S_1 c_{n-1} + n c_n &= 0 \\ S_{n+1} + S_n c_1 + \cdots + S_1 c_n + (n+1) c_{n+1} &= 0 \end{aligned} \tag{1.5}$$

can be used to compute S_k for $S_1, S_2, \cdots, S_n$. Now, once these first n values are obtained, the recurrence relation

$$S_{n+1} + S_n c_1 + S_{n-1} c_2 + \cdots + S_1 c_n = 0 \tag{1.6}$$

will allow one to get the next value S_{n+1} and all subsequent values of S_m are determined by recursion.

Returning now to the polynomial $P(x)$,

$$P(x) = (x - a_1^k)(x - a_2^k)(x - a_3^k) \cdots (x - a_n^k) = x^n + Q_1 x^{n-1} + Q_2 x^{n-2} + \cdots + Q_n , \tag{1.7}$$

where

$$Q_1 = a_1^k + a_2^k + \cdots + a_n^k = S_k$$

and it is desired to find the $Q_1, Q_2, Q_3, \cdots, Q_n$. Clearly, one now uses the Newton identities (1.5) again, since $S_k, S_{2k}, S_{3k}, \cdots, S_{nk}$ can be found from the recurrence for S_m, where we know $S_k, S_{2k}, S_{3k}, \cdots, S_{nk}$ and

wish to find the recurrence for the k-sected sequence. Before, we had the auxiliary polynomial for S_m and computed the $S_1, S_2, \cdots, S_n$. Here, we have $S_k, S_{2k}, \cdots, S_{nk}$ and wish to calculate the coefficients of the auxiliary polynomial $P(x)$. Given a sequence S_m and that it satisfies a *linear recurrence* of order n, one can use Newton's identities to obtain that recurrence. This requires only that $S_1, S_2, S_3, \cdots, S_n$ be known. If

$$S_{n+1} + (S_n c_1 + S_{n-1} c_2 + \cdots + S_1 c_n) + (n+1)c_{n+1} = 0$$

is used, then $S_{n+1} = -(S_n c_1 + \cdots + S_1 c_n)$ and $c_{n+1} = 0$.

Suppose that we know that $L_1, L_2, L_3, L_4, \cdots$, the Lucas sequence, satisfies a linear recurrence of order two. Then $L_1 + c_1 = 0$ yields $c_1 = -1$; $L_2 + L_1 c_1 + 2c_2 = 0$ yields $c_2 = -1$; and $L_3 + L_2 c_1 + L_1 c_2 + 3c_3 = 0$ yields $c_3 = 0$. Thus, the recurrence for the Lucas numbers is

$$L_{n+2} - L_{n+1} - L_n = 0.$$

We next seek the recurrence for $L_k, L_{2k}, L_{3k}, \cdots$. $L_{nk} = \alpha^{nk} + \beta^{nk}$ is a Lucas-type sequence and $L_k + Q_1 = 0$ yields $Q_1 = -L_k$; $L_{2k} + c_1 L_k + 2c_2 = 0$ yields $L_{2k} - L_k^2 + 2c_2 = 0$, but $L_k^2 = L_{2k} + 2(-1)^k$ so that

$$L_{2k} - L_k^2 + 2c_2 = 0$$

gives $c_2 = (-1)^k$. Thus, the recurrence for L_{nk} is

$$L_{(n+2)k} - L_k L_{(n+1)k} + (-1)^k L_{nk} = 0.$$

This one was well known. Suppose as a second example we deal with the generalized Lucas sequence associated with the Tribonacci sequence. Here, $S_1 = 1$, $S_2 = 3$, and $S_3 = 7$, so that $S_1 + c_1 = 0$ yields $c_1 = -1$;

$$S_2 + c_1 S_2 + 2c_2 = 0 \quad \text{yields} \quad c_2 = -1,$$

and

$$S_3 + c_1 S_2 + c_2 S_1 + 3c_3 = 0 \quad \text{yields} \quad c_3 = -1.$$

Here,

$$S_k = \alpha^k + \beta^k + \gamma^k,$$

where α, β, γ are roots of

$$x^3 - x^2 - x - 1 = 0.$$

Suppose we would like to find the recurrence for S_{nk}. Using Newton's identities,

$$S_k + Q_1 = 0 \qquad Q_1 = -S_k$$

$$S_{2k} + S_k(-S_k) + 2Q_2 = 0 \qquad Q_2 = \tfrac{1}{2}(S_k^2 - S_{2k})$$

$$S_{3k} + S_{2k}(-S_k) + S_k[\tfrac{1}{2}(S_k^2 - S_{2k})] + 3Q_3 = 0 \qquad Q_3 = \tfrac{1}{6}(S_k^3 - 3S_k S_{2k} + 2S_{2k})$$

This is, of course, correct, but it doesn't give the neatest value. What is Q_2 but the sum of the product of roots taken two at a time,

$$Q_2 = (\alpha\beta)^k + (\alpha\gamma)^k + (\beta\gamma)^k = \frac{1}{\gamma^k} + \frac{1}{\beta^k} + \frac{1}{\alpha^k} = S_{-k}$$

and $Q_3 = (\alpha\beta\gamma)^k = 1$. Thus, the recurrence for S_{nk} is

$$S_{(n+3)k} - S_k S_{(n+2)k} + S_{-k} S_{(n+1)k} + S_{nk} = 0. \tag{1.8}$$

This and much more about the Tribonacci sequence and its associated Lucas sequence is discussed in detail by Trudy Tong [3].

2. DISCUSSION OF E-2487

A problem in the Elementary Problem Section of the *American Mathematical Monthly* [2] is as follows:

If $S_k = a_1^k + a_2^k + \cdots + a_n^k$ and $S_k = k$ for $1 \leqslant k \leqslant n$, find S_{n+1}.

From $S_k = a_1^k + \cdots + a_n^k$, we know that the sequence S_m obeys a linear recurrence of order n. From Newton's Identities we can calculate the coefficients of the polynomial whose roots are $a_1, a_2, \cdots, a_n$. (We do not need to know the roots themselves.) Thus, we can find the recurrence relation, and hence can find S_{n+1}. This is for an arbitrary but fixed n.

Let

(2.1) $$S(x) = S_1 + S_2x + S_3x^2 + \cdots + S_{n+1}x^n + \cdots,$$

where $S_1, S_2, S_3, \cdots, S_n$ are given. In our case, $S(x) = 1/(1-x)^2$.

Let

(2.2) $$C(x) = c_1x + c_2x^2 + \cdots + c_nx^n + \cdots.$$

These coefficients c_n are to be calculated from the $S_1, S_2, \cdots, S_n$.

From Newton's Identities (1.5),

$$S_{n+1} + S_nc_1 + S_{n-1}c_2 + \cdots + S_1c_n + (n+1)c_{n+1} = 0.$$

These are precisely the coefficients of x^n in

$$S(x) + S(x)C(x) + C'(x) = 0.$$

The solution to this differential equation is easily obtained by using the integrating factor. Thus

$$C(x)e^{\int S(x)dx} = \int e^{\int S(x)dx}(-S(x))dx + C$$

so that

$$C(x) = -1 + ce^{-\int S(x)dx} = -1 + e^{-(S_1x + S_2x^2/2 + \cdots + S_nx^n/n + \cdots)}$$

since $C(0) = 0$.

In this problem, $S(x) = 1/(1-x)^2$ so that

$$C(x) = -1 + e^{-x/(1-x)}.$$

If one writes this out,

$$-1 + e^{-x/(1-x)} = -1 + 1 - \frac{x}{1!(1-x)} + \frac{x^2}{2!(1-x)^2} - \frac{x^3}{3!(1-x)^3} + \cdots.$$

From Waring's Formula (See Patton and Burnside, *Theory of Equations,* etc.)

$$C_n = \sum \frac{(-1)^{r_1+r_2+\cdots+r_n}S_1^{r_1}S_2^{r_2}\cdots S_n^{r_n}}{r_1!\,r_2!\,r_3!\cdots r_n!\,1^{r_1}2^{r_2}\cdots n^{r_n}},$$

where the summation is over all non-negative solutions to

$$r_1 + 2r_2 + 3r_3 + \cdots + nr_n = n.$$

In our case where $S_k = k$ for $1 \leqslant k \leqslant n$, this becomes

$$C_n = \sum \frac{(-1)^{r_1+r_2+\cdots+r_n}}{r_1!\,r_2!\cdots r_n!}$$

over all nonnegative solutions to

$$r_1 + 2r_2 + 3r_3 + \cdots + nr_n = n,$$

so that

$$\sum_{r_1+2r_2+\cdots+nr_n=n} \frac{(-1)^{r_1+r_2+\cdots+r_n}}{r_1!\,r_2!\,r_3!\cdots r_n!} = \sum_{k=1}^{n} \frac{(-1)^k\binom{n-1}{k-1}}{k!}.$$

Then

$$c_1 = \frac{-1}{1!} = -1$$

$$c_2 = \frac{-1}{1!} + \frac{1}{2!} = -1/2$$

$$c_3 = \frac{-1}{1!} + \frac{2}{2!} - \frac{1}{3!} = -1/6$$

$$c_4 = \frac{-1}{1!} + \frac{3}{2!} - \frac{3}{3!} + \frac{1}{4!} = 1/24$$

$$c_n = -\frac{\binom{n-1}{0}}{1!} + \frac{\binom{n-1}{1}}{2!} - \frac{\binom{n-1}{2}}{3!} + \cdots + \frac{(-1)^n\binom{n-1}{n-1}}{n!}$$

so that

$$c_n = \sum_{k=1}^{n} \frac{(-1)^k\binom{n-1}{k-1}}{k!} \tag{2.3}$$

Here we have an explicit expression for the c_n for $S_k = k$ for $1 \leqslant k \leqslant n$.

We now return to the problem E-2487. From the Newton-Identity equation

$$S_{n+1} + c_1S_n + \cdots + c_nS_1 + (n+1)c_{n+1} = 0.$$

We must make a careful distinction between the solution to E-2487 for n and values of the S_m sequence for larger n. Let S_n^* be the solution to the problem; then

$$S_n^* + c_1S_n + c_2S_{n-1} + \cdots + c_nS_1 = 0,$$

where $S_k = k$ for $1 \leqslant k \leqslant n$ and the c_k for $1 \leqslant k \leqslant n$ are given by the Newton Identities using these S_k. We note two diverse things here. Suppose we write the next Newton-Identity for a higher value of n,

$$S_{n+1} + c_1S_n + \cdots + c_nS_1 + (n+1)c_{n+1} = 0;$$

then

$$(n+1) - S_n^* + (n+1)c_{n+1} = 0$$

so that

$$S_n^* = (n+1)(1+c_{n+1}) = (n+1)\left[1 + \sum_{k=1}^{n+1} \frac{(-1)^k\binom{n}{k-1}}{k!}\right]. \tag{2.4}$$

We can also get a solution in another way.

$$S_n^* = -[c_1S_n + \cdots + c_nS_1]$$

is the nth coefficient in the convolution of $S(x)$ and $C(x)$ which was used earlier (2.1), (2.2). Thus

$$S^*(x) = -C(x)S(x) = [1 - e^{-x/(1-x)}]/(1-x)^2 = \frac{x}{1!\,(1-x)^3} - \frac{x^2}{2!\,(1-x)^4} + \frac{x^3}{3!\,(1-x)^5} - \cdots$$

$$S_1^* = 1/1! = 1$$

$$S_2^* = 3/1! - 1/2! = 5/2$$

$$S_3^* = 6/1! - 4/2! + 1/3! = 25/6$$

and

$$S_n^* = \sum_{k=1}^{n} \frac{(-1)^{k+1}\binom{n+1}{k+1}}{k!}. \tag{2.5}$$

It is not difficult to show that the two formulas (2.4) and (2.5) for S_n^* are the same.

3. A GENERALIZATION OF E-2487

If one lets $S(x) = 1/(1-x)^{m+1}$, then

$$C(x) = -1 + e^{\frac{1}{m}[1-1/(1-x)^m]} \tag{3.1}$$

and

$$S^*(x) = \frac{1 - e^{\frac{1}{m}[1-1/(1-x)^m]}}{(1-x)^{m+1}}. \tag{3.2}$$

We now get explicit expressions for S_n, c_n, and S_n^*.

First,

$$S(x) = \frac{1}{(1-x)^{m+1}} = \sum_{n=0}^{\infty} \binom{n+m}{n} x^n,$$

so that

$$S_{n+1} = \binom{n+m}{n}. \tag{3.3}$$

We shall show that

Theorem 3.1.

$$c_n = \sum_{k=1}^{n} \frac{1}{k!\,m^k} \sum_{\alpha=1}^{k} (-1)^k \binom{k}{\alpha} \binom{\alpha m+n-1}{n}$$

and

$$S_n^* = \binom{n+m}{n} + (n+1)c_{n+1} = \binom{n+m}{n} + (n+1) \sum_{k=1}^{n+1} \frac{1}{k!\,m^k} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} \binom{m\alpha+n}{n+1}$$

Proof. From Schwatt [4], one has the following. If $y = g(u)$ and $u = f(x)$, then

$$\frac{d^n y}{dx^n} = \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} u^{k-\alpha} \frac{d^n u^\alpha}{dx^n} \frac{d^k y}{du^k}.$$

We can find the Maclaurin expansion of

$$y = e^{1/m} e^{-1/m(1-x)^m} = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n y}{dx^n}\right|_{x=0} x^n.$$

Let $y = e^{1/m} e^u$, where $u = -1/m(1-x)^m$; then $u^\alpha = (-1)^\alpha/m^\alpha(1-x)^{m\alpha}$ and

$$\frac{d^n u^\alpha}{dx^n} = \frac{(-1)^\alpha}{m^\alpha} \frac{(m\alpha)(m\alpha+1)\cdots(m\alpha+n-1)}{(1-x)^{m\alpha+n}},$$

$$\frac{d^k y}{du^k} = e^{1/m} e^u, \qquad \text{and} \qquad \left.\frac{d^k y}{dx^k}\right|_{x=0} = 1.$$

Thus,

$$\frac{1}{n!} \left.\frac{d^n y}{dx^n}\right|_{x=0} = \sum_{k=1}^{k} \frac{(-1)^k}{k!} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} \frac{(-1)^{k-\alpha}}{m^{k-\alpha}} \frac{(-1)^\alpha}{m^\alpha} \binom{m\alpha+n-1}{n}$$

so that

$$c_n = \sum_{k=1}^{n} \frac{1}{k!\,m^k} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} \binom{m\alpha+n-1}{n}.$$

Thus, since $S_n^* = S_{n+1} + (n+1)c_{n+1}$, then

$$S_n^* = \binom{n+m}{n} + (n+1) \sum_{k=1}^{n+1} \frac{1}{k!\,m^k} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} \binom{m\alpha+n}{n+1}$$

which concludes the proof of Theorem 3.1.

But

$$S^*(x) = -C(x)/(1-x)^{m+1}$$

so that we can get yet another expression for S_n^*,

$$S_n^* = -\sum_{j=1}^{n} (S_j c_{n-j+1}) = -\sum_{j=1}^{n} S_{n-j+1} c_j \,, \tag{3.4}$$

where c_n is as above and

$$S_n = \binom{n+m-1}{m} = \binom{n+m-1}{n-1}.$$

4. RELATIONSHIPS TO PASCAL'S TRIANGLE

An important special case deserves mention. If we let $S_k = m$ for $1 \leqslant k \leqslant n$, then $S(x) = m/(1-x)$ and

$$C(x) = -1 + e^{-\int [m/(1-x)]\,dx} = -1 + (1-x)^m \,.$$

Therefore,

$$c_k = (-1)^k \binom{m}{k}$$

for $1 \leqslant k \leqslant m \leqslant n$ or for $1 \leqslant k \leqslant n < m$, and $c_k = 0$ for $n < k \leqslant m$, and $c_k = 0$ for $k > n$ in any case. Now, let $S_k = -m$ for $1 \leqslant k \leqslant n$; then

$$S(x) = -m/(1-x) \qquad \text{and} \qquad C(x) = -1 + 1/(1-x)^m,$$

and we are back to columns of Pascal's triangle.

If we return to

$$c_k = \frac{(-1)^k}{k!} \begin{vmatrix} m & 1 & 0 & 0 & 0 & \cdots \\ m & m & 2 & 0 & 0 & \cdots \\ m & m & m & 3 & 0 & \cdots \\ m & m & m & m & 4 & \cdots \\ m & m & m & m & m & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{k\times k}$$

then we have rows of Pascal's triange, while with

$$c_k = \frac{(-1)^k}{k!} \begin{vmatrix} -m & 1 & 0 & 0 & 0 & \cdots \\ -m & -m & 2 & 0 & 0 & \cdots \\ -m & -m & -m & 3 & 0 & \cdots \\ -m & -m & -m & -m & 4 & \cdots \\ -m & -m & -m & -m & -m & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{k\times k}$$

we have columns of Pascal's triangle.

Suppose that we have this form for c_k in terms of general S_k but that the recurrence is of finite order. Then, clearly, $c_k = 0$ for $k > n$. To see this easily, consider, for example, $S_1 = 1$, $S_2 = 3$, $S_3 = 7$,

$$S_{n+3} = S_{n+2} + S_{n+1} + S_n \,.$$

$$c_k = \frac{(-1)^k}{k!} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 2 & 0 & 0 & 0 & \cdots \\ 7 & 3 & 1 & 3 & 0 & 0 & \cdots \\ 11 & 7 & 3 & 1 & 4 & 0 & \cdots \\ 21 & 11 & 7 & 3 & 1 & 5 & \cdots \\ 39 & 21 & 11 & 7 & 3 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{k\times k}$$

$$\begin{aligned} 1 - 1 &= 0 \\ 3 - 1 - 2 &= 0 \\ 7 - 3 - 1 - 3 &= 0 \\ 11 - 7 - 3 - 1 &= 0 \\ 21 - 11 - 7 - 3 &= 0 \\ 39 - 21 - 11 - 7 &= 0, \text{ etc.} \end{aligned}$$

Thus, in this case, we can get the first column all zero with multipliers c_1, c_2, c_3, each of which is -1.

5. THE GENERAL CASE AND SOME CONSEQUENCES

Returning now to

$$C(x) = -1 + e^{-(S_1 x + S_2 x^2/2 + S_3 x^3/3 + \cdots + S_n x^n/n + \cdots)} \tag{5.1}$$

which was found in Riordan [6], we can see some nice consequences of this neat formula.

It is easy to establish that the regular Lucas numbers have generating function

$$\frac{1+2x}{1-x-x^2} = S(x) = \sum_{n=0}^{\infty} L_{n+1} x^n \tag{5.2}$$

$$e^{-[(1+2x)/(1-x-x^2)]\,dx} = e^{\ln(1-x-x^2)} = 1 - x - x^2 = 1 + C(x).$$

Here we know that $c_1 = -1$, $c_2 = -1$, and $c_m = 0$ for all $m > 2$. This implies that the Lucas numbers put into the formulas for c_m $(m > 2)$ yield zero, and furthermore, since $L_k, L_{2k}, L_{3k}, \cdots$, obey $1 - L_k x + (-1)^k x^2$, then it is true that $S_n = L_{nk}$ put into those same formulas yield non-linear identities for the k-sected Lucas number sequence. However, consider

$$e^{(L_1 x + L_2 x^2/2 + \cdots + L_n x^n/n + \cdots)} = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n \tag{5.3}$$

and

$$e^{(L_k x + L_{2k} x^2/2 + \cdots + L_{nk} x^n/n + \cdots)} = \frac{1}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} \frac{F_{(n+1)k}}{F_k} x^n .$$

Let us illustrate. Let $S_1, S_2, S_3, \cdots$ be generalized Lucas numbers,

$$\begin{aligned} c_1 &= -S_1 \\ c_2 &= \tfrac{1}{2}(S_1^2 - S_2) \\ c_3 &= \tfrac{1}{6}(S_1^3 - 3S_1S_2 + 2S_3) \\ c_4 &= \tfrac{1}{24}(S_1^4 - 6S_1^2S_2 + 8S_1S_3 + 3S_2^2 - 6S_4) \\ &\cdots \quad \cdots \end{aligned}$$

Let $S_n = L_{nk}$ so that $c_m = 0$ for $m > 2$.

$$\frac{1}{6}[L_k^3 - 3L_k L_{2k} + 2L_{3k}] = 0$$

while

$$\frac{1}{6}[L_k^3 + 3L_k L_{2k} + 2L_{3k}] = F_{4k}/F_k .$$

In Conkwright [1] was given

$$c_m = \frac{(-1)^m}{m!} \begin{vmatrix} S_1 & 1 & 0 & 0 & 0 & \cdots \\ S_2 & S_1 & 2 & 0 & 0 & \cdots \\ S_3 & S_2 & S_1 & 3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{m-1} & \cdots & \cdots & \cdots & \cdots & m-1 \\ S_m & S_{m-1} & S_{m-2} & \cdots & S_2 & S_1 \end{vmatrix}$$

which was derived in Hoggatt and Bicknell [5].

Thus for $m > 2$

$$c_m = \frac{(-1)^m}{m!} \begin{vmatrix} L_k & 1 & 0 & 0 & 0 & \cdots \\ L_{2k} & L_k & 2 & 0 & 0 & \cdots \\ L_{3k} & L_{2k} & L_k & 3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{(m-1)k} & L_{(m-2)k} & \cdots & \cdots & \cdots & k-1 \\ L_{mk} & L_{(m-1)k} & \cdots & \cdots & L_{2k} & L_k \end{vmatrix} = 0 \tag{5.5}$$

for all $k > 0$, where L_k is the kth Lucas number. This same formula applies, since $c_m = 0$ for $m > 3$, if $S_m = \mathcal{L}_{mk}$ where

$$\mathcal{L}_1 = 1, \quad \mathcal{L}_2 = 3, \quad \mathcal{L}_3 = 7, \quad \text{and} \quad \mathcal{L}_{m+3} = \mathcal{L}_{m+2} + \mathcal{L}_{m+1} + \mathcal{L}_m$$

are the generalized Lucas numbers associated with the Tribonacci numbers T_n

$$(T_1 = T_2 = 1, \quad T_3 = 2, \quad \text{and} \quad T_{n+3} = T_{n+2} + T_{n+1} + T_n.)$$

If $\mathcal{L}_m$ are the Lucas numbers associated with the generalized Fibonacci numbers F_n whose generating function is

$$\frac{1}{1 - x - x^2 - x^3 - \cdots - x^r} = \sum_{n=0}^{\infty} F_{n+1}x^n, \tag{5.6}$$

then if $S_m = \mathcal{L}_{mk}$, then the corresponding $c_m = 0$ for $m > r$, yielding (5.5) for $m > r$ with L_{mk} everywhere replaced by $\mathcal{L}_{mk}$.

Further, let

$$F(x) = 1 - x - x^2 - x^3 - \cdots - x^r;$$

then

$$F'(x) = -1 - 2x - 3x^2 - \cdots - rx^{r-1}$$

and

$$-\frac{F'(x)}{F(x)} = \frac{1 + 2x + 3x^2 + \cdots + rx^{r-1}}{1 - x - x^2 - x^2 - \cdots - x^r} = \sum_{n=0}^{\infty} \mathcal{L}_{n+1}x^n, \tag{5.7}$$

where $\mathcal{L}_n$ is the generalized Lucas sequence associated with the generalized Fibonacci sequence whose generating function is $1/F(x)$. Thus, any of these generalized Fibonacci sequences is obtainable as follows:

$$e^{-\int [F'(x)/F(x)]\,dx} = \frac{1}{1 - x - x^2 - x^3 - \cdots - x^r} = \sum_{n=0}^{\infty} F_{n+1}x^n$$

and we have

Theorem 5.1.

$$e^{\mathcal{L}_1 x + \mathcal{L}_2 x^2/2 + \cdots + \mathcal{L}_n x^n/n + \cdots} = 1/F(x) = \sum_{n=0}^{\infty} F_{n+1}x^n.$$

The generalized Fibonacci numbers F_n generated by (5.6) appear in Hoggatt and Bicknell [7] and [8] as certain rising diagonal sums in generalized Pascal triangles.

