COMBINATIONS, COMPOSITIONS AND OCCUPANCY PROBLEMS

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INTRODUCTION

Let $r \leq k$ be positive integers. By a composition of k into r parts (an r-composition of k) we mean an ordered sequence of r positive integers (called the parts of the composition) where sum is k, i.e.,

$$a_1 + a_2 + \cdots + a_r = k \ . \tag{1}$$

The length of the part a_i in (1) is a_i, $i = 1, \cdots, r$. We call k integers

$$x_1 < x_2 < \cdots < x_k \tag{2}$$

chosen from $\{1, 2, \cdots, n\}$ a k-combination (choice) from n. A part of (2) is a sequence of consecutive integers not contained in a longer sequence of consecutive integers. The length of such a part is the number of integers contained in it. For example, the 6-combination 2, 3, 4, 6, 8, 9 from $n \geq 9$ consists of 3 parts (2, 3, 4), (6), and (8, 9) of lengths 3, 1, and 2, respectively.

A great deal of literature exists on restricted compositions and may be found in most standard texts, for example [7]. However, there does not seem to be much literature on restricted combinations, in particular on the notion of parts with respect to combinations. The notion of parts has been used in [2] and [6] (in disguised form) as preliminaries to solve certain permutation problems. A treatment of restricted combinations in itself seems worthwhile for the following reasons. First, as noted in paragraph 4, the study of certain occupancy problems (like objects into unlike cells) is shown to follow immediately from the study of restricted combinations. Although, of course, many occupancy problem results are well known, many of the results obtained in paragraphs 1 and 2 by elementary combinatorial methods are believed new and might not otherwise be readily obtained. In particular, they are relevant to the development of tests of randomness in two-dimensional

arrays. Also, restricted combinations are useful in dealing with certain restricted sequences of Bernoulli trials. In paragraph 1, the simple connection between restricted compositions and restricted combinations is given. Although the results contained herein are perhaps of a technical and specialized nature, the approach is completely elementary.

Throughout this note, we take

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{(n-k)!k!} & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases} .$$

1. Consider the following six symbols, each denoting the number of compositions of k into r parts with further restrictions where indicated.

List 1

1. $C(k,r)$, no other restrictions.
2. $C(k,r;\ w)$, each part $\le w$.
3. $C_e(k,r)$, each part of even length.
4. $C_e(k,r;\ w)$, each part of even length and each part $\le w$ (even).
5. $C_0(k,r)$, each part of odd length.
6. $C_0(k,r;\ w)$, each part of odd length and each part $\le w$ (odd).

Expressions for the above numbers are well known and may be obtained by combinatorial arguments or by considering the appropriate enumerator generating function in each case, as described by Riordan [7, p. 124].

Corresponding to the 6 restricted combination symbols given in List 1, we have the following six restricted combination symbols, each denoting the number of k-combinations from n with exactly r parts and with further restrictions as indicated.

List 2

1. $g(n,k;\ r)$, no further restrictions.
2. $g(n,k;\ r,w)$, each part $\le w$.
3. $g_e(n,k;\ r)$, each part of even length.
4. $g_e(n,k;\ r,w)$, each part of even length and each part $\le w$ (even).
5. $g_0(n,k;\ r)$, each part of odd length.
6. $g_0(n,k;\ r,w)$, each part of odd length and each part $\le w$ (odd).

Denote by C^i the restricted composition symbol in the ith row of List 1, and g^i the ith restricted combination symbol of List 2. Then

$$g^i = \binom{n-k+1}{r} C^i, \qquad i = 1, \cdots, 6 . \tag{3}$$

To establish (3), note that a k-combination from n can be represented by $n-k$ symbols 0 and k symbols 1 arranged along a straight line, the symbol 0 representing an integer not chosen and a symbol 1 representing an integer chosen. Now place $n-k$ symbols 0 along a straight line forming $n-k+1$ cells including one before the first zero and one after the last. Choose r of these cells in

$$\binom{n-k+1}{r}$$

ways. Now distribute the k symbols 1 into these cells with none empty in C^i ways. The result follows.

In fact, corresponding to a specified r-composition of k with r parts we have

$$\binom{n-k+1}{r}$$

k-combinations of n consisting of r parts with the same specifications and clearly

$$g(n,k;\, b_1, b_2, \cdots, b_u) = \binom{n-k+1}{r} C(k;\, b_1, b_2, \cdots, b_u) , \tag{4}$$

where $g(n,k;\, b_1, b_2, \cdots, b_u)$ denotes the number of k-combinations from n, $C(k;\, b_1, b_2, \cdots, b_u)$ denotes the number of compositions of k, each consisting of exactly b_i parts of length i, $i = 1, 2, \cdots, n$ with

$$\sum_{i=1}^{u} ib_i = r .$$

A succession of a k-combination (2) is a pair x_i, x_{i+1} with $x_{i+1} - x_i = 1$. It is easy to see that a k-combination from n contains exactly s successions if and only if it contains exactly k - s parts. Hence, instead of describing the restricted choices by their parts, we may use succession conditions. The numbers

$$g(n,k;\ r) = \binom{n-k+1}{r}\binom{k-1}{r-1}$$

and $g(n,k;\ k-s)$ are used in [2] and [6]. The numbers

$$\sum_{r=1} g^i, \qquad i = 1, \cdots, 6$$

give the number of combinations with the same restrictions as on the combinations counted in g^i, $i = 1, \cdots, 6$, but with the number of parts not being specified. Of course, the numbers

$$\sum_{r=1} g^i$$

may also be found by considering the appropriate generating function.