Write the left-justified polynomial coefficient array generated by expansions of

$$(1 + x + x^2 + \cdots + x^{r-1})^n, \quad n = 0, 1, 2, 3, \cdots, r \geqslant 2.$$

Then the generalized Fibonacci numbers $u(n; p, q)$ are given sequentially by the sum of the element in the left-most column and the nth row and the terms obtained by taking steps p units up and q units right through the array. The simple rising diagonal sums which occur for $p = q = 1$ give

$$u(n; 1, 1) = F_{n+1}, \quad n = 0, 1, 2, \cdots.$$

The special case $r = 2$, $p = q = 1$ is the well known relationship between rising diagonal sums in Pascal's triangle and the ordinary Fibonacci numbers,

$$\sum_{i=0}^{[(n+1)/2]} \binom{n-i}{i} = F_{n+1}$$

while

$$\sum_{i=0}^{[(n+1)/2]} \binom{n-i}{i}_r = F_{n+1}$$

where

$$\binom{n-i}{i}_r$$

is the polynomial coefficient in the ith column and $(n-i)^{st}$ row of the left-adjusted array.

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[Continued from p. 122.]

From this we have that

$$L(F(n)) = \frac{f(n+1) - (-1)^{F(n+2)} f(n-2)}{f(n-1)} \tag{3}$$

Now, letting $a = F(n)$, $b = F(n+1)$ in (2), we have

$$5f(n)f(n+1) = L(F(n+2)) - (-1)^{F(n)} L(F(n-1)). \tag{4}$$

Finally, substituting (3) for each term on the right of (4) and rearranging gives the required recursion.

It is interesting to note that a 5th order recursion for $f(n)$ exists, but it is much more complicated.

Proposition.

$$f(n) = \frac{(5f(n-2)^2 + 2(-1)^{F(n+1)} f(n-3)^2 f(n-4) + f(n-2)(f(n-2) - (-1)^{F(n-1)} f(n-5))(f(n-1) - (-1)^{F(n)} f(n-4))}{2f(n-4)f(n-3)}$$

Proof. Use Equation (2) and the identity

$$L(a)L(b) = L(a+b) + (-1)^a L(b-a), \tag{5}$$

to obtain

$$5f(n)f(n+1) = 2L(F(n+2)) - L(F(n))L(F(n+1)).$$

Using (3) on the right-hand side and rearranging gives the required recursion.

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AN APPLICATION OF TRIBONACCI NUMBERS

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An interesting application of the Tribonacci numbers appeared unexpectedly in the solution of the following problem. Begin with 4 nonnegative integers, for example, 9, 4, 6, 7. Take cyclic differences of pairs of numbers (the smaller number from the larger) where the fourth difference is always the difference between the last number (7 in the above example) and the first number (9 in the above example). Repeat this process on the differences. For the example above, we have

1st row	9		4		6		7					
2nd row		5		2		1		2				
3rd row			3		1		1		3			
4th row				2		0		2		0		
5th row					2		2		2		2	
6th row						0		0		0		0.

Starting with the numbers 9, 4, 6, 7 and following the procedure described, the process terminates in the 6th row with all zeros.

Problem. Are there 4 starting numbers that will terminate with all zeros in the 7th row, the 8th row, $\cdots$, the nth row?

Various sequences of numbers were tried but they were found unsatisfactory. One development that leads to a solution is outlined below.

(a) Begin with 4 numbers, not all zero,

(1) $$a \quad b \quad c \quad d$$

which are assumed to be known and then try to get the 4 numbers in the row directly above $a, b, c, d,$ namely, the numbers

(2) $$x_1 \quad x_2 \quad x_3 \quad x_4 \ .$$

Thus,

(3) $$\begin{array}{lccccccc} 2^{nd}\text{ row} & x_1 & & x_2 & & x_3 & & x_4 \\ 1^{st}\text{ row} & & a & & b & & c & & d \end{array}$$

(b) Now, rather than try to solve the problem for arbitrary numbers $a, b, c, d,$ we will take the special case where

(4) $$d = a + b + c.$$

In place of (3), we have

(5) $$\begin{array}{lccccccc} 2^{nd}\text{ row} & x_1 & & x_1 + a & & x_1 + a + b & & x_1 + a + b + c \\ 1^{st}\text{ row} & & a & & b & & c & & d = a + b + c. \end{array}$$

At this point, one can select x_1 to be any nonnegative integer. However, this procedure proves rather unproductive. We now assume that the summability pattern for the 4 known starting numbers

$$a \quad b \quad c \quad d = a + b + c$$

also holds for

(6) $$x_1 \quad x_1 + a \quad x_1 + a + b \quad x_1 + a + b + c.$$

ɹr the above assumption, we have

$$x_1 + (x_1 + a) + (x_1 + a + b) = x_1 + a + b + c.$$

ɹing for x_1, we get

$$x_1 = \frac{c-a}{2},$$

ɜre now x_1 is determined in terms of the known numbers a and c. Note that $c - a$ must be even for x_1 to be nteger.

) For a given set of 4 numbers $a, b, c, d = a + b + c$, once x_1 is determined, we can get the 2nd row in (5). umably, the procedure can then be repeated on the 2nd row to get a 3rd, 4th, etc. row. The following ex- ɹle shows that another slight modification is necessary.

ɜample 1. Begin with the four numbers 1, 1, 1, 3. These numbers satisfy the summability condition $a + b + c$. Using the condition in (8) with $a = 1, c = 1$, we have

$$x_1 = \frac{c-a}{2} = 0.$$

stituting in (5), we get

2nd row	0		1		2		3
1st row		1		1		1		3.

2nd row now serves as our 4 known numbers $a, b, c, d = a + b + c$. Here $a = 0, c = 2$ and from (8), we have

) $$x_1 = \frac{c-a}{2} = 1$$

ng (5) and (9), we now have

3rd row	$\boxed{1}$		$\boxed{1}$		$\boxed{2}$		$\boxed{4}$
ı) 2nd row		0		1		2		3
1st row			1		1		1		3.

Ve now go on to the 4th row. However, if we take the 3rd row 1, 1, 2, 4 in (11) as our 4 known numbers, ɜn $a = 1, c = 2$ and from (8)

?) $$x_1 = \frac{c-a}{2} = \frac{1}{2}$$

ich is not an integer. Apparently, we cannot get the 4th row from our present method.

Ve pause to point out several items of interest in the example above.

l. We began the example 1 with the 4 starting numbers 1, 1, 1, 3. This was a rather arbitrary selection. If we ɹ started with the 4 numbers 0, 0, 2, 2 we could have calculated the 4th row but the numbers here would ɹe been 1, 1, 2, 4 precisely the same as in our present example where again we would have been stopped. ɜre appears to be no marked advantage in selecting other starting numbers rather than 1, 1, 1, 3.

. In (11) the numbers in the 3rd row are the first four numbers of the classical Tribonacci sequence

) $$\begin{matrix} 1 & 1 & 2 & 4 \\ T_1 & T_2 & T_3 & T_4 \end{matrix}.$$

If we start with the Tribonacci numbers in (13), we have for the cyclic differences

1.	1		1		2		4						
2.		0		1		2		3					
3.			1		1		1		3				
4.				0		0		2		2			
5.					0		2		0		2		
6.						2		2		2		2	
7.							0		0		0		0

ɜt all zeros in the seventh row.

Let us now return to (11) where our procedure was stopped. Multiply each element in each row of (11) by 2. We have

$$
\begin{array}{lccccccccc}
3^{rd}\text{ row} & & 2 & & 2 & & 4 & & 8 & \\
2^{nd}\text{ row} & & & 0 & & 2 & & 4 & & 6 \\
1^{st}\text{ row} & & & & 2 & & 2 & & 2 & & 6\,.
\end{array} \tag{15}
$$

In the third row of (15), $a = 2$, $c = 4$ and using (8), we have

$$x_1 = \frac{c-a}{2} = 1 \tag{16}$$

We can now get the 4th row. From the 4th row, we can get the 5th row and from the 5th row, we can get the 6th row before we are stopped by a non-integral value of x_1. The cyclic differences are shown below.

$$
\begin{array}{lccccccccccc}
6. & \boxed{2 & & 4 & & 7 & & 13} & & & & \\
5. & & 2 & & 3 & & 6 & & 11 & & & \\
4. & & & 1 & & 3 & & 5 & & 9 & & \\
3. & & & & 2 & & 2 & & 4 & & 8 & \\
2. & & & & & 0 & & 2 & & 4 & & 6 \\
1. & & & & & & 2 & & 2 & & 2 & & 6
\end{array} \tag{17}
$$

As in (11) so in (17), the four numbers in row 6 (where we are stopped) are consecutive Tribonacci numbers T_3 to T_6. A list of the first seventeen Tribonacci numbers is given below.

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \qquad n = 4, 5, 6, \cdots. \quad T_1 = T_2 = 1 \quad T_3 = 2.$$

$$
\begin{array}{cccccccccc}
1 & 1 & 2 & 4 & 7 & 13 & 24 & 44 & 81 & 149 \\
T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 & T_{10} \\
274 & 504 & 927 & 1705 & 3136 & 5768 & 10{,}609 & & & \\
T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} & T_{17} & & &
\end{array} \tag{18}
$$

If we return to (17) and multiply each element in each row by 2, we can get rows 7, 8, 9 before we are stopped. The 4 numbers in row 9 are the 4 Tribonacci numbers 7, 13, 24, 44 (T_5 to T_8, see (18)).

The procedure is now clear. From (11), (15) and (17), whenever we are stopped, we multiply each element in each row by 2. This will allow us to go 3 rows upward. We are then stopped at a set of 4 Tribonacci numbers where the first two Tribonacci numbers overlap with the last two Tribonacci numbers of the preceding stopping point. If in (11) and (17), we take the cyclic differences from row 1 downward, we get 4 more rows before terminating in all zeros. We summarize the results.

(19)

Starting Tribonacci numbers	Rows upward counting from		Rows downward not counting		Total rows
T_1 to T_4	row 1, 1, 1, 3	3	row 1, 1, 1, 3	4	7
T_3 to T_6	row 2, 2, 2, 6	6	row 2, 2, 2, 6	4	10
T_5 to T_8	row 4, 4, 4, 12	9	row 4, 4, 4, 12	4	13
T_7 to T_{10}	row 8, 8, 8, 24,	12	row 8, 8, 8, 24	4	16
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T_{2n+1} to T_{2n+4}	row $2^n, 2^n, 2^n, (3)2^n,$	$3(n+1)$	row $2^n, 2^n, 2^n, (3)2^n,$	4	$3(n+2)+1$

where $n = 0, 1, 2, 3, \cdots$.

If we take the four consecutive Tribonacci numbers T_{2n+1} to T_{2n+4}, $n = 0, 1, 2, 3, \cdots$ we get all zeros in the $3(n+2)+1$ row.

The starting Tribonacci numbers above begin with an odd-numbered term such as T_1, T_3, T_5, and so on. What happens if we start with an even-numbered term of the sequence, say T_2, T_4, T_6, and so on? Actually,

we get all zeros at precisely the same row as we did when we started with the odd-numbered Tribonacci sequence T_1, T_3, T_5, and so on. The summary is given below.

	Starting Tribonacci numbers	Rows upward counting from		Rows downward not counting		Total rows
(20)	T_2 to T_5	row 1, 1, 3, 5	3	row 1, 1, 3, 5,	4	7
	T_4 to T_7	row 2, 2, 6, 10	6	row 2, 2, 6, 10	4	10
	T_6 to T_9	row 4, 4, 12, 20	9	row 4, 4, 12, 20	4	13
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	T_{2n} to T_{2n+3}	$2^{n-1}, 2^{n-1}, (3)2^{n-1}, (5)2^{n-1}$	$3n$	(see column 2)	4	$3(n+1)+1$

where $n = 1, 2, 3, \cdots$.

We can rewrite the results in (19) to agree with the values of n in (20). Thus, for $n = 1, 2, 3, \cdots$

(21) Odd numbered starting Tribonacci numbers $T_{2n-1}, \quad T_{2n}, \quad T_{2n+1}, \quad T_{2n+2}$

(22) Even numbered starting Tribonacci numbers $T_{2n}, \quad T_{2n+1}, \quad T_{2n+2}, \quad T_{2n+3}$

will give all zeros for the $3(n+1)+1$ row.

Conclusion. What are 4 starting numbers which give all zeros at precisely row m, where $m = 1, 2, 3, \cdots$?

	Number of rows for which we get all zeros	4 starting numbers
(23)	$m = 1$	0, 0, 0, 0
	$m = 2$	1, 1, 1, 1
	$m = 3$	2, 0, 2, 4
	$m = 4$	0, 2, 2, 4
	$m = 5$	1, 1, 3, 5

For $m \geqslant 6$, note that the numbers 6, 7, 8, 9, $\cdots$, are

a. multiples of 3, so that $m = 3(n+1)$, $\quad n = 1, 2, 3, \cdots$,
b. multiples of 3 plus 1, so that $m = 3(n+1)+1$, $\quad n = 1, 2, 3, \cdots$,
c. multiplies of 3 plus 2, so that $m = 3(n+1)+2$, $\quad n = 1, 2, 3, \cdots$.

Actually, we have already solved the problem for the case where $m = 3(n+1)+1$, $n = 1, 2, 3, \cdots$ (for m equal to a multiple of 3 plus 1) in (21) and (22). If we take the solution (21), we can easily get the row above (21) which will be the solution for $m = 3(n+1)+2$, $n = 1, 2, 3, \cdots$. Moreover, if we go downward from (21) by taking the cyclic differences, we will have the solution for the case $m = 3(n+1)$, $n = 1, 2, 3, \cdots$. Thus,

		Starting Tribonacci Numbers				Solution for
(24)	Upward from (21)	0	T_{2n-1}	$T_{2n+1} + T_{2n}$	T_{2n+2}	$m = 3(n+1)+2$
	Relation (21)	T_{2n-1}	T_{2n}	T_{2n+1}	T_{2n+2}	$m = 3(n+1)+1$
(25)	Downward from (21)	$T_{2n} - T_{2n-1}$	$T_{2n+1} - T_{2n}$	$T_{2n+2} - T_{2n+1}$	$T_{2n+2} - T_{2n-1}$	$m = 3(n+1)$

Example 2. Find the 4 starting numbers that give all zeros for precisely the 8th row.

Solution. Here $m = 8$ and m is a multiple of 3 plus 2. From $m = 3(n+1)+2$ we have $8 = 3(n+1)+2$ or $n = 1$. From (24) the 4 starting Tribonacci numbers are $0, T_1, T_1 + T_2, T_4$ and concretely from (18) 0, 1, 2, 4.

Now

(26)

1.	0		1		2		4								
2.		1		1		2		4							
3.			0		1		2		3						
4.				1		1		1		3					
5.					0		0		2		2				
6.						0		2		0		2			
7.							2		2		2		2		
8.								0		0		0		0	

Using (21), (24) and (25) we have constructed the following table.

Table

m	n	4 Starting Tribonacci Numbers
6	1	0, 1, 2, 3
7	1	1, 1, 2, 4
8	1	0, 1, 2, 4
9	2	2, 3, 6, 11
10	2	2, 4, 7, 13
11	2	0, 2, 6, 13
12	3	6, 11, 20, 37
13	3	7, 13, 24, 44
14	3	0, 7, 20, 44

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[Continued from page 116.]

where

$$q = [k/2], \quad r = k, \bmod 2, \quad 1 \leqslant j \leqslant k,$$

$$P_i(x) = (\tfrac{1}{2})\ln[x^2 - 2x\cos((2i+1)\pi/k) + 1],$$

$$Q_i(x) = \arctan[(x - \cos((2i+1)\pi/k)/\sin((2i+1)\pi/k)] .$$

Proof. The G function has the series and integral representation [4, p. 20]

$$G(z) = 2\sum_{n=0}^{\infty}(-1)^n/(z+n) = 2\int_0^1 x^{z-1}dx/(1+x)$$

from which the first part of (2) is immediate. The integration formula is recorded in [5, p. 20].

Lemma 2.

(3) $$\omega(j; k_1, k_2) = (1/S)[\psi((j+k_1)/S) - \psi(j/S)] ,$$

where the psi (digamma) function is the logarithmic derivative of the gamma function and has integral representation for rational argument u/v, $0 < u < v$,

(4) $$\psi(u/v) = -C + v\int_0^1 (x^{v-1} - x^{u-1})dx/(1-x^v)$$

$$= -C - \ln v - (\pi/2)\cot(u\pi/v)$$

$$+ \sum_{i=1}^{q}\cos(2ui\pi/v)\ln(4\sin^2 i\pi/v) + (-1)^u\delta_0^r \ln 2 ,$$

where $q = [(v-1)/2]$, $r = u/2 - [u/2]$, C is Euler's constant.

[Continued on page 149.]

AMATEUR INTERESTS IN THE FIBONACCI SERIES IV
CALCULATION OF GROUP SIZES OF RESIDUES OF MODULI

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As indicated in a previous paper [1], the statement that the residues of any modulus M of the Fibonacci Series are periodic was investigated. It was found that, in dividing consecutive F_n by M, residues were formed in a Fibonacci-type series until a residue of zero was reached. The succession of residues so formed may be called a *group* and the number of residues in the group, including the terminal zero is the *group size.* (Note: "Group size" is identical numerically to "entry point" found in [2]. Editor.)

If the residue immediately preceding the terminal zero is unity, the next residue will be an exact repetition of the first residues calculated. Therefore, the group ending 1, 0 marks the end of the group and the period. The period may contain 1, 2, or 4 groups. For example, when the modulus $M = 5$, the period contains four groups:

GROUP	RESIDUES
1	1, 1, 2, 3, 0
2	3, 3, 1, 4, 0
3	4, 4, 3, 2, 0
4	2, 2, 4, 1, 0

Note that each group ends in a zero and that the last group (and the period) ends in a 1, 0. Succeeding residues will merely recapitulate the residues in the order shown, starting with the first residue, 1, in the first group.

After calculating the group and period sizes for successive moduli from 2 through 200 (see Table 1), certain regularities were noted, though the table apparently shows nothing of the kind. The group size G_M (but not the period size) of any modulus given in Table 1 can be calculated from the following two rules.

Rule 1. Determine the prime factors of the modulus, such that

$$M = A^{\ell}B^{m}C^{n}\cdots, \tag{1.1}$$

where $A, B, C, \cdots$ are primes and $\ell, m, n, \cdots, \geqslant 1$. Then the group size G_M for modulus M is the product of the group sizes of moduli equal to these factors, i.e.,

$$G_M = G_{A^{\ell}}\cdot G_{B^m}\cdot G_{C^n}\cdots, \tag{1.2}$$

except that, if any two of the factor group sizes $G_{A^{\ell}}, G_{B^m}, G_{C^n}, \cdots$, contain some common factor D, divide one or the other of the factor group sizes by D so that the quotient obtained is prime relative to the other factor group size in the pair containing that factor D. Continue until all the quotients are prime relative to each other.

Thus:

$$G_{132} = G_{2^2}\cdot G_3\cdot G_{11} = 6\times 4\times 10.$$

The numbers 6, 4, 10 have common factor $D = 2$. Divide 6 by 2, giving quotient 3 which is prime relative to 4. (Note that dividing the 4 by 2 is incorrect because the quotient 2 is not prime relative to 6.) This leaves

$$G_{132} = 3\times 4\times 10.$$

Now, taking the pair 4 and 10, divide 10 by 2, getting 5 which is prime to 4. The final result is

$$G_{132} = 3\times 4\times 5 = 60$$

which will be found to be correct.

Table 1

M	$A^{\ell}B^{m}C^{n}$	$G_{A^{\ell}}G_{B^{m}}G_{C^{n}}$	G_M
2	P	$M+1$	3
3	P	$M+1$	4
4	2^2	2×3	6
5	P	M	5*
6	2×3	3×4	12
7	P	$M+1$	8
8	2^3	2×6	6*
9	3^2	3×4	12
10	2×5	3×5	15
11	P	$M-1$	10
12	$2^2\times 3$	6×4	12
13	P	$(M+1)/2$	7
14	2×7	3×8	24
15	3×5	4×5	20
16	2^4	2×6	12
17	P	$(M+1)/2$	9
18	2×3^2	3×12	12
19	P	$M-1$	18
20	$2^2\times 5$	6×5	30
21	3×7	4×8	8
22	2×11	3×10	30
23	P	$M+1$	24
24	$2^3\times 3$	6×4	12
25	5^2	5×5	25
26	2×13	3×7	21
27	3^3	3×12	36
28	$2^2\times 7$	6×8	24
29	P	$(M-1)/2$	14
30	$2\times 3\times 5$	$3\times 4\times 5$	60
31	P	$M-1$	30
32	2^5	2×12	24
33	3×11	4×10	20
34	2×17	3×9	9
35	5×7	5×8	40
36	$2^2\times 3^2$	6×12	12
37	P	$(M+1)/2$	19
38	2×19	3×18	18
39	3×13	4×7	28
40	$2^3\times 5$	6×5	30
41	P	$(M-1)/2$	20
42	$2\times 3\times 7$	$3\times 4\times 8$	24
43	P	$M+1$	44
44	$2^2\times 11$	6×10	30
45	$3^2\times 5$	12×5	60
46	2×23	3×24	24
47	P	$(M+1)/3$	16
48	$2^4\times 3$	12×4	12
49	7^2	7×8	56
50	2×5^2	3×25	75
51	3×17	4×9	36
52	$2^2\times 13$	6×7	42
53	P	$(M+1)/2$	27
54	2×3^3	3×36	36
55	5×11	5×10	10
56	$2^3\times 7$	6×8	24
57	3×19	4×18	36
58	2×29	3×14	42
59	P	$M-1$	58
60	$2^2\times 3\times 5$	$6\times 4\times 5$	60
61	P	$(M-1)/4$	15
62	2×31	3×30	30
63	$3^2\times 7$	12×8	24
64	2^6	2×24	48
65	5×13	5×7	35
66	$2\times 3\times 11$	$3\times 4\times 10$	60
67	P	$M+1$	68
68	$2^2\times 17$	6×9	18
69	3×23	4×24	24
70	$2\times 5\times 7$	$3\times 5\times 8$	120
71	P	$M-1$	70
72	$2^3\times 3^2$	6×12	12
73	P	$(M+1)/2$	37
74	2×37	3×19	57
75	3×5^2	4×25	100
76	$2^2\times 19$	6×18	18
77	7×11	8×10	40
78	$2\times 3\times 13$	$3\times 4\times 7$	84
79	P	$M-1$	78
80	$2^4\times 5$	12×5	60
81	3^4	3×36	108
82	2×41	3×20	60
83	P	$M+1$	84
84	$2^2\times 3\times 7$	$6\times 4\times 8$	24
85	5×17	5×9	45
86	2×43	3×44	132
87	3×29	4×14	28
88	$2^3\times 11$	6×10	30
89	P	$(M-1)/8$	11
90	$2\times 3^2\times 5$	$3\times 12\times 5$	60
91	7×13	8×7	56
92	$2^2\times 23$	6×24	24
93	3×31	4×30	60
94	2×47	3×16	48
95	5×19	5×18	90
96	$2^5\times 3$	24×4	24
97	P	$(M+1)/2$	49
98	2×7^2	3×56	168
99	$3^2\times 11$	12×10	60
100	$2^2\times 5^2$	6×25	150

M = Modulus
P = Prime Number
*Exceptions

Table 1 (Cont'd.)

M	$A^{\ell}B^{m}C^{n}$	$G_{A^{\ell}}G_{B^{m}}G_{C^{n}}$	G_M
101	P	$(M-1)/2$	50
102	$2\times3\times17$	$3\times4\times9$	36
103	P	$M+1$	104
104	$2^3\times13$	6×7	42
105	$3\times5\times7$	$4\times5\times8$	40
106	2×53	3×27	27
107	P	$(M+1)/3$	36
108	$2^2\times3^3$	6×36	36
109	P	$(M-1)/4$	27
110	$2\times5\times11$	$3\times5\times10$	30
111	3×37	4×19	76
112	$2^4\times7$	12×8	24
113	P	$(M+1)/6$	19
114	$2\times3\times19$	$3\times4\times18$	36
115	5×23	5×24	120
116	$2^2\times29$	6×14	42
117	$3^2\times13$	12×7	84
118	2×59	3×58	174
119	7×17	8×9	72
120	$2^3\times3\times5$	$6\times4\times5$	60
121	11^2	1110	110
122	2×61	3×15	15
123	3×41	4×20	20
124	$2^2\times31$	6×30	30
125	5^3	5×25	125
126	$2\times3^2\times7$	$3\times12\times8$	24
127	P	$M+1$	128
128	2^6	2×48	96
129	3×43	4×44	44
130	$3\times5\times13$	$4\times5\times7$	105
131	P	$M-1$	130
132	$2^2\times3\times11$	$6\times4\times10$	60
133	7×19	8×18	72
134	2×67	3×68	204
135	$3^3\times5$	36×5	180
136	$2^3\times17$	6×9	18
137	P	$(M+1)/2$	69
138	$2\times3\times23$	$3\times4\times24$	24
139	P	$(M-1)/3$	46
140	$2^2\times5\times7$	$6\times5\times8$	120
141	3×47	4×16	16
142	2×71	3×70	210
143	11×13	10×7	70
144	$2^4\times3^2$	12×12	12
145	5×29	5×14	70
146	2×73	3×37	111
147	3×7^2	4×56	56
148	$2^2\times37$	6×19	114
149	P	$(M-1)/4$	37
150	$2\times3\times5^2$	$3\times4\times25$	300
151	P	$(M-1)/3$	50
152	$2^3\times19$	6×18	18
153	$3^2\times17$	12×9	36
154	$2\times7\times11$	$3\times8\times10$	120
155	5×31	5×30	30
156	$2^2\times3\times13$	$6\times4\times7$	84
157	P	$(M+1)/2$	79
158	2×79	3×78	78
159	3×53	4×27	108
160	$2^5\times5$	24×5	120
161	7×23	8×24	24
162	2×3^4	3×108	108
163	P	$M+1$	164
164	$2^2\times41$	6×20	60
165	$3\times5\times11$	$4\times5\times10$	20
166	2×83	3×84	84
167	P	$M+1$	168
168	$2^3\times3\times7$	$6\times4\times8$	24
169	13^2	13×7	91
170	$2\times5\times17$	$3\times5\times9$	45
171	$3^3\times19$	36×18	36
172	$2^2\times43$	6×44	132
173	P	$(M+1)/2$	87
174	$2\times3\times29$	$3\times4\times14$	84
175	$5^2\times7$	25×8	200
176	$2^4\times11$	12×10	60
177	3×59	4×58	116
178	2×89	3×11	33
179	P	$M-1$	178
180	$2^2\times3^2\times5$	$6\times12\times5$	60
181	P	$(M-1)/2$	90
182	$2\times7\times13$	$3\times8\times7$	168
183	3×61	4×15	60
184	$2^3\times23$	6×24	24
185	5×37	5×19	95
186	$2\times3\times31$	$3\times4\times30$	60
187	11×17	10×9	90
188	$2^2\times47$	6×16	48
189	$3^3\times7$	36×8	72
190	$2\times5\times19$	$3\times5\times18$	90
191	P	$M-1$	190
192	$2^6\times3$	48×4	48
193	P	$(M+1)/2$	97
194	2×97	3×49	147
195	$3\times5\times13$	$4\times5\times7$	140
196	$2^2\times7^2$	6×56	168
197	P	$(M+1)/2$	99
198	$2\times3^2\times11$	$3\times12\times10$	60
199	P	$(M-1)/9$	22
200	$2^3\times5^2$	6×25	150

As a second example of application of Rule 1, calculate

$$G_{126} = G_2 \cdot G_{3^2} \cdot G_7 = 3 \times 12 = 8.$$

The pair 8 and 12 contain $D = 4$. Divide 12 (not the 8) by 4 to get a quotient of 3 which is prime to 8. Notice that the quotient 3 is not prime to the first factor 3. However, the requirement is that the quotient must be prime to the other number in the pair, not to all the other factor group sizes. So there remains

$$G_{126} = 3 \times 8 \times 3.$$

The two 3's taken as a pair contain $D = 3$ and one of them is reduced by division to 1, making

$$G_{126} = 1 \times 8 \times 3 = 24,$$

which will be found to be correct.

Rule 2. Powers. If M contains only one prime factor A^{ℓ}, then $\ell = 1$.

(2.1) (i) If the final digit in M is 3 or 7, $G_M = (M + 1)/a$;
(ii) If the final digit in M is 1 or 9, $G_M = (M - 1)/a$;

where a is some integer, $a \geqslant 1$; and when $\ell > 1$, then

(2.2) $$G_M = AG_{A^{\ell-1}}.$$

At least up to $M = 200$, there are only two exceptions to Rule 2. For $M = 5$, $G_M = M = 5$. Here, (2.1) does not apply, since 5 is not a final digit mentioned. However, since 5 is the only prime whose terminal digit is 5, this exception is easy to bear. It is interesting to note that G_5 is the average of $(M + 1)/a$ and $(M - 1)/2$ if $a = 1$. The second exception is that for $M = 8$, $G_M = 6$. Going by (2.2), G_{2^3} should be equal to $2G_{2^2} = 2 \times 6 = 12$. If rule (1.1) is applied, which Rule 2 specifically forbids, G_8 comes out as $2 \times 3 = 6$. This exception cannot be explained.

While these rules will enable one to calculate group size, one sould not deprive himself of the pleasure of calculating and recording the individual residues as described in [1]. Of particular interest is the examination of corresponding residues in successive groups. Look for equality of corresponding residues or for two residues whose sum is M. These will normally occur at the aG^{th} residue, where G is one of the factor group sizes and a is an integer, $a \geqslant 1$. Thus, for $M = 200$,

$$G_{2^3} G_{5^2} = 6 \times 25,$$

the 75th residue in each of the groups is 50.

	Group 1			Group 2		
	25th	75th	125th	25th	75th	125th
Residue:	25	50	125	125	50	25

Note the mirror image characteristic. This is again shown in the residues which occur in every sixth place of both groups. These residues always add up to $M = 200$ and are arranged symmetrically about the 75th residue already identified as 50. Thus:

Group 1	Group 2	Group 1	Group 2
8	192	50	50
144	56	64	136
184	16	88	112
168	32	120	80
40	160	72	128
152	48	176	24
96	104	96	104
176	24	152	48
72	128	40	160
120	80	168	32
88	112	184	16
64	136	144	56
50	50	8	192

Note that no residue in Group 1 occurs in Group 2 but that corresponding residues in the two groups add up to $M = 200$. Also, these numbers have other unusual characteristics. Add any two and the sum will be some one or the other of the numbers or, if the sum is greater than 200, subtract $M = 200$ and the remainder will be found somewhere in the list. Subtract any two numbers with the same result. Of course, the reader's inspection has already noted that the numbers above the central 50 are arranged as mirror images of those below.

It is interesting to note that mirror-image molecules (stereoisomers) are of the utmost importance in biochemical considerations and in heredity. Since the connection between the Fibonacci Series and certain facts in heredity has long been noted, perhaps further investigation of the self-reproductive nature of the Fibonacci Series and of its tendency to form mirror images would be fruitful.

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Proof. By pairwise association and use of the relationship [3, p. 285],

$$\psi'(x/S) = \sum_{i=0}^{\infty} 1/(i + x/S)^2$$

which is uniformly convergent for $x \geqslant 1$, one establishes

$$\omega(j; k_1, k_2) = \sum_{i=0}^{\infty} \int_j^{j+k_1} dx/(x + iS)^2$$

$$= (1/S)^2 \int_j^{j+k_1} \psi'(x/S)dx$$

which integrates into the statement (3). The integral form of the psi function occurring in (4) is listed in [4, p. 16] and the integral evaluation is a celebrated theorem of Gauss [3, p. 286; 4, p. 18].

Corollary. Formula (4) can be extended to an arbitrary positive rational argument via the identity [4, p. 16],

$$\psi(n + z) = \psi(z) + \sum_{i=0}^{n-1} 1/(z + i).$$

An ω-series with an arbitrary even number of k_i parameters can be grouped into a series of successive cycles of parametric incrementation within which the terms are pairwise associated. This procedure leads to an expression in terms of the biparameter ω-series, and application of Lemma 2 yields an explicit summation formula in terms of the psi function.

Theorem 1.

$$\omega(j; k_1, \cdots, k_{2n}) = \sum_{i=0}^{n-1} \omega(j + s_{2i}; k_{2i+1}, S - k_{2i+1})$$

$$= (1/S) \sum_{i=0}^{2n-1} (-1)^{i+1} \psi((j + s_{2i})/S).$$

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PERIODIC LENGTHS OF THE GENERALIZED FIBONACCI SEQUENCE MODULO p

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INTRODUCTION

This paper concerns the periodic lengths of the Generalized Fibonacci Sequence modulo p, where p is a prime integer. The GF sequence will be denoted by H_n, $n = 1, 2, \cdots$, for which

(1) $$H_1 = P, \quad H_2 = bP + cQ, \quad H_n = bH_{n-1} + cH_{n-2} \quad (n > 2)$$

and its periodic length reduced modulo p, i.e., the periodic length of the recurring series

(2) $$H_n \pmod p, \quad n = 1, 2, \cdots,$$

will be represented by $k(H,p)$. Clearly for $P = 1$, $Q = 0$ the periodic length of the series

(3) $$U_1 = 1, \quad U_2 = b, \quad U_n = bU_{n-1} + cU_{n-2} \quad (n > 2)$$

is given by $k(U,p)$. We prove the following theorems.

2. NATURE OF $k(H,p)$

Theorem a. For primes whose quadratic residue is $b^2 + 4c$, if $(b,c,P,Q) = 1$, then $k(H,p) | (p - 1)$.

Proof. In the known formula,

(4) $$H_n = (1r^n - ms^n)/(r - s), \quad (r + s = b, \quad rs = -c, \quad 1 = P - sQ \text{ and } m = P - pQ),$$

let $r,s = (b \pm \sqrt{(b^2 + 4c)})/2$ so that it may be simplified by the use of binomial theorem to obtain

(5) $$2^n H_n = \left\{ b^n(1 - m) + \binom{n}{1} b^{n-1}\sqrt{(b^2 + 4c)}(1 + m) + \binom{n}{2} b^{n-2} (\sqrt{(b^2 + 4c)})^2(1 - m) + \cdots + \binom{n}{n} (\sqrt{(b^2 + 4c)})^n(1 - (-1)^n m) \right\} / (\sqrt{(b^2 + 4c)}).$$

Then it is easy to show for $n = p$ and $p + 1$ that

(6) $$H_p \equiv P \pmod p, \quad H_{p+1} \equiv bP + cQ \pmod p,$$

if $(b^2 + 4c)^{(p-1)/2} \equiv 1 \pmod p$ and $(b,c,P,Q) = 1$. Hence the desired result follows.

Theorem b. For primes whose quadratic nonresidue is $b^2 + 4c$, if $(b,c,P,Q) = 1$, then $k(H,p) | (p^2 - 1)$.

Proof. On using the known formula $H_n = PU_n + cQU_{n-1}$, $(b^2 + 4c)^{(p-1)/2} \equiv -1 \pmod p$ and the following set of congruences, viz.,

(7) $$\begin{aligned} &U_p \equiv -1, \quad U_{p+1} \equiv 0, \quad U_{p+2} \equiv -c, \\ &U_{2p+1} \equiv 1, \quad U_{2p+2} \equiv 0, \quad U_{2p+2} \equiv (-c)^2 \\ &\vdots \\ &U_{p(p-1)+p-2} \equiv 1, \quad U_{p(p-1)+p-1} \equiv 0, \quad U_{p(p-1)+p} \equiv (-c)^{p-1}, \end{aligned}$$

it is easy to show that

(8)
$$H_{p+1} \equiv -cQ, \quad H_{p+2} \equiv -cP, \quad H_{p+3} \equiv -c(bP + cQ),$$
$$H_{2p+2} \equiv cQ, \quad H_{2p+3} \equiv (-c)^2P, \quad H_{2p+4} \equiv (-c)^2bP + c(cQ),$$
$$\vdots$$
$$H_{p(p-1)+p-1} \equiv cQ, \quad H_{p(p-1)+p} \equiv (-c)^{p-1}P, \quad H_{p^2+1} \equiv (-c)^{p-1}bP + c(cQ),$$
$$H_{p(p+1)} \equiv -cQ, \quad H_{p(p+1)+1} \equiv (-c)^pP, \quad H_{p(p+1)+2} \equiv (-c)^pbP + c(-cQ).$$

Clearly $(-c)^p \equiv -c \pmod p$ and (8) shows that $k(H,p) \mid (p^2 - 1)$.

Theorem c. For primes of the form $2g(2t + 1) + 1$, where $t \equiv h \pmod{10}$ and $4gh + 2g + 1 \equiv \pm 1 \pmod{10}$, if
$U_{\{(p-1)/2g\}+1} + cU_{\{(p-1)/2g\}-1} \equiv 0 \pmod p$ and $c^{(p-1)/2g} \equiv 1 \pmod{U_{\{(p-1)/2g\}+1} + cU_{\{(p-1)/2g\}-1}}$,
then $k(H,p) = (p - 1)/g$.

Proof. From the well known formulas,

(9) $U_{2n+1} = U_{n+1}(U_{n+1} + cU_{n-1}) + (-1)^{n-1}c^n$, $U_{2n} = U_n(U_{n+1} + cU_{n-1})$ and $H_n = PU_n + cQU_{n-1}$,

let us set

(10)
$$U_{(p-1)/g} \equiv 0 \pmod{U_{\{(p-1)/2g\}+1} + cU_{\{(p-1)/2g\}-1}},$$
$$U_{(p-1)/g+1} \equiv (-1)^{\{(p-1)/2g\}-1}c^{(p-1)/2g} \pmod{U_{\{(p-1)/2g\}+1} + cU_{\{(p-1)/2g\}-1}},$$

It is then easy to show that

(11)
$$U_{(p-1)/g} \equiv 0 \pmod p, \qquad U_{\{(p-1)/g\}+1} \equiv 1 \pmod p$$

when it follows

(12)
$$H_{(p-1)/g} \equiv Q \pmod p \quad \text{and} \quad H_{\{(p-1)/g\}+1} \equiv P \pmod p.$$

Hence, $k(H,p) = (p - 1)/g$.

Theorem d. For primes of the form $4gt + 1$, where $t \equiv h \pmod{10}$ and $4gh + 1 \equiv \pm 1 \pmod{10}$, if
$$U_{(p-1)/2g} \equiv 0 \pmod p \quad \text{and} \quad (-c)^{(p-1)/2g} \equiv 1 \pmod p,$$
then $k(H,p) = (p - 1)/g$.

Proof. From the known formulas,

(13) $U_{2n} = U_n(U_{n+1} + cU_{n-1})$, $U_{2n+1} = U_{n+1}(U_{n+1} + cU_{n-1}) + (-1)^{n-1}c^n$ and $U_n^2 - U_{n+1}U_{n-1} = (-c)^{n-1}$,

it is easy to show that

(14) $U_{(p-1)/g} \equiv 0 \pmod{U_{(p-1)/2g}}$, $U_{\{(p-1)/g\}+1} \equiv (-c)^{(p-1)/2g} \pmod{U_{(p-1)/2g}}$.

when it follows

(15)
$$H_{(p-1)/g} \equiv Q \pmod p, \qquad H_{\{(p-1)/g\}+1} \equiv P \pmod p.$$

Hence $k(H,p) = (p - 1)/g$.

Theorem e. For primes of the form $2g(2t+2) + 1$, where $t \equiv h \pmod{10}$ and $4g + 4gh + 1 \equiv \pm 1 \pmod{10}$, if
$$U_{\{(p-1)/2g\}+1} + cU_{(p-1)/2g-1} \equiv 0 \pmod p \quad \text{and} \quad (-c)^{(p-1)/2g} \equiv 1 \pmod p,$$
then $k(H,p) = 2(p - 1)/g$.

Proof. We have from (14), $U_{(p-1)/g} \equiv 0 \pmod p$ and $U_{\{(p-1)/g\}+1} \equiv -1 \pmod p$ so that

(16)
$$H_{(p-1)/g} \equiv -Q \pmod p \quad \text{and} \quad H_{\{(p-1)/g\}+1} \equiv P \pmod p.$$

Hence the desired result follows.

Theorem f. For primes of the form $2g(2t + 1) + 1$, where $t \equiv h \pmod{10}$ and $4gh + 2g + 1 \equiv \pm 1 \pmod{10}$, if

$$U_{(p-1)/2g} \equiv 0 \pmod p \quad \text{and} \quad (-c)^{(p-1)/2g} \equiv 1 \pmod p,$$

then $k(H,p) = 2(p-1)/g$.

Proof. Let us use (13) to obtain

$$U_{(p-1)/g} \equiv 0 \pmod p \quad \text{and} \quad U_{\{(p-1)/g\}+1} \equiv -1 \pmod p.$$

Then it is easy to show that

$$U_{2(p-1)/g} \equiv 0 \pmod p \quad \text{and} \quad U_{\{2(p-1)/g\}+1} \equiv \pmod p \tag{17}$$

when we get

$$H_{2(p-1)/g} \equiv Q \pmod p \quad \text{and} \quad H_{\{2(p-1)/g\}+1} \equiv P \pmod p \tag{18}$$

and the desired result follows.

Analogously, we state the following theorems.

Theorem g. For primes of the form $2g(2t+1)-1$, where $t \equiv h \pmod{10}$ and $4gh+2g-1 \equiv \pm 3 \pmod{10}$, if

$$U_{\{(p+1)/2g\}+1} + cU_{\{(p+1)/2g\}-1} \equiv 0 \pmod p \quad \text{and} \quad c^{(p+1)/2g} \equiv 1 \pmod p,$$

then $k(H,p) = (p+1)/g$.

Theorem h. For primes of the form $4gt-1$, where $t \equiv h \pmod{10}$ and $4gh-1 \equiv \pm 3 \pmod{10}$, if

$$U_{(p+1)/2g} \equiv 0 \pmod p \quad \text{and} \quad (-c)^{(p+1)/2g} \equiv 1 \pmod p,$$

then $k(H,p) = (p+1)/g$.

Theorem i. For primes of the form $2g(2t+2)-1$, where $t \equiv h \pmod{10}$ and $4g+4gh-1 \equiv \pm 3 \pmod p$, if

$$U_{\{(p+1)/2g\}-1} + cU_{\{(p+1)/2g\}-1} \equiv 0 \pmod p \quad \text{and} \quad (-c)^{(p+1)/2g} \equiv 1 \pmod p,$$

then $k(H,p) = 2(p+1)/g$.

Theorem j. For primes of the form $2g(2t+1)-1$, where $t \equiv h \pmod{10}$ and $4gh+2g-1 \equiv \pm 3 \pmod{10}$, if

$$H_{(p+1)/2g} \equiv 0 \pmod p \quad \text{and} \quad (-c)^{(p+1)/2g} \equiv 1 \pmod p,$$

then $k(H,p) = 2(p+1)/g$.

The proofs for Theorems g–j are left to the reader.

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Therefore,

$$F(0,1) = [1, 1, 1, \cdots] = \frac{1+\sqrt{4+1}}{2} \tag{7}$$

or

$$\lim_{\alpha \to \infty} \frac{I_{\alpha-1}(2a)}{I_\alpha(2a)} = \frac{1+\sqrt{5}}{2} = \phi \text{ (the "golden" ratio)}. \tag{8}$$

Expressing ϕ in this manner as the limit of a ratio of modified Bessel Functions appears to be new [2].

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ON A PROPERTY OF CONSECUTIVE FAREY-FIBONACCI FRACTIONS

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Krishnaswami Alladi [1] defined the Farey sequence of Fibonacci numbers of order F_n (where F_n is the nth Fibonacci number) as the set of all possible fractions F_i/F_j, $i = 0, 1, \cdots, n-1; j = 1, 2, \cdots, n; (i < j)$ arranged in ascending order of magnitude, with the last item $1 (= F_1/F_2)$ and the first term $0\ (= F_0/F_{n-1})$.

Now, the necessary and sufficient condition that the fractions h/k, h'/k', of $\mathcal{F}_n$, the nth ordinary Farey section, be consecutive is that

$$|kh' - hk'| = 1 \tag{1}$$

and the fraction

$$(h + h')/(k + k') \tag{2}$$

is not in $\mathcal{F}_n$.

All terms in $\mathcal{F}_{n+1}$ which are not in $\mathcal{F}_n$ are of the form $(h + h')/(k + k')$, where h/k and h'/k' are consecutive terms of $\mathcal{F}_n$. (Proofs of these results are given in Hardy and Wright [3].)

The usefulness of this result in the description of continued fractions in terms of Farey sections (Mack [5]) is an incentive to determine its Fibonacci analogue. (Also relevant are Alladi [2] and Mack [4].)

In the notation of Alladi where $f{\cdot}f_n$ denotes a Farey sequence of order F_n, the analogue of (2) above is:

All terms of $f{\cdot}f_{n+1}$ which are not already in $f{\cdot}f_n$ are of the form $(F_i + F_j)/(F_k + F_{k+1})$ where F_i/F_k and F_j/F_{k+1} are consecutive terms of $f{\cdot}f_n$ (with the exception of the first term which equals $0/F_n$).

The result follows from Alladi's definition of "generating fractions" and it can be illustrated by

$$f{\cdot}f_5: \quad 0/3,\ 1/5,\ 1/3,\ 2/5,\ 1/2,\ 3/5,\ 2/3,\ 1/1$$

and

$$f{\cdot}f_6: \quad 0/5,\ 1/8,\ 1/5,\ 2/8,\ 1/3,\ 3/8,\ 2/5,\ 1/2,\ 3/3,\ 5/8,\ 2/3,\ 1/1;$$

the terms of $f{\cdot}f_6$ which are not in $f{\cdot}f_5$ are

$$\frac{0}{5}, \frac{1}{8} = \frac{0+1}{3+5}, \quad \frac{2}{8} = \frac{1+1}{3+5}, \quad \frac{3}{8} = \frac{1+2}{3+5}, \quad \frac{5}{8} = \frac{3+2}{5+3}.$$

It is of interest to consider the analogue of (1) and here we have a result similar to Theorem 2.3 of Alladi [1]. Our problem is the following:

If $f_{(r)n} = h/k$, and $f_{(r+1)n} = h/k$ then to find $kh' - hk'$ purely in terms of r and n. We have the following theorem to this effect.

Theorem: Let $f_{(r)n} = h/k$ and $f_{(r+1)n} = h'/k'$. Then

$$kh' - hk' = \begin{cases} F_{n-1} & \text{for } r = 1 \\ F_{n-m} & \text{for } 1 < r \leqslant (n^2 - 7n + 14)/2 \\ 1 & \text{for } r > (n^2 - 7n + 14)/2 \end{cases},$$

where

$$m = 2 + [(\sqrt{8r - 15} - 1)/2]$$

in which $[\cdot]$ is the greatest integer function.

Proof. The theorem follows if we combine Theorems 2.3 and 3.1a of Alladi [1]. By Theorem 2.3, if h/k and h'/k' are consecutive in $f \cdot f_n$ and satisfy

(3) $$\frac{1}{F_i} \leqslant \frac{h}{k} < \frac{h'}{k'} \leqslant \frac{1}{F_{i-1}}$$

then

(4) $$h' - h' = F_{i-2} .$$

So we first need to find the position of $1/F_i$ in $f \cdot f_n$. By Theorem 3.1a, if $f_{(r)n} = 1/F_{n-m}$ then

(5) $$r = 2 + \{1 + 2 + 3 + \cdots + \} .$$

So by (3) and (4) if $f_{(r)n} = h/k$, and $f_{(r+1)} = h'/k'$ then

$$kh' - hk' = F_{n-m}$$

if and only if

(6) $$\frac{1}{F_{n-m+2}} \leqslant f_{(r)n} \leqslant f_{(r+1)n} \leqslant \frac{1}{F_{n-m+1}} .$$

Now (6) and (5) combine to give

(7) $$2 + \{1 + 2 + \cdots + m - 2\} = \frac{m^2 - 3m + 6}{2} \leqslant r < r + 1 \leqslant 2 + \{1 + 2 + \cdots + m - 1\} = \frac{m^2 - m + 4}{2} .$$

Now the first inequality of (7) is essentially

(8) $$\begin{aligned} m^2 - 3m + 6 \leqslant 2r &\iff \left(m - \frac{3}{2}\right)^2 + \frac{15}{4} \leqslant 2r \iff (2m - 3)^2 + 15 \leqslant 8r \\ &\iff m \leqslant 2 + \frac{\sqrt{8r - 15} - 1}{2} = \frac{\sqrt{8r - 15} + 3}{2} . \end{aligned}$$

Similarly the second inequality in (7) may be expressed as

(9) $$\begin{aligned} r + 1 \leqslant \frac{m^2 - m + 4}{2} &\iff r \leqslant \frac{m^2 - m + 2}{2} \iff 2r \leqslant (m - \tfrac{1}{2})^2 + \frac{7}{4} \\ &\iff 8r \leqslant (2m - 1)^2 + 7 \iff \frac{\sqrt{8r - 7} + 1}{2} \leqslant m . \end{aligned}$$

Now consider for $r \geqslant 2$

(10) $$0 < \frac{\sqrt{8r - 15} + 3}{2} - \frac{\sqrt{8r - 7} + 1}{2} = \frac{2 + \sqrt{8r - 15} - \sqrt{8r - 1}}{2} < 1 .$$

Now (10), (9) and (8) together imply

$$m = \left[\frac{\sqrt{8r - 15} + 3}{2}\right] = 2 + \left[\frac{\sqrt{8r - 15} - 1}{2}\right]$$

and that proves the theorem for $r \geq 2$. For $r = 1$, the first statement is trivially true.

Since it is of interest if $kh' - hk' = 1$, let us determine when this occurs. This will happen if and only if (by (6) and (4))

(11) $$\frac{1}{F_4} \leqslant f_{(r)n} .$$

By (5) and (11) we have

$$r \geqslant 2 + \{1 + 2 + \cdots + n - 4\} = \frac{n^2 - 7n + 16}{2}$$

which is for

$$r > \frac{n^2 - 7n + 14}{2}$$

and that completes the proof.

REMARK. Note, in our theorem, if $f_{(r)n} = h/k$, and $f_{(r+1)n} = h'/k'$, we need not know the values of h/k, and h'/k' to determine $kh' - hk'$. This is determined purely in terms of r and n.

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SUMS OF PRODUCTS INVOLVING FIBONACCI SEQUENCES

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DEDICATED TO JANE LEGRANGE

Definition. $\{H_n\}$ is Fibonacci if $H_n = H_{n-1} + H_{n-2}$, $n > 1$. Every Fibonacci sequence $\{H_n\}$ can be written as $H_n = A\alpha^n + B\beta^n$, where α, β are the roots of $x^2 - x - 1 = 0$. Thus

Theorem.

$$\sum_{i,j=0}^{n} a_{ij} H_i K_j = 0$$

for any two Fibonacci sequences if and only if

$$P(z,w) = \sum_{i,j=0}^{n} a_{ij} z^i w^j$$

vanishes on $\{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\}$.

Example. (Berzsenyi [1]): If n is even, prove that

$$\sum_{k=0}^{n} H_k K_{k+2m+1} = H_{m+n+1} K_{m+n+1} - H_{m+1} K_{m+1} + H_0 K_{2m+1}.$$

The corresponding $P(z,w)$ is easily seen to satisfy the hypothesis of the theorem (using $\alpha\beta = -1$, $\alpha^2 - \alpha - 1 = 0$).

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CONVERGENT GENERALIZED FIBONACCI SEQUENCES

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1. INTRODUCTION

In this note we consider sequences of numbers defined by the recursion formula

$$a_{n+2} = \alpha a_{n+1} + \beta a_n, \qquad n = 1, 2, \cdots, \tag{1}$$

with real parameters α, β and arbitrary real numbers a_1, a_2. The sequence $\{a_n\}$ will be called *generalized Fibonacci sequence* and its elements a_n the nth *generalized Fibonacci number.* Sequences like these have been introduced previously by, for example, Bessel-Hagen [1] and Tagiuri [4]. Special cases of (1) are known as the classical Fibonacci sequence with $\alpha = \beta = 1$, $a_1 = a_2 = 1$, the Lucas sequence with $\alpha = \beta = 1$, $a_1 = 1$, $a_2 = 3$, the Pell sequence with $\alpha = 2$, $\beta = 1$, $a_1 = 1$, $a_2 = 2$ and the Fermat sequences with $\alpha = 3$, $\beta = -2$, $a_1 = 1$, $a_2 = 3$ or $a_1 = 2$, $a_2 = 3$. Basic properties of the generalized Fibonacci sequences have been given by A. F. Horadam [3]. In this paper we consider generalized Fibonacci sequences from an analytic point of view. We start with a real representation of the generalized formula of Binet in the second section. In the third section we repeat and complete some properties of finite sums of generalized Fibonacci numbers [3]. With these preparations we are able to characterize convergent generalized Fibonacci sequences in the fourth section and finally in the fifth section we give some limits of *Fibonacci series.*

2. BINET'S FORMULA

For the generalized Fibonacci numbers defined by (1) the (generalized) formula of Binet holds.

$$a_n = \frac{a_2 - a_1 q_2}{q_1 - q_2} q_1^{n-1} + \frac{a_1 q_2 - a_2}{q_1 - q_2} q_2^{n-1}, \qquad n = 1, 2, \cdots, \tag{2}$$

with q_1, q_2 defined by

$$q_1 = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \beta}, \qquad q_2 = \frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} + \beta}.$$

The proof of (2) can be given by induction.

Theorem 1. Binet's formula (2) has the following real representations

$$a_n = \begin{cases} \dfrac{a_2 - a_1 q_2}{q_1 - q_2} q_1^{n-1} + \dfrac{a_1 q_1 - a_2}{q_1 - q_2} q_2^{n-1}, & \dfrac{\alpha^2}{4} + \beta > 0, \quad (3a) \\[2ex] \left(\dfrac{\alpha}{2}\right)^{n-2} \left[(n-1)a_2 - \dfrac{\alpha}{2}(n-2)a_1 \right], & \dfrac{\alpha^2}{4} + \beta = 0, \quad (3b) \\[2ex] \dfrac{r^{n-2}}{\sin\phi} [a_2 \sin(n-1)\phi - a_1 r \sin(n-2)\phi], & \dfrac{\alpha^2}{4} + \beta < 0, \quad (3c) \\ r := \sqrt{-\beta} > 0, \qquad 0 < \phi := 2 \arctan \dfrac{\sqrt{-\beta} - (\alpha/2)}{\sqrt{-\dfrac{\alpha^2}{4} + \beta}} < \pi, & \end{cases} \quad n = 1, 2, \cdots,$$

with q_1, q_2 defined as in (2).

Proof. Setting $R := (\alpha^2/4) + \beta$ the case $R > 0$ follows immediately from (2). If $R < 0$, then q_1 and q_2 are the conjugate complex numbers

$$q_{1/2} = \frac{\alpha}{2} \pm i\sqrt{-R}$$

or in polar form $q_{1/2} = re^{\pm i\phi}$ with $r = \sqrt{-\beta} > 0$ and

$$\tan\frac{\phi}{2} = \frac{1-\cos\phi}{\sin\phi} = \frac{\sqrt{-\beta} - \frac{\alpha}{2}}{\sqrt{-R}} \quad \text{or} \quad 0 < \phi = 2\arctan\frac{\sqrt{-\beta} - \frac{\alpha}{2}}{\sqrt{-R}} < \pi,$$

respectively. Further rewriting (2) with $q_2 = \bar{q}_1$

$$a_n = a_1 q_1 \bar{q}_1 \frac{\bar{q}_1^{n-2} - q_1^{n-2}}{q_1 - \bar{q}_1} + a_2 \frac{q_1^{n-1} - \bar{q}_1^{n-1}}{q_1 - \bar{q}_1},$$

employing the polar form mentioned above and using

$$\frac{q_1^m - \bar{q}_1^m}{q_1 - \bar{q}_1} = r^{m-1}\frac{\sin m\phi}{\sin\phi}, \qquad m = 1, 2, \cdots,$$

we conclude statement (3c). (3b) follows from (3c) as limit for $\phi \to 0$. From (3a) and (3b) we get two special cases, which will be useful in the following discussion.

First let $\alpha + \beta = 1$. Then

$$a_n = \begin{cases} \dfrac{a_2 - a_1}{\alpha - 2}(\alpha - 1)^{n-1} + \dfrac{a_1(\alpha - 1) - a_2}{\alpha - 2}, & \alpha \neq 2, \\ (n-1)a_2 - (n-2)a_1, & \alpha = 2, \end{cases} \qquad n = 1, 2, \cdots. \tag{4}$$

Let be $\beta - \alpha = 1$. Then

$$a_n = \begin{cases} \dfrac{a_2 + a_1}{\alpha + 2}(\alpha + 1)^{n-1} + \dfrac{a_1(\alpha + 1) - a_2}{\alpha + 2}(-1)^{n-1}, & \alpha \neq -2, \\ (-1)^n[(n-1)a_2 + (n-2)a_1], & \alpha = -2, \end{cases} \qquad n = 1, 2, \cdots. \tag{5}$$

3. SUMS OF GENERALIZED FIBONACCI NUMBERS

In this chapter we consider some simple properties of finite sums of generalized Fibonacci numbers.

Property 1. The sum of the first n generalized Fibonacci numbers is given by

$$\sum_{\nu=1}^{n} a_\nu \frac{1}{\alpha + \beta - 1}[a_{n+1} + \beta a_n - a_2 - (1 - \alpha)a_1], \qquad n = 1, 2, \cdots, \tag{6a}$$

if $\alpha + \beta \neq 1$ and by

$$\sum_{\nu=1}^{n} a_\nu = \begin{cases} n\dfrac{a_1(\alpha - 1) - a_2}{\alpha - 2} + \dfrac{a_1 - a_2}{(\alpha - 2)^2}[1 - (\alpha - 1)^n], & \alpha \neq 2, \\ \dfrac{n}{2}[n(a_2 - a_1) + 3a_1 - a_2], & \alpha = 2, \end{cases} \quad n = 1, 2, \cdots, \tag{6b}$$

if $\alpha + \beta = 1$.

Repeated use of the recursion formula yields statement (6a).

If $\alpha + \beta = 1$, $\alpha \neq 2$, we get the first part of (6b) from (4) using the formula of the finite geometric series. The second part in (6b) follows immediately from (3b) with $\alpha = 2$. Since the following properties can be shown in a similar way, we omit their proofs.

Property 2. The sum of generalized Fibonacci numbers with odd suffixes is given by

$$\sum_{\nu=1}^{n} a_{2\nu-1} = a_1 + \frac{1}{\alpha^2 - (\beta - 1)^2}[a_{2n} + \beta(1 - \beta)a_{2n-1} - \alpha a_2 - \beta(1 - \beta)a_1], \tag{7a}$$

$n = 1, 2, \cdots$, if $\alpha + \beta \neq 1$, $\beta - \alpha \neq 1$, and by

$$\text{(7b)}\quad \sum_{\nu=1}^{n} a_{2\nu-1} = \left\{ \begin{array}{ll} n\,\dfrac{a_2 + a_1(1-\alpha)}{2-\alpha} + \dfrac{a_1 - a_2}{\alpha(2-\alpha)^2}\,[1-(\alpha-1)^{2n}], & \alpha \neq 2, \\ n[(n-1)a_2 - (n-2)a_1], & \alpha = 2, \end{array} \right\} \quad n = 1, 2, \cdots,$$

if $\alpha + \beta = 1$ and by

$$\text{(7c)}\quad \sum_{\nu=1}^{n} a_{2\nu-1} = \left\{ \begin{array}{ll} n\,\dfrac{a_1(1+\alpha) - a_2}{2+\alpha} - \dfrac{a_1 + a_2}{\alpha(2+\alpha)^2}\,[1-(\alpha+1)^{2n}], & \alpha \neq -2, \\ -n[(n-1)a_2 + (n-2)a_1], & \alpha = -2, \end{array} \right\} \quad n = 1, 2, \cdots,$$

if $\beta - \alpha = 1$.

Property 3. The sum of generalized Fibonacci numbers with even suffixes is given by

$$\text{(8a)}\quad \sum_{\nu=1}^{n} a_{2\nu} = \frac{1}{\alpha^2 - (\beta-1)^2}\,[\alpha a_{2n+1} + \beta(1-\beta)a_{2n} + (\beta-1)a_2 - \alpha\beta a_1], \quad n = 1, 2, \cdots,$$

if $\alpha + \beta \neq 1$, $\beta - \alpha \neq 1$, and by

$$\text{(8b)}\quad \sum_{\nu=1}^{n} a_{2\nu} = \left\{ \begin{array}{ll} n\,\dfrac{a_2 - a_1(\alpha-1)}{2-\alpha} + \dfrac{(\alpha-1)(a_1 - a_2)}{\alpha(2-\alpha)^2}\,[1-(\alpha-1)^{2n}], & \alpha \neq 2, \\ n[na_2 - (n-1)a_1], & \alpha = 2, \end{array} \right\} \quad n = 1, 2, \cdots,$$

if $\alpha + \beta = 1$ and by

$$\text{(8c)}\quad \sum_{\nu=1}^{n} a_{2\nu} = \left\{ \begin{array}{ll} n\,\dfrac{a_2 - a_1(1+\alpha)}{2+\alpha} - \dfrac{(1+\alpha)(a_2 + a_1)}{\alpha(2+\alpha)^2}\,[1-(1+\alpha)^{2n}], & \alpha \neq -2, \\ n[na_2 + (n-1)a_1], & \alpha = -2, \end{array} \right\} \quad n = 1, 2, \cdots,$$

if $\beta - \alpha = 1$.

Property 4. The sum of generalized Fibonacci numbers with alternating signs is given by

$$\text{(9a)}\quad \sum_{\nu=1}^{n} (-1)^{\nu-1} a_\nu = \frac{1}{\alpha - \beta + 1}\,[(-1)^{n+1}(a_{n+1} - \beta a_n) - 2 + (\alpha+1)a_1],$$

$n = 1, 2, \cdots$, if $\beta - \alpha \neq 1$ and by

$$\text{(9b)}\quad \sum_{\nu=1}^{n} (-1)^{\nu-1} a_\nu = \left\{ \begin{array}{ll} n\,\dfrac{a_1(1+\alpha) - a_2}{2+\alpha} + \dfrac{a_1 + a_2}{(2+\alpha)^2}\,[1 + (-1)^{n-1}(\alpha+1)^n], & \alpha \neq -2, \\ -\dfrac{n}{2}\,[(n-1)a_2 + (n-3)a_1], & \alpha = -2, \end{array} \right\}$$

$n = 1, 2, \cdots$, if $\beta - \alpha = 1$.

We terminate this section with one nonlinear property.

Property 5. The sum of squares of the generalized Fibonacci numbers is given by

$$\text{(10)}\quad \sum_{\nu=1}^{n} a_\nu^2 = \frac{1}{1+\beta}\,[a_1\sigma_n + (a_2 - \alpha a_1)\tau_{n-1} + \beta a_n^2], \quad \beta \neq -1, \quad n = 1, 2, 3, \cdots,$$

with σ_n and τ_n defined by

$$\sigma_n := \sum_{\nu=1}^{n} a_{2\nu-1}, \qquad \tau_n := \sum_{\nu=1}^{n} a_{2\nu}.$$

The explicit form of (10) may be found with the formulas (7) and (8).

4. CONVERGENT FIBONACCI SEQUENCES

Using Binet's formula (2) we are able to characterize the convergent Fibonacci sequences.

Theorem 2. Generalized Fibonacci sequences are convergent if and only if the parameters α, β are points of the region (see Fig. 1)

$$D := \left\{ (\alpha,\beta) \in R^2 \mid \alpha+\beta \leqslant 1, \quad \beta-\alpha < 1, \quad \beta > -1 \right\}. \tag{11}$$

In the interior $\underline{D}$ of the region D the generalized Fibonacci sequences converge to zero. On the bounday

$$\alpha+\beta = 1, \quad 0 < \alpha < 2, \quad -1 < \beta < 1,$$

the limit a of the generalized Fibonacci sequences is given by

$$a := \lim_{n\to\infty} a_n = \frac{a_2 + a_1\beta}{1+\beta}. \tag{12}$$

Proof. With the representations (3a)–(3c) for Binet's formula we conclude the following necessary and sufficient conditions for the convergence of the generalized Fibonacci sequences

$$-1 < q_1,\ q_2 \leqslant 1,\ \frac{\alpha^2}{4}+\beta > 0, \quad \text{from (3a)},$$

$$\left|\frac{\alpha}{2}\right| < 1,\ \frac{\alpha^2}{4}+\beta = 0, \quad \text{from (3b)},$$

$$r = \sqrt{-\beta} < 1,\ \frac{\alpha^2}{4}+\beta < 0, \quad \text{from (3c)}.$$

This means in detail in (3a)

$$-1 < \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4}+\beta} \leqslant 1,$$

which leads together with $\frac{\alpha^2}{4}+\beta > 0$ to $\alpha+\beta \leqslant 1$, $\alpha < 2$, and in an analogous way from

$$-1 < \frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4}+\beta} \leqslant 1$$

to $\beta-\alpha < 1$, $\alpha > -2$, by (3b) we have $\alpha^2 = -4\beta$, $-2 < \alpha < 2$, and from (3c) $\alpha^2 < -4\beta$, $\beta > -1$. All these conditions yield the required convergence domain D for the parameters α, β. In D it follows from

$$\lim_{n\to\infty} q_{1/2}^n = 0, \quad |q_{1/2}| < 1, \quad \text{from} \quad \lim_{n\to\infty} n\left(\frac{\alpha}{2}\right)^n = 0, \quad \left|\frac{\alpha}{2}\right| < 1,$$

and from

$$\lim_{n\to\infty} r^n = 0, \quad 0 < r < 1,$$

that all limits vanish. On the boundary of D, $\alpha+\beta = 1$, $|\alpha| < 2$, we get from (4) for $n \to \infty$ the required result[1]

$$a := \lim_{n\to\infty} a_n = \frac{a_1(\alpha-1) - a_2}{\alpha-2} = \frac{a_2+\beta a_1}{1+\beta}.$$

5. FIBONACCI SERIES

Finally we will consider some Fibonacci series, which are defined as convergent series with generalized Fibonacci numbers as terms. Since terms of convergent series necessarily converge to zero, we have to choose the parameters α, β from the interior $\underline{D}$ of the convergence domain D (11). Tending n to infinity and using Theorem 2 we get the following limits from the properties 1–5:

$$\sum_{\nu=1}^{\infty} a_\nu = \frac{a_2 + (1-\alpha)a_1}{1-\alpha-\beta}, \qquad (\alpha,\beta) \in \underline{D}, \tag{13}$$

[1] In [2] a special case of this general result is mentioned with $\alpha = \beta = 1/2$, $a = (a_1 + 2a_2)/3$. This result is obtained by the only use of the recurrence relation (1).

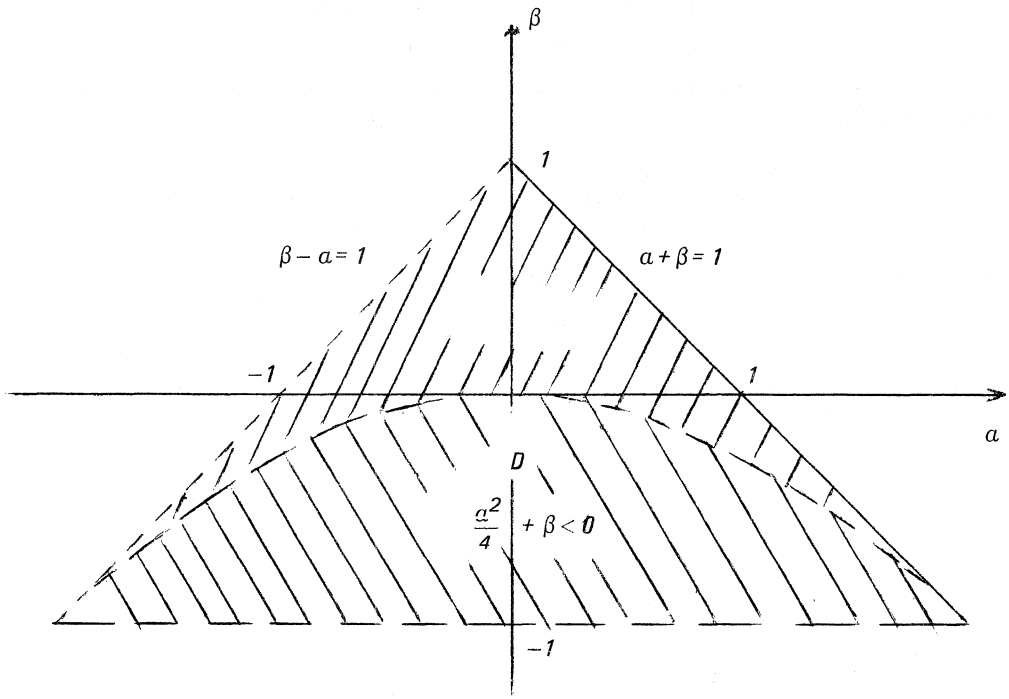

Fig. 1 Region D of Convergence of Generalized Fibonacci Sequences

(14) $$\sum_{\nu=1}^{\infty} a_{2\nu-1} = \frac{(\alpha^2+\beta-1)a_1-\alpha a_2}{\alpha^2-(\beta-1)^2}, \qquad (\alpha,\beta)\in \underline{D},$$

(15) $$\sum_{\nu=1}^{\infty} a_{2\nu} = \frac{(\beta-1)a_2-\alpha\beta a_1}{\alpha^2-(\beta-1)^2}, \qquad (\alpha,\beta)\in \underline{D},$$

(16) $$\sum_{\nu=1}^{\infty} (-1)^{\nu-1}a_\nu = \frac{(\alpha+1)a_1-a_2}{\alpha-\beta+1}, \qquad (\alpha,\beta)\in \underline{D},$$

(17) $$\sum_{\nu=1}^{\infty} a_\nu^2 = \frac{a_2^2(\beta-1)-2\alpha\beta a_1a_2+[\alpha^2(1+\beta)+\beta-1]a_1^2}{(1+\beta)[\alpha^2-(\beta-1)^2]}, \qquad (\alpha,\beta)\in \underline{D}.$$

Naturally this list can be extended to other, e.g., cubic or binomial, sums using Theorem 2.

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ON GENERATING FUNCTIONS

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Theorem. Consider the following three statements:

(1) $$\psi(x,t) = \sum_{n=0}^{\infty} \phi_n(x)t^n ,$$

(2) $$\ln \psi(x,t) = \sum_{n=1}^{\infty} \frac{A_n(x)t^n}{n} ,$$

(3) $$n\phi_n(x) = \sum_{k=1}^{n} A_k(x)\phi_{n-k}(x) .$$

Any two of these statements imply the third.

Proof. For convenience in sum manipulation, let us define $A_0 = 1$ so that (3) becomes

(4) $$(n+1)\phi_n(x) = \sum_{k=0}^{n} A_k(x)\phi_{n-k}(x).$$

We also normalize the $\phi_n(x)$ so that $\phi_0(x) = 1$.

Now assume that (1) and (4) are true; then from (4) we have

$$\sum_{n=0}^{\infty} (n+1)\phi_n t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_k \phi_{n-k} t^n ,$$

or

$$\frac{d}{dt}\left[t \sum_{n=0}^{\infty} \phi_n t^n \right] = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} A_k \phi_n t^{n+k} .$$

Hence by (1)

$$\frac{d}{dt}\left[t\psi(x,t) \right] = \sum_{n=0}^{\infty} \phi_n t^n \sum_{k=0}^{\infty} A_k t^k ,$$

or

$$\frac{d}{dt}\left[t\psi(x,t) \right] = \psi(x,t) \sum_{k=0}^{\infty} A_k t^k .$$

Therefore

$$\frac{\frac{d}{dt}[t\psi(x,t)]}{t\psi(x,t)} = \sum_{k=0}^{\infty} A_k t^{k-1} ,$$

or, by integration,

$$\ln [t\psi(x,t)] = \sum_{k=1}^{\infty} \frac{A_k t^k}{k} + \ln t + K(x).$$

Hence

$$\ln \psi(x,t) = \sum_{k=1}^{\infty} \frac{A_k t^k}{k} + K(x) .$$

We may assume $K(x) = 0$ since we assume the $\phi_k(x)$ do not all have a common factor.

$$\therefore \ln \psi(x,t) = \sum_{k=1}^{\infty} \frac{A_k t^k}{k} ,$$

which is statement (2).

If we assume (2) and (4) are true, then we have from (4)

$$\sum_{n=0}^{\infty} (n+1)\phi_n t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_k \phi_{n-k} t^n = \sum_{n=0}^{\infty} \phi_n t^n \sum_{k=0}^{\infty} A_k t^k ,$$

or

$$\frac{d}{dt}\left[t \sum_{n=0}^{\infty} \phi_n t^n \right] = t \sum_{n=0}^{\infty} \phi_n t^n \sum_{k=0}^{\infty} A_k t^{k-1} .$$

Divide and integrate, and we obtain

$$\ln \left[t \sum_{n=0} \phi_n t^n \right] = \sum_{k=1} \frac{A_k t^k}{k} + \ln t + \ln K(x).$$

Therefore, using (2),

$$\sum_{n=0}^{\infty} \phi_n(x) t^n = K(x)\psi(x,t) . \tag{5}$$

From (2), $\ln \psi(x,0) = 0$, so that $\psi(x,0) = 1$. Let $t \to 0$ in (5) and we get $\phi_0(x) = K(x)$, so $K(x) = 1$ since $\phi_0(x) = 1$. Hence

$$\sum_{n=0}^{\infty} \phi_n t^n = \psi(x,t) ,$$

which is statement (1).

If we assume (1) and (2) are true, we get

$$\ln [t\psi(x,t)] = \sum_{k=1}^{\infty} \frac{A_k t^k}{k} + A_0 \ln t$$

by adding $\ln t$ to both sides of (2) and remembering that $A_0 = 1$. Replacing $\psi(x,t)$ by its sum given in (1) and differentiating with respect to t,

$$\frac{d}{dt} \sum_{n=0}^{\infty} \phi_n t^{n+1} = \sum_{n=0}^{\infty} \phi_n t^{n+1} \sum_{k=1}^{\infty} A_k t^{k-1} + \frac{A_0}{t} .$$

$$\sum_{n=0}^{\infty} (n+1)\phi_n t^n = \sum_{n=0}^{\infty} \phi_n t^n \sum_{k=0}^{\infty} A_k t^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \phi_{n-k} A_k t^n .$$

Equating coefficients of t^n,

$$(n+1)\phi_n = \sum_{k=0}^{n} \phi_{n-k} A_k ,$$

which is (4).

By rewording the previous theorem, we obtain this rendition:
If $\psi = \Sigma\phi_n t^n$, so that $t\psi = \Sigma\phi_n t^{n+1}$, then

$$e^{t\psi} = \sum \Theta_n t^n, \quad \text{where} \quad n\Theta_n = \sum_{k=1}^{n} k\phi_{k-1}\phi_{n-k}.$$

This naturally leads to all manner of strange generating functions. Omitting the trivial intervening steps, we list a small sample and note it is mildly surprising that the left-hand side should generate such a nice set of coefficients.

1| $$\exp\{t\}\exp\{xt\}\exp\{J_0(t\sqrt{1-x^2})\} = \sum \phi_n t^n,$$

where $$n\phi_n = \sum_{k=1}^{n}\left(\frac{k}{(k-1)!}\right)P_{k-1}\phi_{n-k}.$$

2| $$\exp\left\{t(1-2xt+t^2)^{-1/2}\right\} = \sum \phi_n t^n,$$

where $$n\phi_n = \sum_{k=1}^{n} kP_{k-1}\phi_{n-k}.$$

3| $$\exp\{t(1-t)^{1-\alpha-\beta}\}\exp\left\{2\,{}^{F}{}_{1}\left[\frac{1+a+\beta}{2}, 1+\frac{a+\beta}{2}; 1+a; \frac{2t(x-1)}{(1-t)^2}\right]\right\} = \sum \phi_n t^n,$$

where $$n\phi_n = \sum_{k=1}^{n}\frac{k(a+\beta+1)_{k-1}}{(1+a)_{k-1}}P_{k-1}^{\alpha,\beta}\phi_{n-k}.$$

4| $$\exp\{t^2(e^t-1)^{-1}\} = \sum \phi_n t^n,$$

where $$n\phi_n = \sum_{k=1}^{n}\left(\frac{k}{(k-1)!}\right)B_{k-1}\phi_{n-k}.$$

In these equations, P_n and $P_n^{\alpha,\beta}$ are the Legendre and Jacobi polynomials, respectively, and B_n are the Bernoulli numbers. The ϕ_n are polynomials of degree n except in 4|.

The class of integrals easily obtained from these generating functions should delight any collector of the esoteric.

We close with two direct applications of the Theorem. Both are known, but the derivation is quite simplified.

5| Since $$(1-t)^{-1-\alpha}\exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{\alpha}t^n,$$

and $$-(1+a)\ln(1-t) - \frac{xt}{1-t} = \sum_{n=0}^{\infty}\left[\frac{1+a-x(n+1)}{n+1}\right]t^{n+1},$$

then $$nL_n = \sum_{k=1}^{n}(1+a-kx)L_{n-k}^{\alpha},$$

where L_n^{α} are the Laguerre polynomials.

6| Since $$(1-2t\cos x+t^2)^{-1/2} = \sum_{n=0}^{\infty}P_n(\cos x)t^n \quad \text{and} \quad -\tfrac{1}{2}\ln(1-2t\cos x+t^2) = \sum_{r=1}^{\infty}\frac{t^r\cos rx}{r},$$

then $$(n+1)P_n(\cos x) = \sum_{k=0}^{n}\cos kx P_{n-k}(\cos x).$$

This work was supported in part by a grant from the Research Grant Committee of the University of Alabama, Project 763.

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A RESULT IN ANALYTIC NUMBER THEORY

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The purpose of this note is to state and prove a result in analytic number theory that seems largely to have been overlooked. The usefulness of this result is illustrated by applying it to obtain an extremely simple proof of an estimate for a certain set of integers.

Let the letter p be used to denote primes.

Theorem 1. If f is multiplicative, then a necessary and sufficient condition that

$$\sum_{n=1}^{\infty} f(n)$$

converge absolutely is that

$$\prod_{p} \sum_{n=0}^{\infty} |f(p^n)|$$

converge. Furthermore, in the case of convergence,

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \left(\sum_{n=0}^{\infty} f(p^n) \right) .$$

Before we prove the theorem a few comments seem to be in order. The necessity is proved by Hardy and Wright [7, Theorem 286]. However, Hardy and Wright do not prove or even state the sufficiency condition above. Both necessary and sufficient conditions are stated by Ayoub [1, Theorem 1.5], but his statement of the sufficiency condition is careless and the proof given is not adequate.

Proof of Sufficiency. Let the increasing sequence of positive primes be denoted $p_1, p_2, \cdots$ and let t be a fixed integer. Then the general term in the product

$$\prod_{i=1}^{t} \left(\sum_{k=0}^{\infty} |f(p_i^k)| \right)$$

is of the form

$$|f(p_1^{\alpha_1})||f(p_2^{\alpha_2})| \cdots |f(p_t^{\alpha_t})| = |f(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t})| ,$$

where

$$a_i \geqslant 0 \quad (1 \leqslant i \leqslant t).$$

The last equality is true because f is multiplicative. An integer n will appear in this product (as argument of f) if and only if it has no prime factors other than $p_1, p_2, \cdots, p_t$. By the unique factorization theorem it will then appear only once. Thus

$$\prod_{i \leqslant t} \sum_{k=0}^{\infty} |f(p_i^k)| = \sum_{(t)} |f(n)| ,$$

where the last summation is over all integers n whose only prime factors are in the set $p_1, p_2, \cdots, p_t$. Thus

$$\prod_{p} \sum_{k=0}^{\infty} |f(p^k)| = \lim_{t \to \infty} \prod_{i \leqslant t} \sum_{k=0}^{\infty} |f(p_i^k)| = \lim_{t \to \infty} \sum_{(t)} |f(n)| .$$

Now

$$A_t \equiv \sum_{n=1}^{p_t} |f(n)| \leqslant \sum_{(t)} |f(n)| \equiv B_t ,$$

since the summation on the right includes at least those on the left. Since $\{B_t\}$ converges, it is bounded, and therefore $\{A_t\}$ is a bounded, non-decreasing sequence. The fundamental theorem on monotone sequences applies and hence $\{A_t\}$ converges. But $\{A_t\}$ is a subsequence of the partial sums $\{s_n\}$ of the series

$$\sum_{n=1}^{\infty} |f(n)| .$$

It follows that $\{s_n\}$ converges and the proof is complete.

Before we obtain the asymptotic result mentioned above we need the following definition. Let L represent the set of positive integers n with the property that p divides n implies that p^2 divides n. An integer in L is called a square-full integer. The characteristic function of L will be denoted by $1(n)$ and the summatory function of $1(n)$ will be denoted $L(x)$, so that

$$L(x) = \sum_{n \leqslant x} 1(n).$$

The proof of our result depends upon a famous theorem on series due to Kronecker (cf. [9, p. 129]). We give it in arithmetical form.

Lemma 1. If f is an arithmetical function and

$$\sum_{n=1}^{\infty} f(n)/n$$

is a convergent series, then f has mean value 0, that is,

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leqslant x} f(n) = 0.$$

We now prove that L has density 0.

Theorem 2. The set L has density 0; that is,

$$\lim_{x \to \infty} \frac{L(x)}{x} = 0 .$$

Proof. By Lemma 1 we need only show that $\Sigma 1(n)/n$ converges. But by Theorem 1 and the multiplicativity of $1(n)$, it suffices to show that

$$\prod_p \left(\sum_{n=0}^{\infty} \frac{1(p^n)}{p^n} \right)$$

is convergent. By definition of $1(n)$

$$\prod_p \left(\sum_{n=0}^{\infty} \frac{1(p^n)}{p^n} \right) = \prod_p \left(1 + \frac{1(p)}{p} + \frac{1(p^2)}{p^2} + \cdots \right) = \prod_p (1 + 1/p^2 + 1/p^3 + \cdots) = \prod_p \left(1 + \frac{1}{p(p-1)} \right)$$

which is convergent.

Earlier proofs of this result were given by Feller and Tournier [6, §9] and Schoenberg [10, §12]. In addition Erdos and Szekeres [5], Hornfack [8], and Cohen [2], [3] have considered generalizations of the above problem. For a discussion of previous results including refinements of Theorem 2, see [3] and [4].

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ADDITIVE PARTITIONS I

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David Silverman in July 1976 found the following property of the Fibonacci Numbers. This Theorem I was subsequently proved by Ron Evans, Harry L. Nelson, David Silverman, and Krishnaswami Alladi with myself, all independently.

Theorem I. The Fibonacci Numbers uniquely split the positive integers, N, into two sets A_0 and A_1 such that

$$A_0 \cup A_1 = N$$
$$A_0 \cap A_1 = \phi$$

and so that no two members of A_0 nor two members of A_1 add up to a Fibonacci number.

Theorem. (Hoggatt) Every positive integer $n \neq F_k$ is the sum of two members of A_0 or the sum of two members of A_1.

Theorem. (Hoggatt) Using the basic ideas above the Fibonacci Numbers uniquely split the Fibonacci Numbers, the Lucas Numbers uniquely split the Lucas Numbers and uniquely split the Fibonacci Numbers, and $\{5F\}_{n=2}^{\infty}$ uniquely splits the Lucas Sequence.

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PROOF OF A SPECIAL CASE OF DIRICHLET'S THEOREM

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For any prime p I give a simple proof that there are infinitely many primes $q \equiv -1 \bmod p$, a special case of Dirichlet's Theorem that if g.c.d. $(a,m) = 1$ there are infinitely many primes $\equiv a \pmod m$. The proof is of interest in that it utilizes several number-theoretic properties of the Fibonacci Numbers, which are also developed herein.

In this paper F_n represents the Pseudo-Fibonacci Numbers, defined as $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = aF_n + bF_{n-1}$, where a and b are non-zero relatively prime integers.

F_n may then be written non-recursively as

$$F_n = \frac{\left(\frac{a+\sqrt{a^2+4b}}{2}\right)^n - \left(\frac{a-\sqrt{a^2+4b}}{2}\right)^n}{\sqrt{a^2+4b}} \quad . \tag{1}$$

For a derivation of this result see Niven and Zuckerman [1].

We will need the following lemmas:

Lemma 1. For any positive integer r that divides F_n for some n, let h be the smallest positive integer such that r divides F_h. Then h is a divisor of n.

Lemma 2. For any positive integer n, g.c.d. $(F_n, b) = 1$.

These results are noted in a paper by Hoggatt and Long [2].

Lemma 3. For any odd prime q,

$$F_q \equiv (a^2+4b)^{\frac{q-1}{2}} \pmod q \tag{2}$$

$$2F_{q+1} \equiv a(a^2+4b)^{\frac{q-1}{2}} + a \pmod q \tag{3}$$

$$2bF_{q-1} \equiv -a(a^2+4b)^{\frac{q-1}{2}} + a \pmod q. \tag{4}$$

Proof of Lemma 3. Replacing n by q in (1), expanding the right-hand side by the binomial expansion, and multiplying by 2^{q-1} we get modulo q,

$$2^{q-1}F_q \equiv (a^2+4b)^{\frac{q-1}{2}} .$$

This gives (2) because $2^{q-1} \equiv 1 \bmod q$.

Similarly, if we replace n by $q+1$ in (1) and expand, noting that $\binom{q+1}{i} \equiv 0 \bmod q$ for $2 \leqslant i \leqslant q-1$, and then multiply by 2^q, we get

$$2^qF_{q+1} \equiv (q+1)a(a^2+4b)^{\frac{q-1}{2}} + (q+1)a^q \pmod q .$$

this reduces to (3) by use of $a^q \equiv a \bmod q$. Then (4) follows from (2) and (3) and the equality

$$2F_{q+1} = 2aF_q + 2bF_{q-1} .$$

Theorem (Dirichlet). For any prime p there exist infinitely many primes $q \equiv -1 \pmod p$.

Proof. If $p = 2$ every odd prime satisfies $q \equiv -1 \pmod 2$. So henceforth let p be a fixed odd prime. Suppose

there are only finitely many primes $q_1, q_2, \cdots, q_m$ satisfying the congruence. By Theorem 2.27, Chapter 2 of Niven and Zuckerman [3], there exist $(p-1)/2$ positive integers $k \leqslant p-1$ satisfying $k^{(p-1)/2} \equiv 1 \bmod p$. Hence there also exist $(p-1)/2$ positive integers $j \leqslant p-1$ satisfying $j^{(p-1)/2} \equiv -1 \bmod p$. Let λ be one of these positive integers j and define the positive integers $a = 2$,

$$\theta = \lambda \prod_{j=1}^{m} q_j^2, \qquad b = 4\theta - 1.$$

It follows that

$$a^2 + 4b = 16\theta, \qquad \frac{a \pm \sqrt{a^2+4b}}{2} = 1 + 2\sqrt{\theta}\ . \tag{5}$$

Using these values of a and b in (1) and using (2) from Lemma 3 with q replaced by p, we see that

$$F_p \equiv (a^2+4b)^{\frac{p-1}{2}} \equiv (16\theta)^{\frac{p-1}{2}} \equiv 4^{p-1}(\Pi q_j)^{p-1}\lambda^{\frac{p-1}{2}} \equiv -1 \pmod p. \tag{6}$$

Also from (1) and (5) we see that

$$F_p = \frac{(1+2\sqrt{\theta})^p - (1-2\sqrt{\theta})^p}{4\sqrt{\theta}}, \qquad F_p \equiv p \pmod{4\theta}, \tag{7}$$

where the second result here is obtained by expanding the first result and taking everything modulo 4θ.

Now let q be a prime factor of F_p. From (6) we see that $q \neq p$, and from the second part of (7) we see that q is not a divisor of 4θ, so q is different from the primes $2, q_1, q_2, \cdots, q_m$.

We note that

$$(a^2+4b)^{\frac{q-1}{2}} \equiv (16\theta)^{\frac{q-1}{2}} \equiv 4^{q-1}(\Pi q_j)^{q-1}\lambda^{\frac{q-1}{2}} \equiv \lambda^{\frac{q-1}{2}} \equiv \epsilon \bmod q,$$

where $\epsilon = +1$ or $\epsilon = -1$.

If $\epsilon = +1$ we use (4) from Lemma 3 to conclude that q is a divisor of $2bF_{q-1}$. But q is odd and by Lemma 2 is not a divisor of b, since $(F_p, b) = 1$ and q is a divisor of F_p, and so q is a divisor of F_{q-1}. By Lemma 1, with n replaced by $q-1$, h replaced by p, and r by q, we see that p is a divisor of $q-1$ and so $q \equiv 1 \bmod p$. Now if this congruence holds for every prime divisor q of F_p it would follow from the multiplication of such congruences that $F_p \equiv 1 \bmod p$, contrary to (6). Hence we must have $\epsilon = -1$ for at least one prime divisor q of F_p.

In the case $\epsilon = -1$ we use (3) from Lemma 3 to conclude that q is a divisor of $2F_{q+1}$, and so a divisor of F_{q+1}. By Lemma 1 we see that p is a divisor of $q+1$, so $q \equiv -1 \pmod p$, contrary to the assumption that $q_1, q_2, \cdots, q_m$ are the only primes satisfying this congruence. Q.E.D.

Corollary. From the same analysis used to establish the above result, with $a = 2$ and $b = 4\lambda - 1$ substituted into (1), $p \cdot 1$, for any prime p

$$F_p = \frac{(1+2\sqrt{\lambda})^p - (1-2\sqrt{\lambda})^p}{4\sqrt{\lambda}}$$

is divisible by a prime $q \equiv -1 \pmod p$. Since $\lambda \leqslant p-1$, a prime

$$q \equiv -1 \pmod p < (2\sqrt{p-1}+1)^p.$$

For a proof of the existence of infinitely many primes $q \equiv -1 \pmod m$, (m any positive integer $\geqslant 2$) using polynomial theory, see Nagell [4]. For a simple proof of the existence of infinitely many primes $q \equiv 1 \pmod m$ see Ivan Niven and Barry Powell [5].

ADDITIONAL RESULTS

Theorem: Consider any odd prime p which does not divide (a^2+4b), where $(a,b) = 1$ as in (1), $p \cdot 1$. Then $F_p \equiv 0 \bmod q$, q prime, $\rightarrow q \equiv 1 \bmod p$ or $q \equiv -1 \bmod p$ if and only if

$$(a^2+4b)^{\frac{q-1}{2}} \equiv 1 \bmod q \qquad \text{or} \qquad (a^2+4b)^{\frac{q-1}{2}} \equiv -1 \bmod q.$$

[Co-discovered by Professor Verner E. Hoggatt, Jr., per telephone communication.]

Proof. We have, from (1), p. 1,

$$F_p = \frac{\left(\frac{a+\sqrt{a^2+4b}}{2}\right)^p - \left(\frac{a-\sqrt{a^2+4b}}{2}\right)^p}{\sqrt{a^2+4b}}$$

Multiplying both sides by 2^{p-1} and using the binomial expansion, we get

$$2^{p-1}F_p \equiv pa^{p-1} \bmod (a^2+4b). \tag{8}$$

$$F_p \equiv 0 \bmod q \to q \nmid (a^2+4b).$$

Otherwise

$$q \mid (a^2+4b) \to 2^{p-1}F_p \equiv pa^{p-1} \bmod q \quad \text{from (8)},$$

$$\to pa^{p-1} \equiv 0 \bmod q \to q \mid p \text{ or } q \mid a.$$

$$q \mid p \to q = p \to F_p \equiv 0 \bmod p \to p \mid (a^2+4b)$$

by (2) of Lemma 3, contradicting the assumption that $p \nmid (a^2+4b)$. $q \nmid a$, since

$$a = F_2 \equiv 0 \bmod q \to 2 \mid p$$

by Lemma 1, and p is odd.

Thus from Lemma 3, (3) and (4),

$$F_{q+1} \equiv 0 \bmod q \text{ iff } (a^2+4b)^{\frac{q-1}{2}} \equiv -1 \bmod q$$

and

$$2bF_{q-1} \equiv 0 \bmod q \text{ iff } (a^2+4b) \equiv 1 \bmod q.$$

$F_p \equiv 0 \bmod q$ and $F_{q+1} \equiv 0 \bmod q \to q \equiv -1 \bmod p$ by Lemma 1 with h replaced by p. Since

$$p \mid (q+1) \to F_p \mid F_{q+1}$$

Therefore $F_{q+1} \equiv 0 \bmod q$. Thus $F_{q+1} \equiv 0 \bmod q$ iff $q \equiv -1 \bmod p$. Hence $(a^2+4b)^{\frac{q-1}{2}} \equiv -1 \bmod q$ iff $q \equiv -1 \bmod q$.

Similarly $F_{q-1} \equiv 1 \bmod q$ iff $q \equiv 1 \bmod p$ and hence $(a^2+4b)^{\frac{q-1}{2}} \equiv 1 \bmod q$ iff $q \equiv 1 \bmod p$ follows from Lemma 1, Lemma 2, and the fact that $p \mid (q-1) \to F_p \mid F_{q-1}$.

Conjecture. For n any positive integer sufficiently large, there exists at least 1 prime $q \equiv \pm 1 \bmod n$ dividing F_n.

EXAMPLES. F_{15} of the Fibonacci sequence

$$= 610 = 61 \cdot 10 \text{ and } 61 \equiv 1 \bmod 15.$$

$$F_{18} = 136 \cdot 19 \text{ and } 19 \equiv 1 \bmod 18.$$

$$F_{20} = 165 \cdot 41 \text{ and } 41 \equiv 1 \bmod 20.$$

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DIOPHANTINE EQUATIONS INVOLVING THE GREATEST INTEGER FUNCTION

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It is known [1, p. 142] that if λ and μ are fixed positive irrationals such that $\mu\lambda = \mu + \lambda$, then the equation $[n\lambda] = [m\mu]$ has no solution in integers m and n, where $[x]$ denotes the greatest integer less than or equal to x. We prove the following generalization.

Theorem. Let λ and μ be fixed positive irrationals. The equation $[n\lambda] = [m\mu]$ has no solution in integers m and n if and only if $\mu\lambda = b\mu + c\lambda$ for some integers b and c such that $\lambda > b > 0$.

Proof. Let $\boldsymbol{Z}$ denote the set of integers. Suppose first that $\mu\lambda = b\mu + c\lambda$, where $b,c \in \boldsymbol{Z}$, $\lambda > b > 0$. Assume (for the purpose of contradiction) that

$$[n\lambda] = [m\mu] \tag{1}$$

for some $m,n \in \boldsymbol{Z}$. Write $\theta = \mu/\lambda$, $\epsilon = m\theta - [m\theta]$. Since $\mu = b\theta + c$, θ is irrational and thus $0 < \epsilon < 1$. By (1), $n\lambda = m\mu + \sigma$, where $-1 < \sigma < 1$. Thus $n = m\theta + \sigma/\lambda = [m\theta] + (\epsilon + \sigma/\lambda)$. Since $\lambda > 1$, $-1 < (\epsilon + \sigma/\lambda) < 2$. Therefore, $n = [m\theta] + \delta$, where $\delta = 0$ or 1.

We have

$$m\mu = mb\theta + mc = b\epsilon + b[m\theta] + mc. \tag{2}$$

Hence,

$$[m\mu] = [b\epsilon] + b[m\theta] + mc. \tag{3}$$

We have, using (2),

$$\begin{aligned}[n\lambda] &= [(m\theta + \delta - \epsilon)\lambda] = [m\mu + (\delta - \epsilon)\lambda] = [b\epsilon + b[m\theta] + mc + (\delta - \epsilon)\lambda] \\ &= [b\epsilon + (\delta - \epsilon)\lambda] + b[m\theta] + mc.\end{aligned} \tag{4}$$

Since the left sides of (3) and (4) are equal,

$$[b\epsilon] = [b\epsilon + (\delta - \epsilon)\lambda].$$

If $\delta = 0$, then $[b\epsilon] = [(b - \lambda)\epsilon]$, a contradiction, since $b\epsilon > 0$ and $(b - \lambda)\epsilon < 0$. If $\delta = 1$, then

$$b > [b\epsilon] = [b\epsilon + (1 - \epsilon)\lambda] \geqslant [b\epsilon + (1 - \epsilon)b] = b,$$

a contradiction. This proves that there are no integers m,n for which (1) holds.

To prove the converse, it suffices to show that (1) has a solution in each of the following three cases. Case 1: μ, θ, and 1 are linearly independent over the rationals, i.e., if $a\mu\lambda = b\mu + c\lambda$ with $a,b,c \in \boldsymbol{Z}$, then $a = b = c = 0$; Case 2: $a\mu\lambda = b\mu + c\lambda$, where a, b, and c are relatively prime integers, $a \geqslant 0$, and $a \neq 1$; Case 3: $\mu\lambda = b\mu + c\lambda$, where $b,c \in \boldsymbol{Z}$ and either $b < 0$ or $\lambda < b$.

Case 1. By Kronecker's Theorem [2, p. 382], there exist $m, z_1, z_2 \in \boldsymbol{Z}$ such that

$$m\mu = 1/2 + z_1 + E_1$$

and

$$m\theta = 1/3(1 + \lambda) + z_2 + E_2,$$

where $|E_i| < 1/6(1 + \lambda)$ for $i = 1,2$. Then

$$\epsilon = m\theta - [m\theta] = 1/3(\lambda + 1) + E_2$$

and

$$m\mu - \epsilon\lambda = (1/2 - \lambda/3(\lambda + 1)) + z_1 + (E_1 - \lambda E_2).$$

Since $|E_1 - \lambda E_2| < 1/6 < 1/2 - \lambda/3(\lambda + 1)$, we have $[m\mu - \epsilon\lambda] = z_1$. Since $[m\mu] = z_1$, we have

$$[m\mu] = [m\mu - \epsilon\lambda] = [(m\theta - \epsilon)\lambda] = [[m\theta]\lambda],$$

so that Eq. (1) has a solution with $n = [m\theta]$.

Case 2. If $a = 0$, then (1) has the solution $m = b$, $n = -c$. Thus assume $a \geqslant 2$. Since $(a,b,c) = 1$, either $a \nmid b$ or $a \nmid c$. Without loss of generality, we assume $a \nmid b$. Since $\mu = b\theta/a + c/a$, θ is irrational. Thus there exist $p,q \in \mathbf{Z}$ such that $p\theta = \eta + q + E$, where $\eta = 1/a + 1/2a(a\lambda + |b|)$ and $|E| < \eta - 1/a$. Let $m = ap$ and $\epsilon = m\theta - [m\theta]$. Then

$$m\theta = (aq + 1) + (a\eta - 1) + aE,$$

so that

(5) $$[m\theta] = aq + 1.$$

Also, $\epsilon = (a\eta - 1) + aE$, so that

(6) $$0 < \epsilon < 2(a\eta - 1) = 1/(a\lambda + |b|).$$

By (5),

(7) $$m\mu = mb\theta/a + mc/a = b\epsilon/a + b[m\theta]/a + mc/a = b\epsilon/a + b/a + bq + pc.$$

Thus,

(8) $$[m\mu] = [b\epsilon/a + b/a] + bq + pc.$$

Since $b \nmid a$ and since $|b\epsilon/a| < 1/a$ by (6), it follows from (8) that

(9) $$[m\mu] = [b/a] + bq + pc.$$

By (7),

$$m\mu - \epsilon\lambda = (b - a\lambda)\epsilon/a + b/a + bq + pc,$$

so that

(10) $$[m\mu - \epsilon\lambda] = [(b - a\lambda)\epsilon/a + b/a] + bq + pc.$$

Since $|(b - a\lambda)\epsilon/a| < 1/a$ by (6), it follows from (10) that

(11) $$[m\mu - \epsilon\lambda] = [b/a] + bq + pc.$$

By (9) and (11),

$$[m\mu] = [m\mu - \epsilon\lambda] = [(m\theta - \epsilon)\lambda] = [[m\theta]\lambda].$$

Thus (1) has a solution with $n = [m\theta]$.

Case 3. We argue as in Case 2 with $a = 1$. By (8) with $a = 1$,

(12) $$[m\mu] = [b\epsilon] + b + bq + pc.$$

By (10) with $a = 1$,

(13) $$[m\mu - \epsilon\lambda] = [(b - \lambda)\epsilon] + b + bq + pc.$$

By (6), with $a = 1$, $0 < \epsilon < 1/(\lambda + |b|)$. Thus $|b\epsilon| < 1$ and $|(b - \lambda)\epsilon| < 1$. Moreover, by the hypotheses of Case 3, $b\epsilon$ and $(b - \lambda)\epsilon$ have the same sign. Thus, by (12) and (13),

$$[m\mu] = [m\mu - \epsilon\lambda] = [(m\theta - \epsilon)\lambda] = [[m\theta]\lambda].$$

Therefore (1) has a solution with $n = [m\theta]$. Q.E.D.

Corollary 1. Let λ be a positive irrational. Then $[n\lambda] = [m\lambda^2]$ has no solution with $n,m \in \mathbf{Z}$ if and only if $\lambda = (b + (b^2 + 4c)^{1/2})/2$ for some positive integers b and c.

Proof. Note that if $\mu\lambda = b\mu + c\lambda$ with $b,c \in \mathbf{Z}$ and $\lambda > b > 0$, then $(\lambda - b)(\mu - c) = bc$, so that $c > 0$. Hence Corollary 1 follows from the Theorem with $\mu = \lambda^2$. Q.E.D.

Corollary 2. Let λ be a positive irrational. Then $[n\lambda] = [m\lambda] + m$ has no solution with $n,m \in \mathbf{Z}$ if and only if

$$\lambda = ((b + c - 1) + ((b + c - 1)^2 + 4b)^{1/2})/2$$

for some positive integers b and c.

Proof. This follows from the Theorem with $\mu = \lambda + 1$.

Corollary 3. Let σ be a positive irrational. Then $[n\sigma] + n = [m/\sigma] + m$ has no solution with $n,m \in \mathbf{Z}$.

Proof. This follows from the Theorem with $\mu = 1 + 1/\sigma$, $\lambda = \sigma + 1$, and $b = c = 1$. Q.E.D.
(Corollary 3 is part of Problem 22 in [3, p. 84].)

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[Continued from page 149.]

For an ω-series with an arbitrary odd number of k_i parameters two cycles of parametric incrementation are required to bring the series into alignment for grouping. Use of the identity

$$G(z) = \psi(z/2 + 1/2) - \psi(z/2),$$

[4, p. 20], and Lemma 1 render the following summation expression.

Theorem 2.

$$\omega(j; k_1, \cdots, k_{2n+1}) = \sum_{i=0}^{2n} (-1)^i \omega(j + s_i; S) = (1/2S) \sum_{i=0}^{2n} (-1)^i G((j + s_i)/S).$$

3. EXAMPLES

Some calculations for the uniparameter ω-series are to be found in [1] and for the biparameter series in [2]. The above theorems and their proofs can be illustrated with the following triparameter ω-series:

$$\begin{aligned}
\omega(1; 1, 1, 2) &= [(1 - 1/2) + (1/3 - 1/5) + (1/6 - 1/7)] + [(1/9 - 1/10) + (1/11 - 1/13) + \cdots] \\
&\qquad + [(1/17 - 1/18) + \cdots] + \cdots \\
&= (1 - 1/2) + (1/9 - 1/10) + (1/17 - 1/18) + \cdots + (1/3 - 1/5) + (1/11 - 1/13) + \cdots \\
&\qquad + (1/6 - 1/7) + \cdots \\
&= \omega(1; 1, 7) + \omega(3; 2, 6) + \omega(6; 1, 7) \\
&= (1/8)[G(3/4) - G(1/2) + G(1/4)] \\
&= (1/8)[\sqrt{2}(\pi - 2\ln(1 + \sqrt{2}) - \pi + \sqrt{2}(\pi + 2\ln(1 + \sqrt{2}))] \\
&= (\pi/8)[2\sqrt{2} - 1].
\end{aligned}$$

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THE FIBONACCI SERIES AND THE PERIODIC TABLE OF ELEMENTS

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The ratio, 0.6180, of the short, *a*, to the long, *b*, part of a line, divided so that $a/b = b/a + b$, is common in nature; often is called the "golden mean." An analogous line for the chemical elements is the distance between the centers of atoms in a compound. The alkali metal halide salts, which form from atoms at the extremes of reactivity in each series of the periodic table, should serve as reference compounds. If the short, *a*, part of the line is the covalent radius of the halogen atom, *X*, and the long part, *b*, is the corresponding radius of the alkali metal atom, *M*, in the same series, the mean of the ratio, *X/M*, for the five series is 0.605 ± 0.043, an understandable variation of 7% within itself as Table 1 shows. This mean is within 2.2% of the golden mean and possibly should be the same within experimental error. Only the covalent radii (see the Table) give this result. Calculations based on the ionic radii show a ratio as high as 2.27 and a 36% decrease from the first to the fourth series. Data are lacking for calculations based on the atomic radii, but in the present case the atoms in a compound, not the separate atoms, are under consideration.

Table 1
Covalent Radii of the Halide Salts and Calculation of the Factor, *R*, in the Fibonacci Equation

Column 1	2		3	4		5	
Halide pair	Observed* covalent radus, Å		Ratio *X/M*	Summation of observed radii		Correction Factor, *R*, Ratio Obs./Sum.	
	X	*M*		*X*	*M*	*X*	*M*
FLi	0.72	1.23	0.585				
ClNa	0.99	1.54	0.643	2.26	1.95	0.44	0.79
BrK	1.14	2.03	0.562	3.02	2.53	0.38	0.80
IRb	1.33	2.16	0.616	3.30	3.17	0.40	0.68
AtCs	1.45	2.35	0.618	3.68	3.49	0.39	0.67
?Fr					3.80		
Avg. or Min.		Avg.	0.605		Min.	0.39	0.67
Theory			0.618				
	Calculated†						
? Fr	1,56	2,55	0.610				

*Data are from the Sargent-Welch Company table commonly used by students.
†Calculated for the unknown francium halide as described in the text.

To approximate the position of the periodic table in the Fibonacci series we first use the lengths in angstrom units, Å, of the lithium and fluorine covalent radii for the construction in Fig. 1 of the smallest rectangle with dimensions of $a + b$ by *b*. From that rectangle, and the *b* by *b* square, larger and still larger rectangles and squares can be constructed in the usual manner [1, 3]. The centers of each square are marked with an alkali metal in increasing order, Li to Fr. Thus the pattern for a Fibonacci series is evident. A curved line, rather than the straight lines shown, could connect the centers between which the symbols for the other elements could be written (not done here for lack of space). In that way the periodic table would appear as a spiral, analogous to other spirals [1], the galaxies, the whirls in some flowers and plants, the horns of some animals, and the spirals in shells, all called the "golden horn." A simple calculation back to zero angstrom units suggests that the first series in the periodic table is roughly at the ninth number in the Fi bonacci series.

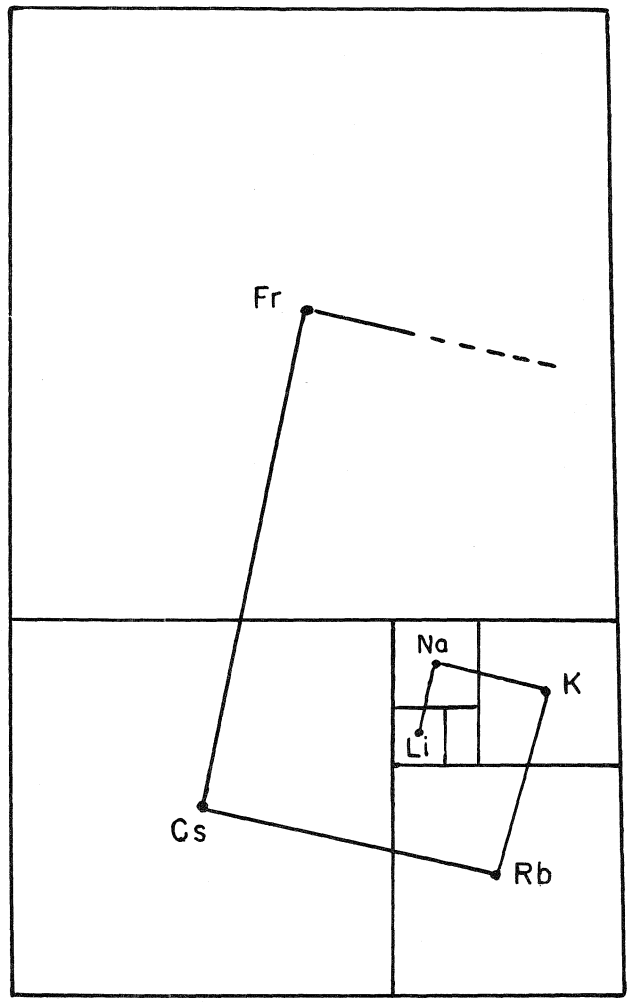

Fig. 1 Geometrical Arrangement According to the Fibonacci Pattern of the Five-Plus Series in the Periodic Table

The numbers in Column 2 of the Table, however, do not follow exactly the simple Fibonacci way where each succeeding number is the sum of the two preceding ones as in

$$U_n = U_{n-1} + U_{n-2}.$$

This situation is seen in Column 4, where the summation for chloride is 2.26 instead of the observed 0.99; and that for sodium is 1.95 instead of 1.54. Such abnormality probably results from the fact that the line (the sum of the two radii) does not pass through uniformly similar territory, for the specific volume of the halogen is much less and the atomic weight is much more than for the metal of each pair. To compensate for this situation the ratios of the observed to the summation for each radius are recorded in Column 5 for the X and M component of each pair. These values appear to attain minima-- 0.39 for the halogen and 0.67 for the metal. In other words in the formula $U_n = R(U_{n-1} + U_{n-2})$ the value of R is 0.39 when U_n is a halogen and 0.67 when a metal.

With these ratios, the value for the unknown halogen which would be paired with francium can be estimated as 1.56, and for francium would be 2.55. Then the ratio, X/M for that undiscovered salt would be 0.610, within 1.3% of the golden mean.

Whether that unknown halogen will ever be prepared may be doubted. Its atomic number would be 117 if the number of elements in the sixth series is the same as in the fifth. Wlodorski [4] has used the Fibonacci series to estimate the limiting stability of the nucleus in the transuranium elements and has concluded that efforts [5] to extend the series above number 114 cannot succeed. No objection to that prediction is here intended. However, attention should be drawn to a recent paper by Anders and co-workers [6] about the possibility of elements 115 (or 114, 113) having been found in a meterorite.

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ON FIBONACCI AND TRIANGULAR NUMBERS

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The infinite sequence discovered by the author in [1], namely the numerators of C_k, i.e.,

(1) $$F_{2k}C_k = (1 + L_k + F_{2k-1})$$

are related to the Triangular numbers $\{T_n\}$, where $T_{-1} = 0 = T_0$ and

(2) $$T_n = n(n+1)/2 \quad \text{for all integral } n,$$

in general. It is interesting that four members of the sequence defined by T_{-1+F_n} are zero, namely those for $n = -1, 0, 1, 2$. It will be shown that

(3) $$F_{2k}C_k = T_{1+F_{k+1}} + T_{-1+F_{k-2}}$$

for all natural numbers k. The first term on the right-hand side merely picks off the 2, 3, 4, 6, 9th ··· terms of $\{T_n\}$.

Proof. The proof is direct and easy considering that (3) is not obvious. We first need

(4) $$3F_{k+1} - F_{k-2} = 2L_k$$

which is easily derived from $F_{k+1} + F_{k-1} = L_k$. Next we need

$$F_{k+1}^2 = F_{2k} + F_{k-1}^2 \qquad \text{and} \qquad F_{k-1}^2 = F_{2k-3} - F_{k-2}^2$$

which are (I_{10}) and (I_{11}) of Hoggatt [2] which enables us to write

(5) $$F_{k+1}^2 + F_{k-2}^2 = 2F_{2k-1}\,.$$

First we write

$$2T_{1+F_{k+1}} + 2T_{-1+F_{k-2}} = (1+F_{k+1})(2+F_{k+1}) + (-1+F_{k-2})F_{k-2} = 2 + 3F_{k+1} + F_{k+1}^2 + F_{k-2}^2 - F_{k-2}$$

which via (4) and (5)

$$= 2 + 2L_k + 2F_{2k-1}$$

as was to be shown.

Table of $C_k F_{2k}$ Numbers and Triangular Numbers

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$C_k F_{2k}$	4	3	6	10	21	46	108	263	658	1674	4305	11146	28980
$T_{1+T_{k+1}}$	3	3	6	10	21	45	105	253	630	1596	4095	10585	27495
$T_{-1+F_{k-2}}$	1	0	0	0	0	1	3	10	28	78	210	561	1485

Now it would be nice if a generalization obtained for the generalized $C_{j,k}$ in the author's second paper on sums of Fibonacci reciprocals [3]. Such *is* the case. First we must define generalized Triangular numbers

(6) $$T_{n,j} = n(n+j)/2$$

which may *not* always be integers. Let $\{P_n\}$ be any generalized sequence such that

(7) $$P_{n+1} = jP_n + P_{n-1}\,,$$

where j is an integer; then using the general Binet formula one can show that

(8) $$P_{2n+1} = P_{n+1}^2 + P_n^2$$

and it definitely is equally obvious that we can show

(9) $$jP_{2n} = P_{n+1}^2 - P_{n-1}^2 \ .$$

Using (8) and (9), we may show that

(10) $$P_{k+1}^2 + P_{k-2}^2 = jP_{2k} + P_{2k-3} = (j^2+1)P_{2k-1}$$

which corresponds to (5) in the Fibonacci case. Now the author [3, (9)] has shown that the numerators of C_{jk} are

(11) $$P_{2k}C_{j,k} = (1 + P_k^* + P_{2k-1}) \ .$$

The j subscript has been dropped from the P's for neatness but they are still a function of j and ideally we should write $P_{j,k}$.

Theorem.

(12) $$(1 + P_k^* + P_{2k-1}) = (1 + 2T_{P_k,j} + 2T_{P_{k-2},2}).$$

The proof is straightforward and note that $P_k^* = P_{k+1} + P_{k-1}$ is by definition the Lucas complement of P_k. From (6) Eq. (12) becomes

(13) $$(1 + P_k(P_k + j) + P_{k-1}(P_{k-1} + 2) = (1 + jP_k + 2P_{k-1} + P_k^2 + P_{k-1}^2) = (1 + P_{k+1} + P_{k-1} + P_{2k-1})$$

by using (8). Note that we did not use (9) and that has led to (12) being different from (3). I illustrate this by taking $C_{3,4} = 1309/3927$. Now $\{P_{3,k}\}$ is 0, 1, 3, 10, 33, 109, 360, 1189, $\cdots$. According to (11) and (12) the numerator of $C_{3,4}$ is $1 + 33(33+3) + 10(10+2) = 1309$ as it should. In (12) be careful to note that j and 2 are subscripts of T and not of P.

H. W. Gould has called my attention to a known theorem [4] that an integer m is the sum of two triangular numbers if and only if $4m + 1$ is the sum of two squares, say $4m + 1 = u^2 + v^2$, where $(u - v) \geqslant 3$. Hence for the sequence $G_k = C_k F_{2k}$ we have the following table.

k	$(1 + 4C_kF_{2k}) = (u^2 + v^2)$	k	$(1 + 4C_kF_{2k}) = (u^2 + v^2)$
0	$17 = 4^2 + 1^2$	5	$185 = 8^2 + 11^2$
1	$13 = 2^2 + 3^2$	6	$433 = 12^2 + 17^2$
2	$25 = 4^2 + 3^2$	7	$1053 = 18^2 + 29^2$
3	$41 = 4^2 + 5^2$	8	$2633 = 28^2 + 43^2$
4	$85 = 6^2 + 7^2$	9	$6697 = 44^2 + 69^2$

We noticed that the differences between adjacent u numbers seems to be twice the Fibonacci numbers and that a similar relation holds for the v numbers. V. E. Hoggatt, Jr., in a letter dated Jan. 22, 1977, has found the following closed form.

(14) $$1 + 4G_k = 1 + 4C_kF_{2k} = (2(1 + F_{k-1}))^2 + (1 + 2F_k)^2 = u^2 + v^2 \ .$$

Now Sloane [5] contains the sequence $N^2 + (N-1)^2$, his No. 1567, which generates a lot of primes. The sequence above may also be prime rich since 17, 13, 41, 433, 2633 are primes. Also G numbers for negative k values may be found in the recently submitted [6]. Then the sequence $(1 + 4G_{-k})$ for $k = 0, 1, 2, \cdots$ gives: 17, 9, 37, 41, 169, 317, 1009, 2329, 6581, $\cdots$ all of which are primes but 2329 and the perfect squares 9 and 169.

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THE PERIODIC GENERATING SEQUENCE

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Given an integer sequence $S = \{a_1, a_2, \cdots\}$, $a_i > 0$. Form a new sequence $\{r_n\}$ by first choosing two integers r_{-1} and r_0, then setting

$$r_m = r_{m-1}a_m + r_{m-2}, \quad a_m \in S.$$

We call S a *Generating Sequence.*

Notice that for each $r_k \in \{r_n\}$, we can reduce r_k to $r_k = A(k)r_0 + B(k)r_{-1}$, where $A(k)$ and $B(k)$ are integers. Hence $\{r_0, r_{-1}\}$ can be viewed as a "basis" for $\{r_n\}$. Then,

$$r_{-1} = A(-1)r_0 + B(-1)r_{-1} \Rightarrow A(-1) = 0, \qquad B(-1) = 1,$$

$$r_0 = A(0)r_0 + B(0)r_{-1} \Rightarrow A(0) = 1, \qquad B(0) = 0.$$

Theorem 1. Suppose two sequences $\{r'_n\}$ and $\{r''_n\}$ are generated from the same sequence with different choices of r'_{-1}, r'_0 and r''_{-1}, r''_0, then

$$\begin{vmatrix} r'_{k-1} & r'_k \\ r''_{k-1} & r''_k \end{vmatrix} = (-1)^k \begin{vmatrix} r'_{-1} & r'_0 \\ r''_{-1} & r''_0 \end{vmatrix}.$$

Proof. By induction.

Notation: Let

$$L = \begin{bmatrix} A(k) & B(k) \\ A(k-1) & B(k-1) \end{bmatrix}.$$

Notice that

$$\begin{bmatrix} r_k \\ r_{k-1} \end{bmatrix} = L \begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}$$

Lemma. $\det(L) = (-1)^k$.

Proof.

$$\begin{vmatrix} r'_{k-1} & r'_k \\ r''_{k-1} & r''_k \end{vmatrix} = \begin{vmatrix} A(k-1)r'_0 + B(k-1)r'_{-1} & A(k)r'_0 + B(k)r'_{-1} \\ A(k-1)r''_0 + B(k-1)r''_{-1} & A(k)r''_0 + B(k)r''_{-1} \end{vmatrix}$$

$$= \{A(k)B(k-1) - A(k-1)B(k)\} \begin{vmatrix} r'_{-1} & r'_0 \\ r''_{-1} & r''_0 \end{vmatrix}$$

$$= \det(L) \begin{vmatrix} r'_{-1} & r'_0 \\ r''_{-1} & r''_0 \end{vmatrix}$$

$$\Rightarrow \det(L) = (-1)^k.$$

Theorem 2. Let

$$S = \{a_1, a_2, \cdots\}$$

be the generating sequence for $\{r_n\}$, then

$$A(m) = A(m-1)a_m + A(m-2)$$

$$B(m) = B(m-1)a_m + B(m-2), \quad a_m \in S.$$

Proof. We have

$$r_m = r_{m-1}a_m + r_{m-2} \Rightarrow [A(m)B(m)]\begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix} = [A(m-1)B(m-1)]\begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix} a_m + [A(m-2)B(m-2)]\begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}$$

$$\Rightarrow [A(m)B(m)] = [A(m-1)a_m + A(m-2)B(m-1)a_m + B(m-2)].$$

Remark: The above theorem shows that $\{A(n)\}$ and $\{B(n)\}$ are also sequences generated by S. Recall that

$$A(-1) = 0, \qquad A(0) = 1; \qquad B(-1) = 1, \qquad B(0) = 0.$$

We shall now investigate what happens when the generating sequence is an infinite periodic sequence

$$P = \{\overline{a_1, \cdots, a_k}\}.$$

We will let k be the period of P for the rest of our work.

Theorem 3. If $\{r_n\}$ is generated from P, then

$$[A(nk+u)B(nk+u)] = [A(u)B(u)]L^n.$$

Proof. Recall

$$L = \begin{bmatrix} A(k) & B(k) \\ A(k-1) & B(k-1) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} r_k \\ r_{k-1} \end{bmatrix} = L\begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}.$$

Then

$$r_u = [A(u)B(u)]\begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}$$

$$r_{k+u} = [A(u)B(u)]\begin{bmatrix} r_k \\ r_{k-1} \end{bmatrix} = [A(u)B(u)]L\begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}$$

$$r_{2k+u} = [A(u)B(u)]\begin{bmatrix} r_{2k} \\ r_{2k-1} \end{bmatrix} = [A(u)B(u)]L\begin{bmatrix} r_k \\ r_{k-1} \end{bmatrix}$$

$$= [A(u)B(u)]L^2\begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}.$$

It is easy to see that

$$r_{nk+u} = [A(u)B(u)]L^n\begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix} \Rightarrow [A(nk+u)B(nk+u)]\begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix} = [A(u)B(u)]L^n\begin{bmatrix} r_0 \\ r_{-1} \end{bmatrix}$$

$$\Rightarrow [A(nk+u)B(nk+u)] = [A(u)B(u)]L^n.$$

Corollary.

$$\begin{vmatrix} A(nk+u) & B(nk+u) \\ A(nk+v) & B(nk+v) \end{vmatrix} = (-1)^{nk}\begin{vmatrix} A(u) & B(u) \\ A(v) & B(v) \end{vmatrix}$$

Proof. By Theorem 3, we get

$$\begin{bmatrix} A(nk+u) & B(nk+u) \\ A(nk+v) & B(nk+v) \end{bmatrix} = \begin{bmatrix} A(u) & B(u) \\ A(v) & B(v) \end{bmatrix}L^n \Rightarrow \begin{vmatrix} A(nk+u) & B(nk+u) \\ A(nk+v) & B(nk+v) \end{vmatrix} = \begin{vmatrix} A(u) & B(u) \\ A(v) & B(v) \end{vmatrix} \det(L^n).$$

Theorem 4. If a sequence $\{r_n\}$ is generated from an infinite periodic sequence P with period k, then

$$r_{n+2k} - C(k)r_{n+k} + (-1)^k r_n = 0,$$

where $C(k)$ is a positive integer independent of the choice of r_{-1} and r_0.

Proof. Consider

$$r_{n+2k} + xr_{n+k} + yr_n = 0.$$

Assume the theorem is true except for the existence of x and y. We have

$$r_{n+2k} + xr_{n+k} + yr_n = 0 \Rightarrow \{[A(n+2k)B(n+2k)] + x[A(n+k)B(n+k)] + y[A(n)B(n)]\}\begin{bmatrix} r_0 \\ r_{-1}\end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} A(n+2k) + xA(n+k) + yA(n) = 0 \\ B(n+2k) + xB(n+k) + yB(n) = 0 \end{cases}.$$

These are solvable iff

$$D = \begin{vmatrix} A(n+k) & B(n+k) \\ A(n) & B(n) \end{vmatrix} \neq 0.$$

Then by Theorem 3,

$$[A(n+k)B(n+k)] = [A(n)B(n)]L = [A(n)A(k) + A(k-1)B(n)A(n)B(k) + B(n)B(k-1)]$$

$$\Rightarrow D = \begin{vmatrix} A(n+k) & B(n+k) \\ A(n) & B(n) \end{vmatrix}$$

$$= A(n)A(k)B(n) + A(k-1)B(n)^2 - A(n)^2B(k) - A(n)B(n)B(k-1).$$

The only possibilities for making D vanish are either $n = k-1$ or $n = k$.

When $n = k-1$,

$$D = A(k)A(k-1)B(k-1) - A(k-1)^2B(k) = A(k-1)\det(L) \neq 0.$$

When $n = k$,

$$D = A(k-1)B(k)^2 - A(k)B(k)B(k-1) = -B(k)\det(L) \neq 0.$$

Hence x and y exist. Then let $n = 0$, we have

$$A(2k) + xA(k) + yA(0) = 0, \qquad B(2k) + xB(k) + yB(0) = 0.$$

Since $A(0) = 1$, $B(0) = 0$, we get

$$x = -B(2k)/B(k), \qquad y = A(k)[B(2k)/B(k)] - A(2k).$$

By Theorem 3, we obtain

$$[A(2k)B(2k)] = [A(0)B(0)]L^2 = [1\ 0]L^2 = [A(k)^2 + A(k-1)B(k)A(k)B(k) + B(k)B(k-1)].$$

Thus

$$x = -B(2k)/B(k) = -(A(k) + B(k-1)) \Rightarrow C(k) = A(k) + B(k-1)$$

$$y = A(k)[A(k) + B(k-1)] - [A(k)^2 + A(k-1)B(k)]$$

$$= A(k)B(k-1) - A(k-1)B(k) = \det(L) = (-1)^k.$$

<u>Remark</u>. Since $\{A(n)\}$ and $\{B(n)\}$ are also generated from P, then

$$A(n+2k) - C(k)A(n+k) + (-1)^kA(n) = 0 \quad \text{and} \quad B(n+2k) - C(k)B(n+k) + (-1)^kB(n) = 0.$$

By Theorem 3, this leads us to

$$[A(n)B(n)]\{L^2 - C(k)L + (-1)^kI\} = 0 \Rightarrow L^2 - C(k)L + \det(L)I = 0,$$

I is the identity matrix.

What happens when $P = \{\bar{a}\}$ since k can be chosen as large as one desires?

Theorem 5. Suppose $\{r_n\}$ is generated from $P = \{\bar{a}\}$ such that

$$r_{n+2k} - C(k)r_{n+k} + (-1)^kr_n = 0.$$

Then $\{C(n)\}$ is also a sequence generated from P with $C(0) = 2$, $C(-1) = -a$.

Proof. Recall $C(k) = A(k) + B(k-1)$. Then

$$C(k) - C(k-1)a - C(k-2) = \{A(k) - A(k-1)a - A(k-2)\} - \{B(k-1) - B(k-2)a - B(k-3)\}$$

$$= 0 \Rightarrow C(k) = C(k-1)a + C(k-2).$$

Also,

$$C(0) = A(0) + B(-1) = 2, \qquad C(1) = A(1) + B(0) = a.$$

But then

$$C(1) = C(0)a + C(-1) \Rightarrow C(-1) = -a.$$

Remark. Since $\{C(n)\}$ is generated from $P=\{\overline{a}\}$, there exists another sequence $\{C'(n)\}$ such that

$$C(n+2k) - C'(k)C(n+k) + (-1)^k C(n) = 0.$$

Notice that $\{C'(n)\} = \{C(n)\}$. For example, when $P=\{\overline{1}\}$, then

$$\{A(n)\} = \{f_{n+1}\}$$

and $\{B(n)\} = \{f_n\}$, $C(n) = f_{n+1} + f_{n-1}$, $\{f_n\}$ is the Fibonacci sequence. Remember

$$A(n+2k) - C(k)A(n+k) + (-1)^k A(n) = 0 \Rightarrow f_{n+2k+1} - (f_{k+1} + f_{k-1})f_{n+k+1} + (-1)^k f_{n+1} = 0$$

and

$$B(n+2k) - C(k)B(n+k) + (-1)^k B(n) = 0 \Rightarrow f_{n+2k} - (f_{k+1} + f_{k-1})f_{n+k} + (-1)^k f_n = 0.$$

Also from Theorem 5 and the last remark,

$$C(n+2k) - C'(k)C(n+k) + (-1)^k C(n) = 0 \Rightarrow \{f_{n+2k+1} + f_{n+2k-1}\} - (f_{k+1} + f_{k-1})\{f_{n+k+1} + f_{n+k-1}\} + (-1)^k\{f_{n+1} + f_{n-1}\} = 0.$$

Theorem 6. Suppose $\{r_n\}$ is generated from $P=\{\overline{a}\}$, then there exist x and y such that $u \geqslant s > t \geqslant 0$,

$$r_{n+u} + xr_{n+s} + yr_{n+t} = 0,$$

x and y rational.

Proof. Think of n as k since the periodicity can vary.

Then follow the proof for Theorem 4. Carrying out the proof, we also find that

$$x = -\frac{\begin{vmatrix} A(u) & B(u) \\ A(t) & B(t) \end{vmatrix}}{\begin{vmatrix} A(s) & B(s) \\ A(t) & B(t) \end{vmatrix}}, \qquad y = -\frac{\begin{vmatrix} A(s) & B(s) \\ A(u) & B(u) \end{vmatrix}}{\begin{vmatrix} A(s) & B(s) \\ A(t) & B(t) \end{vmatrix}}.$$

In particular, when $P=\{\overline{1}\}$, we get

$$f_{n+u} - \frac{\begin{vmatrix} f_{u+1} & f_u \\ f_{t+1} & f_t \end{vmatrix}}{\begin{vmatrix} f_{s+1} & f_s \\ f_{t+1} & f_t \end{vmatrix}} f_{n+s} - \frac{\begin{vmatrix} f_{s+1} & f_s \\ f_{u+1} & f_u \end{vmatrix}}{\begin{vmatrix} f_{s+1} & f_s \\ f_{t+1} & f_t \end{vmatrix}} f_{n+t} = 0.$$

For example, when $u=9$, $s=6$ and $t=2$,

$$f_{n+9} - (13/3)f_{n+6} + (2/3)f_{n+2} = 0.$$

We are going to relate some of the above results to Continued Fractions.

A simple purely periodic continued fraction is denoted by $c = [\overline{a_1, \cdots, a_k}]$. If we take $P = \{\overline{a_1, \cdots, a_k}\}$, then immediately we see that $A(n)/B(n)$ is the nth convergent of c. We also know that

$$A(n+2k) - C(k)A(n+k) + (-1)^k A(n) = 0 \quad \text{and} \quad B(n+2k) - C(k)B(n+k) + (-1)^k B(n) = 0.$$

If we regard these as second-order difference equations, then the auxiliary quadratic equation for them is

$$x^2 - C(k)x + (-1)^k = 0$$

and

$$x = \{C(k) \pm \sqrt{C(k)^2 - 4(-1)^k}\}/2, \quad C(k)^2 - 4(-1)^k > 0.$$

Let m_1, m_2 be the distinct zeros such that $|m_1| > |m_2|$, then $A(nk+u) = \alpha_1 m_1^n + \beta_1 m_2^n$,

$$B(nk+u) = \alpha_2 m_1^n + \beta_2 m_2^n, \quad u < k.$$

By choosing the appropriate initial conditions for $\{A(n)\}$ and $\{B(n)\}$, respectively, we can solve for α_1, β_1 and α_2, β_2. One can take $A(u)$, $A(k+u)$ to be the initial conditions for $\{A(n)\}$ and $B(u)$, $B(k+u)$ for $\{B(n)\}$. Then the $(nk+u)^{th}$ convergent of c is given by

$$\frac{A(nk+u)}{B(nk+u)} = \frac{\alpha_1 + \beta_1(m_2/m_1)^n}{\alpha_2 + \beta_2(m_2/m_1)^n} .$$

Hence limit of

$$c = \lim_{n\to\infty} \{A(nk+u)/B(nk+u)\} = \alpha_1/\alpha_2 .$$

Notice that α_1 and α_2 are quadratic irrationals. Is the limit unique? Yes, by Theorem 3, we have

$$\begin{vmatrix} A(nk+u) & B(nk+u) \\ A(nk+v) & B(nk+v) \end{vmatrix} = \det(L^n) \begin{vmatrix} A(u) & B(u) \\ A(v) & B(v) \end{vmatrix} = \pm\sigma,$$

σ is a constant. Then

$$\frac{A(nk+u)}{B(nk+u)} - \frac{A(nk+v)}{B(nk+v)} = \frac{\pm\sigma}{B(nk+u)B(nk+v)}$$

As $n \to \infty$,

$$\frac{A(nk+u)}{B(nk+u)} - \frac{A(nk+v)}{B(nk+v)} = 0.$$

If $c = [a_1, \cdots, a_j, \overline{a_{j+1}, \cdots, a_{j+k}}]$, then take

$$P = \{a_1, \cdots, a_j, \overline{a_{j+1}, \cdots, a_{j+k}}\}$$

as the generating sequence, the limit of c is then given by

$$\lim_{n\to\infty} \frac{A(nk+u+j)}{B(nk+u+j)}, \quad u > 0.$$

Remark. Actually we have proved just now a theorem in continued fractions: A continued fraction c is peridic iff α is a quadratic irrational, for which c is the continued fraction expansion.

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ADDITIVE PARTITIONS II

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Theorem (Hoggatt). The Tribonacci Numbers,

$$1, 2, 4, 7, 13, 24, \cdots, T_{n+3} = T_{n+2} + T_{n+1} + T_n ,$$

with 3 added to the set uniquely *split* the positive integers and each positive integer $n \neq 3$ or $\neq T_m$ is the sum of two elements of A_0 or two elements of A_1. (See "Additive Partitions I," page 166.)

Conjecture. Let A split the positive integers into two sets A_0 and A_1 and be such that $p \notin A \cup \{1,2\}$, and p is representable as the sum of two elements of A_0 or the sum of two elements of A_1. We call such a set *saturated* (that is $A \cup \{1, 2\}$). Krishnaswami Alladi asks: "Does a *saturated* set imply a unique *additive* partition?" My conjecture is that the set $\{1, 2, 3, 4, 8, 13, 24, \cdots\}$ is *saturated* but does not cause a unique split of the positive integers. Here we have added 3 and 8 to the Tribonacci sequence and deleted the 7. *Paul Bruckman points out that this fails for 41. EDITOR*

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A RELATIONSHIP BETWEEN PASCAL'S TRIANGLE AND FERMAT'S NUMBERS

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There are many relations known among the entries of Pascal's triangle. In [1], Hoggatt discusses the relation between the Fibonacci numbers and Pascal's triangle. He also gives several references to other related works.

Here, we propose to show a relation between the triangle and the Fermat numbers $f_i = 2^{2^i} + 1$ for $i = 0, 1, 2, \cdots$. Let $c(n,j)$ be Pascal's triangle, where n represents the row index and j the column index, both indices starting at zero. Let $a[n]$ be the sequence of numbers constructed from Pascal's triangle as follows: construct a new Pascal's triangle by taking the residue of $c(n,j)$ modulo base 2, then, consider each horizontal row of the new triangle as a whole number which is written in binary arithmetic. In symbols, let

$$a[n] = \sum_{j=0}^{n} c^*(n,j)2^j \qquad n = 0, 1, 2, \cdots, \tag{1}$$

where $c^*(n,j)$ is the residue modulo base 2 of $c(n,j)$. The first few terms of this sequence are 1, 3, 5, 15, 17, 51, 85, 255, 257, 771, etc., starting with $a[0]$.

Proposition. The sequence of numbers

$$a[n] = \sum_{j=0}^{n} c^*(n,j)2^j \qquad n = 0, 1, 2, \cdots,$$

constructed from Pascal's triangle, is equal to the sequence of numbers

$$b[n] = (f_k)^{\alpha_0}(f_{k-1})^{\alpha_1} \cdots (3)^{\alpha_k} \qquad n = 0, 1, 2, \cdots,$$

where $n = a_0 a_1 a_2 \cdots a_k$ in binary number expansion, and f_i are the Fermat numbers.

Proof. The proof is by induction. For the purpose of starting the induction, let us verify the relation for $a[0]$ through $a[8]$ by means of the following table:

n	n (binary)	$a[n]$ (binary)	$a[n]$ (decimal)	$b[n]$ (Fermat form)
0	000	1	1	1·1·1
1	001	11	3	1·1·3
2	010	101	5	1·5·1
3	011	1111	15	1·5·3
4	100	10001	17	17·1·1
5	101	110011	51	17·1·3
6	110	1010101	85	17·5·1
7	111	11111111	255	17·5·3
8	1000	100000001	257	257·1·1·1

To complete the induction proof, we assume the theorem is true for $n \leqslant 2^k$, and prove the theorem for the range $2^k < n \leqslant 2^{k+1}$. We are performing induction on k, and note that the table proves the induction hypothesis for $k = 2$ and 3. If n is in the range $2^k \leqslant n < 2^{k+1}$, then it has a binary expansion of the form $1a_1a_2 \cdots a_k$. Next, we observe a pattern forming in the binary construction of a_n between the levels 2^k and 2^{k+1}. For example, the above table shows the pattern above $n = 4$ being repeated, in duplicate, side by side, down to level

$n = 7$, but changing at $n = 8$. The reason that this pattern is formed is that Pascal's triangle can be constructed by addition (sums must be reduced modulo 2) with the well known formula

$$c(n-1, r-1) + c(n-1, r) = c(n,r).$$

We will now describe relationship of the numbers below level 2^k to those above 2^k. Since f_k is equal to one plus the number represented by 1 followed by 2^k zeros, we can form $a[2^k + j]$, for $j = 1, 2, \cdots, 2^{k-1}$, by multiplying $a[j]$ by f_k. This multiplication has the effect of repeating the pattern above level 2^k, side by side, down to level $2^{k+1} - 1$, which will then consist of 2^{k+1} "ones." If we now construct $a[2^{k+1}]$ using the addition method, we see that it will consist of one plus the number represented by 1 followed by 2^{k+1} zeros. Thus, we have the two relations

$$a[2^k + j] = a[j] f_k \qquad \text{for} \qquad j = 1, 2, 3, \cdots, 2^{k-1}$$

and

$$a[2^{k+1}] = f_{k+1}.$$

If we apply the induction hypothesis to $a[j]$ for $j < 2^k$, then

$$a[n] = (f_k)^1 (f_{k-1})^{\alpha_1} \cdots (3)^{\alpha_k} \qquad n < 2^{k+1},$$

where

$$n = 1\alpha_1 \cdots \alpha_k, \qquad \text{and} \qquad a[2^{k+1}] = f_{k+1}.$$

This completes the proof.

REMARK. The same proof easily covers the more general case where Pascal's triangle is computed modulo base ℓ. The resulting sequence is then compared to the Fermat numbers to the base ℓ.

REFERENCE

1. V. E. Hoggatt, Jr., "Generalized Fibonacci Numbers in Pascal's Pyramid," *The Fibonacci Quarterly,* Vol. 10, No. 3 (Oct. 1972), pp. 271–276.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-272 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{j=0}^{m} \binom{r}{j}\binom{p}{m-j}\binom{q}{m-j}\binom{p+q-m+j}{j} \equiv C_m(p, q, r)$$

is symmetric in p, q, r.

H-273 Proposed by W. G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Consider, after Hoggatt and H-257, the array D, indicated below in which L_{2n+1} $(n = 0, 1, 2, \cdots)$ is written in staggered columns

1				
4	1			
11	4	1		
29	11	4	1	
76	29	11	4	1

i. Show that the row sums are $L_{2n+2} - 2$.
ii. Show that the rising diagonal sums are $F_{2n+3} - 1$, where L_{2n+1} is the largest element in the sum.
iii. Show that if the columns are multiplied by $1, 2, 3, \cdots$ sequentially to the right then the row sums are $L_{2n+3} - (2n + 3)$.

SOLUTIONS

LOOK-SERIES

H-251 Proposed by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

Prove the identity:

$$\sum_{n=0}^{\infty} \frac{x^{n^2}}{[(x)_n]^2} = \sum_{n=0}^{\infty} \frac{x^n}{(x)_n},$$

where

$$(x)_n = (1-x)(1-x^2)\cdots(1-x^n), \qquad (x)_0 = 1.$$

Solution by the Proposer.

Define $f(z,y)$ by the following:

$$f(z,y) = \prod_{r=1}^{\infty} (1 + y^{2r-1}z). \tag{1}$$

Then we may set

$$f(z,y) = \sum_{m=0}^{\infty} A_m(y)z^m,$$

also observing that $f(0,y) = 1 = A_0(y)$.

Now,

$$f(y^2z,y) = (1+yz)^{-1}f(z,y),$$

which is readily derived from the definition of $f(z,y)$, i.e.,

$$f(z,y) = (1+yz)f(y^2z,y).$$

Translating this relation into series notation, we obtain the following:

$$\sum_{m=0}^{\infty} A_m(y)z^m = \sum_{m=0}^{\infty} A_m(y)y^{2m}z^m + \sum_{m=1}^{\infty} A_{m-1}(y)y^{2m-1}z^m.$$

This yields the simple recursion:

$$(1-y^{2m})A_m(y) = y^{2m-1}A_{m-1}(y),$$

with $A_0(y) = 1$. By an easy induction, we derive the formula:

$$A_m(y) = \frac{y^{m^2}}{(y^2)_m} \qquad (m = 0, 1, 2, \cdots).$$

Hence,

$$f(z,y) = \prod_{r=1}^{\infty} (1+y^{2r-1}z) = \sum_{m=0}^{\infty} \frac{y^{m^2}}{(y^2)_m} z^m. \tag{2}$$

Similarly,

$$f(z^{-1},y) = \prod_{r=1}^{\infty} (1+y^{2r-1}z^{-1}) = \sum_{n=0}^{\infty} \frac{y^{n^2}}{(y^2)_n} z^{-n}. \tag{3}$$

We now employ the well known Jacobi identity:

$$f(z,y)\cdot f(z^{-1},y)\cdot \prod_{r=1}^{\infty} (1-y^{2r}) = \sum_{k=-\infty}^{\infty} y^{k^2}z^k. \tag{4}$$

Let $\theta(y)$ denote the coefficient of z^0 in $f(z,y)\cdot f(z^{-1},y)$. Multiplying the series in (2) and (3), we see that $\theta(y)$ is obtained by letting $m = n$; hence,

$$\theta(y) = \sum_{k=0}^{\infty} \frac{y^{2k^2}}{\{(y^2)_k\}^2}. \tag{5}$$

However, from (4),

$$\theta(y) = \prod_{r=1}^{\infty} (1-y^{2r})^{-1}.$$

Making the substitution $x = y^2$ we obtain the result:

$$\prod_{r=1}^{\infty} (1-x^r)^{-1} = \sum_{n=0}^{\infty} \frac{x^{n^2}}{\{(x)_n\}^2}. \tag{6}$$

Now the infinite product in (6) is the well known generating function for $p(n)$, the number of partitions of n; however, it is also equal to the series:

$$\sum_{n=0}^{\infty} \frac{x^n}{(x)_n}.$$

To establish this, define

$$g(z,x) = \prod_{r=1}^{\infty} (1 - zx^r)^{-1},$$

and set

$$g(z,x) = \sum_{n=0}^{\infty} B_n(x)z^n,$$

observing that $g(0,x) = 1 = B_0(x)$. By inspection of the infinite product definition of $g(z,x)$, we may obtain the relation: $g(zx,x) = (1 - zx)g(z,x)$; as before, translating this into the infinite series expansions, we obtain the recursion:

$$(1 - x^n)B_n(x) = xB_{n-1}(x), \qquad B_0(x) = 1.$$

From this, we readily establish that

$$B_n(x) = x^n/(x)_n, \qquad n = 0, 1, 2, \cdots.$$

Hence, we have derived the following:

$$\prod_{r=1}^{\infty} (1 - x^r)^{-1} = g(1,x) = \sum_{n=0}^{\infty} p(n)x^n = \sum_{n=0}^{\infty} \frac{x^n}{(x)_n} = \sum_{n=0}^{\infty} \frac{x^{n^2}}{\{(x)_n\}^2}, \tag{7}$$

for suitable region of convergence (actually, for $|x| < 1$.) This establishes the result.

Also solved by G. Lord and P. Tracy.

SUB PRODUCT

H-252 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let $A_{n\times n}$ be an $n \times n$ lower semi-matrix and $B_{n\times n}$, $C_{n\times n}$ be matrices such that $A_{n\times n}B_{n\times n} = C_{n\times n}$. Let $A_{k\times k}, B_{k\times k}, C_{k\times k}$ be the $k\times k$ upper left submatrices of $A_{n\times n}, B_{n\times n}$, and $C_{n\times n}$. Show $A_{k\times k}B_{k\times k} = C_{k\times k}$ for $k = 1, 2, \cdots, n$.

Solution by Paul S. Bruckman, University of Illinois at Chicago, Chicago Circle, Illinois.

Let a_{ij}, b_{ij} and c_{ij} denote the entries of A, B and C, respectively $(i,j = 1, 2, \cdots, n)$. By hypothesis,

$$\sum_{r=1}^{n} a_{ir}b_{rj} = c_{ij}, \qquad i,j = 1, 2, \cdots, n; \tag{1}$$

$$a_{ir} = 0 \text{ if } i < r. \tag{2}$$

Combining (1) and (2), we thus have:

$$\sum_{r=1}^{i} a_{ir}b_{rj} = c_{ij}, \qquad i,j = 1, 2, \cdots, n. \tag{3}$$

If we impose the restriction: $i \leqslant k$, where $k \leqslant n$, then in view of (2) we may as well extend the sum in (3) as follows:

$$\sum_{r=1}^{k} a_{ir}b_{rj} = c_{ij}, \qquad i = 1, 2, \cdots, k, \quad j = 1, 2, \cdots, n. \tag{4}$$

In particular,

$$\sum_{r=1}^{k} a_{ir}b_{rj} = c_{ij}, \qquad i,j = 1, 2, \cdots, k. \tag{5}$$

This is equivalent to the desired result.

TRIPLE PLAY

H-253 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{t=0}^{k}\binom{(\beta-1)n+t+1}{t}\sum_{j=0}^{n-k-1}\binom{n-k-1}{j}\sum_{m=0}^{j}(-1)^{n+m+k+1}\binom{j}{m}$$

$$\times\sum_{r=0}^{n+m-t-j-1}\binom{j}{n+m-j-t-r-1}\binom{2j+r-1}{r}=2^{n-k-1}\binom{\beta n}{k}.$$

where β is an arbitrary complex number and n and k are positive integers, $k<n$.

This identity, in the case $\beta=2$, arose in solving a combinatorial problem in two different ways.

Solution by the Proposer.

To prove the identity we replace n by $n+k+1$ and use

$$\sum_{k=0}^{\infty}\binom{\alpha+\beta k}{k}w^k=\frac{x^{\alpha+1}}{(1-\beta)x+\beta}, \tag{1}$$

where $wx^\beta-x+1=0$. This follows from the Lagrange expansion formula (cf. Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis,* I, Berlin, Springer, 1954, p. 125).

From (1) we have

$$\sum_{k=0}^{\infty}2^n\binom{\beta n+\beta k+\beta}{k}w^k=\frac{2^n x^{\beta n+\beta+1}}{(1-\beta)x+\beta}, \tag{2}$$

where $wx^\beta-x+1=0$. Also from (1) we get

$$\sum_{k=0}^{\infty}\sum_{t=0}^{k}\binom{(\beta-1)(n+k)+t+\beta}{t}w^k\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{j}(-1)^{n+m}\binom{j}{m}\sum_{r=0}^{n+k+m-t-j}\binom{j}{n+k+m-j-t-r}\binom{2j+r-1}{r}$$

$$=\sum_{k=0}^{\infty}w^k\sum_{t=0}^{\infty}\binom{(\beta-1)(n+k)+\beta t+\beta}{t}w^t\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{j}(-1)^{n+m}\binom{j}{m}$$

$$\times\sum_{r=0}^{n+k+m-j}\binom{j}{n+k+m-j-r}\binom{2j+r-1}{r}=\sum_{k=0}^{\infty}w^k\left(\frac{x^{(\beta-1)(n+k)+\beta+1}}{(1-\beta)x+\beta}\right)\sum_{j=0}^{n}\binom{n}{j}$$

$$\times\sum_{m=0}^{j}(-1)^{n+j-m}\binom{j}{m}\sum_{r=0}^{n+k-m}\binom{j}{n+k-m-r}\binom{2j+r-1}{r},$$

where $wx^\beta-x+1=0$.

Now

$$\frac{x^{\beta+1}}{(1-\beta)x+\beta}\sum_{k=0}^{\infty}w^k x^{(\beta-1)(n+k)}\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{j}(-1)^{n+j-m}\binom{j}{m}\sum_{r=0}^{n+k-m}\binom{j}{n+k-m-r}\binom{2j+r-1}{r}$$

$$=\frac{x^{\beta+1}w^{-n}}{(1-\beta)x+\beta}\sum_{j=0}^{n}\binom{n}{j}\sum_{r=0}^{\infty}\binom{2j+r-1}{r}\sum_{k=0}^{\infty}(x^{\beta-1}w)^{k+r}\sum_{m=0}^{\min\{k,j\}}(-1)^{n+j+m}\binom{j}{m}\binom{j}{k-m}$$

$$=\frac{x^{\beta+1}w^{-n}}{(1-\beta)x+\beta}\sum_{j=0}^{n}\binom{n}{j}(1-x^{\beta-1}w)^{-2j}\sum_{m=0}^{\infty}(-1)^{n+j+m}\binom{j}{m}\sum_{k=0}^{\infty}\binom{j}{k}(x^{\beta-1}w)^{k+m}=$$

[Continued on page 192.]

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-352 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let S_n be defined by $S_0 = 1, S_1 = 2,$ and

$$S_{n+2} = 2S_{n+1} + cS_n .$$

For what value of c is $S_n = 2^n F_{n+1}$ for all nonnegative integers n?

B-353 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

For k and n integers with $0 \leqslant k \leqslant n$, let $A(k,n)$ be defined by $A(0,n) = 1 = A(n,n), A(1,2) = c + 2,$ and

$$A(k+1, n+2) = cA(k,n) + A(k, n+1) + A(k+1, n+1).$$

Also let $S_n = A(0,n) + A(1,n) + \cdots + A(n,n)$. Show that

$$S_{n+2} = 2S_{n+1} + cS_n .$$

B-354 Proposed by Phil Mana, Albuquerque, New Mexico.

Show that

$$F_{n+k}^3 - L_k^3 F_n^3 + (-1)^k F_{n-k}[F_{n-k}^2 + 3F_{n+k}F_n L_k] = 0.$$

B-355 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Show that

$$F_{n+k}^3 - L_{3k}F_n^3 + (-1)^k F_{n-k}^3 = 3(-1)^n F_n F_k F_{2k} .$$

B-356 Proposed by Herta T. Freitag, Roanoke, Virginia.

Let

$$S_n = F_2 + 2F_4 + 3F_6 + \cdots + nF_{2n} .$$

Find m as a function of n so that F_{m+1} is an integral divisor of $F_m + S_n$.

B-357 Proposed by Frank Higgins, Naperville, Illinois.

Let m be a fixed positive integer and let k be a real number such that

$$2m \leqslant \frac{\log(\sqrt{5}k)}{\log a} < 2m + 1,$$

where $a = (1 + \sqrt{5})/2$. For how many positive integers n is $F_n \leqslant k$?

SOLUTIONS

SUM OF SQUARES AS A. P.

B-328 Proposed by Walter Hansell, Mill Valley, California, and V. E. Hoggatt, Jr., San Jose, California

Show that

$$6(1^2+2^2+3^2+\cdots+n^2)$$

is always a sum

$$m^2+(m^2+1)+(m^2+2)+\cdots+(m^2+r)$$

of consecutive integers, of which the first is a perfect square.

Solution by Bob Prielipp, The University of Wisconsin–Oshkosh.

Since

$$6(1^2+2^2+3^2+\cdots+n^2) = n(n+1)(2n+1) = (2n+1)n^2+[2n(2n+1)]/2$$

and

$$m^2+(m^2+1)+(m^2+2)+\cdots+(m^2+r) = (r+1)m^2+[r(r+1)]/2,$$

the desired result follows upon letting $m=n$ and $r=2n$.

Also solved by Wray G. Brady, Frank Higgins, Mike Hoffman, Herta T. Freitag, Graham Lord, Jeffrey Shallit, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposers.

UNVEILING AN IDENTITY

B-329 Proposed by Herta T. Freitag, Roanoke, Virginia.

Find r, s, and t as linear functions of n such that $2F_r^2-F_sF_t$ is an integral divisor of $L_{n+2}+L_n$ for $n=1, 2, \cdots$.

Solution by Mike Hoffman, Warner Robins, Georgia.

Let

$$\alpha = \tfrac{1}{2}(1+\sqrt{5}) \qquad \text{and} \qquad \beta = \tfrac{1}{2}(1-\sqrt{5}).$$

Then

$$\begin{aligned}
2F_r^2-F_sF_t &= 2\left(\frac{\alpha^r-\beta^r}{\sqrt{5}}\right)^2-\left(\frac{\alpha^s-\beta^s}{\sqrt{5}}\right)\left(\frac{\alpha^t-\beta^t}{\sqrt{5}}\right)\\
&= 2\,\frac{\alpha^{2r}-2(\alpha\beta)^r+\beta^{2r}}{5}-\frac{\alpha^{s+t}-\alpha^s\beta^t-\beta^s\alpha^t+\beta^{s+t}}{5}\\
&= \frac{2\alpha^{2r}+2\beta^{2r}-\alpha^{s+t}-\beta^{s+t}-4(\alpha\beta)^r+\alpha^s\beta^t+\alpha^t\beta^s}{5}\\
&= \frac{2L_{2r}-L_{s+t}-4(\alpha\beta)^r+(\alpha\beta)^t(\alpha^{s-t}+\beta^{s-t})}{5}\\
&= \frac{2L_{2r}-L_{s+t}+L_{s-t}(-1)^t-4(-1)^r}{5},
\end{aligned}$$

where we have used Binet form for the Fibonacci and Lucas numbers, as well as the fact $\alpha\beta=-1$. Now put $r=n+3$, $s=n+3$, and $t=n-1$. The above becomes

$$\begin{aligned}
2F_r^2-F_sF_t &= \frac{2L_{2n+2}-L_{2n+1}+L_3(-1)^{n-1}-4(-1)^{n+1}}{5}\\
&= \frac{L_{2n+2}+L_{2n+2}-L_{2n+1}+4(-1)^{n-1}-4(-1)^{n+1}}{5} = \frac{L_{2n+2}+L_{2n}}{5} = F_{2n+1}.
\end{aligned}$$

Thus we have

$$L_{2n+2} + L_{2n} = 5(2F_r^2 - F_s F_t)$$

for positive integers n.

Also solved by the Proposer.

FINDING A G. C. D.

B-330 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Let

$$G_n = F_n + 29F_{n+4} + F_{n+8}.$$

Find the greatest common divisor of the infinite set of integers $\{G_0, G_1, G_2, \cdots\}$.

Solution by Graham Lord, Universite Laval, Quebec, Canada.

It is easy to show that $G_n = 36F_{n+4}$ by using repeatedly the classical Fibonacci recursion relation. Hence, as two consecutive Fibonacci numbers are relatively prime, the g.c.d. of the numbers $G_0, G_1, G_2, \cdots$, is equal to 36.

Also solved by Wray G. Brady, Herta T. Freitag, Frank Higgins, Mike Hoffman, Bob Prielipp, Jeffrey Shallit, Sahib Singh, Gregory Wulczyn, David Zeitlin, and the Proposer.

SOME FIBONACCI SQUARES MOD 24

B-331 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Prove that $F_{6n+1}^2 \equiv 1 \pmod{24}$.

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

A congruence table of F_n (modulo 24) is

n	1	2	3	4	5	6	7	8	9	10	11	12
F_n (mod 24)	1	1	2	3	5	8	13	21	10	7	17	0

n	13	14	15	16	17	18	19	20	21	22	23	24
F_n (mod 24)	17	17	10	3	13	16	5	21	2	23	1	0

Hence $F_{6n+1} \equiv 1, 13, 17, 5 \pmod{24}$ and $F_{6n+1}^2 \equiv 1 \pmod{24}$.

Also solved by Herta T. Freitag, Frank Higgins, Mike Hoffman, Graham Lord, Bob Prielipp, Sahib Singh, David Zeitlin, and the Proposer.

ONE SINGLE AND ONE TRIPLE PART

B-332 Proposed by Phil Mana, Albuquerque, New Mexico.

Let $a(n)$ be the number of ordered pairs of integers (r,s) with both $0 \leqslant r \leqslant s$ and $2r + s = n$. Find the generating function

$$A(x) = a(0) + xa(1) + x^2a(2) + \cdots.$$

Solution by Graham Lord, Universite Laval, Quebec, Canada.

If s is written as $r + t$, where $t \geqslant 0$ then the decomposition $n = 2r + s$ is the same as $3r + t$, where the only restriction on r and t is that they be nonnegative integers. Thus $a(n)$ counts the number of partitions of n in the form $3r + t$ and so has the generating function

$$A(x) = (1 + x + x^2 + \cdots)\cdot(1 + x^3 + x^6 + x^9 + \cdots) = [(1-x)(1-x)(1-x^3)]^{-1}.$$

Also solved by Wray G. Brady, Frank Higgins, Mike Hoffman, Sahib Singh, Gregory Wulczyn, and the Proposer.

BIJECTION IN $Z^+ \times Z^+$

B-333 Proposed by Phil Mana, Albuquerque, New Mexico.

Let S_n be the set of ordered pairs of integers (a,b) with both $0 < a < b$ and $a + b \leqslant n$. Let T_n be the set of ordered pairs of integers (c,d) with both $0 < c < d < n$ and $c + d > n$. For $n \geqslant 3$, establish at least one bijection (i.e., 1-to-1 correspondence) between S_n and T_{n+1}.

I. Solution by Herta T. Freitag, Roanoke, Virginia; Frank Higgins, Naperville, Illinois; and the Proposer (each separately).

$$c = b \quad \text{and} \quad d = n + 1 - a$$

or inversely,

$$a = n + 1 - d \quad \text{and} \quad b = c.$$

II. Solution by Mike Hoffman, Warner Robins, Georgia; and the Proposer (separately).

$$c = n + 1 - b \quad \text{and} \quad d = n + 1 - a$$

or, inversely,

$$a = n + 1 - d \quad \text{and} \quad b = n + 1 - c.$$

It is straightforward to verify that $a + b \leqslant n$ if and only if $c + d > n$ and hence that each of I and II gives a one-to-one correspondence.

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[Continued from page 188.]

ADVANCED PROBLEMS AND SOLUTIONS

$$= \frac{x^{\beta+1}w^{-n}}{(1-\beta)x+\beta} \sum_{j=0}^{n} \binom{n}{j} (1 - x^{\beta-1}w)^{-2j} \sum_{m=0}^{\infty} (-1)^{n+j+m} \binom{j}{m} (x^{\beta-1}w)^m (1 + x^{\beta-1}w)^j$$

$$= \frac{x^{\beta+1}(-w)^{-n}}{(1-\beta)x+\beta} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(\frac{1 + x^{\beta-1}w}{1 - x^{\beta-1}w} \right)^j = \frac{x^{\beta+1}(-w)^{-n}}{((1-\beta)x+\beta)} \left(\frac{-2x^{\beta-1}w}{1 - x^{\beta-1}w} \right)^n$$

$$= \frac{x^{\beta+1}2^n}{((1-\beta)x+\beta)} \left(\frac{x^{\beta-1}}{1 - x^{\beta-1}w} \right)^n = \frac{2^n x^{\beta n+\beta+1}}{(1-\beta)x+\beta} .$$

Comparing this with (1), it is clear that we have proved the identity.

CORRECTION

H-267 (Corrected)

Show that

$$S(x) = \sum_{n=0}^{\infty} \frac{(kn+1)^{n-1}x^n}{n!}$$

satisfies $S(x) = e^{xS^k(x)}$.

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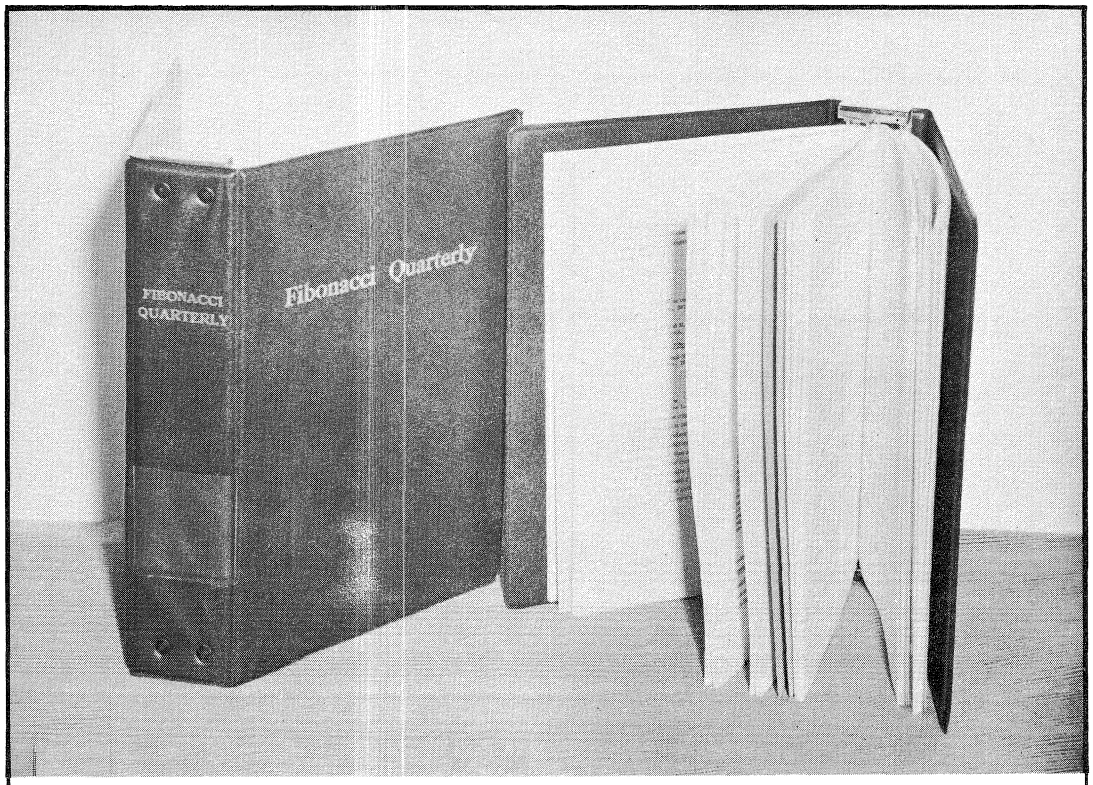

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