Recurrence relations and expressions for $g(n,k;\ r)$ and $g(n,k;\ r,w)$ may be found in [2] and [3]. We consider now some special restricted combinations.

The number of k-combinations from n, all parts even and $\leq w$, is, for k,w even,

$$\begin{aligned} g_e(n,k;\ w) &= \sum_{r=1} g_e(n,k;\ r,w) = \sum_{r=1} \binom{n-k+1}{r} C_e(k,r;\ w) \\ &= \sum_{i=0} (-1)^i \binom{n-k+t-i}{n-k}\binom{n-k+1}{i}, \quad t = \frac{k-iw}{2}. \end{aligned} \tag{5}$$

The number $g_e(n,k;\ w)$ is also the coefficient of x^k in the expression $(1 + x^2 + x^4 + \cdots + x^w)^{n-k+1}$. Taking w sufficiently large in (5), the number of k-combinations from n with all parts even is, for k even

$$g_e(n,k) = \binom{n - k/2}{k/2} \quad \text{with} \quad g_e(n,0) = 1, \tag{6}$$

and

$$g_e(n) = \sum_{r=0} \binom{n - r}{r} \tag{7}$$

is the number of choices from n with all parts even. $[g_e(n) = F_{n+1}]$

In the case of combinations with odd parts only, we have for $w \geq 3$ and w odd,

$$\begin{aligned} g_0(n,k;\ w) &= \sum_{r=1} g_0(n,k;\ r,w) \\ &= \sum_{r=1} \sum_{i=0} (-1)^i \binom{r}{i} \binom{t - 1}{r - 1} \binom{n - k + 1}{r} , \end{aligned} \tag{8}$$

where $r \equiv k \pmod 2$ and

$$t = \frac{k + r - i(w + 1)}{2} .$$

The enumerator generating function of $g_0(n,k;\ w)$ is

$$(1 + x + x^3 + \cdots + x^w)^{n-k+1} .$$

$$g_0(n,k;\ k) = \binom{n - k + 1}{k} \tag{9}$$

is the number of k-combinations from n, no two consecutive, the lemma of Kaplansky [5]. Taking w sufficiently large in (8), the number of k-choices from n, all parts odd, is

$$g_0(n,k) = \sum_{i=0} \binom{k-i-1}{i} \binom{n-k+1}{k-2i}, \quad g_0(n,9) = 1, \tag{10}$$

and the number of choices from n with all parts odd is

$$g_0(n) = \sum_{k=0} g_0(n,k) . \tag{11}$$

2. We also obtain the following relations. For $n \geq w + 3$,

$$\begin{aligned} g_e(n,k;\, w) = g_0(n-1,k;\, w) &+ g_e(n-2,k-2;\, w) \\ &- g_e(n-w-3,\, k-w-2;\, w) . \end{aligned} \tag{12}$$

For, if a k-choice from n with all parts even and $\leq w$ (even):

(i) does not contain n, then it is a k-choice from $n - 1$ with all parts even and $\leq w$, and there are $g_e(n-1), k;\, w)$ of these:

(ii) does contain n, then it must contain $n - 1$. Deleting the $n - 1$ and n we have a (k-2)-choice from $n - 2$ (with all parts even and $\leq w$ (even)) which does not contain all of $n - w - 1$, $n - w, \cdots, n - 2$, and there are $g_e(n-2,\, k-2;\, w) - g_e(n-w-3,\, k-w-2;\, w)$ of these.

Of course,

$$g_e(n,k;\, w) = \begin{cases} 0 & k = w + 2 = n \\ g_e(n,k) & k < w + 2 = n \\ g_e(n,k) & k \leq w + 1 , \end{cases} \tag{13}$$

and hence, from (12) and (13),

$$g_e(n,k) = g_e(n-1,k) + g_e(n-2,\ k-2)\ . \tag{14}$$

The latter is also easily obtained by observing that a k-combination of n with all parts even either does not contain n or does contain n and necessarily n - 1.

In the case $n = w + 3$ and $k = w + 2$, (12) becomes (using (7)),

$$g_e(w+3,\ w+2;\ w) = w/2\ .$$

This is easily verified directly by observing that the $(w + 3)$-choices from $1, 2, \cdots, w + 3$ with all parts even and $\leq w$ (even) are obtained by removing from $1, 2, \cdots, w + 3$ one of the $w/2$ integers $3, 5, \cdots, w + 1$.

Let $g_e(n;\ w)$ denote the total number of combinations from n with all parts even and $\leq w$ (even). Then

$$g_e(n;\ w) = \sum_{k=0} g_e(n,k;\ w)\ . \tag{15}$$

Using (12) and summing over k we have

$$g_e(n;\ w) = \begin{cases} g_e(n-1;\ w) + g_e(n-2;\ w) - g_e(n-w-3;\ w), & n \geq w+3 \\ g_e(n-1;\ w) + g_e(n-2;\ w) - 1, & n = w+2\ . \end{cases} \tag{16}$$

Putting $n \leq w + 1$ in (16) or summing over all k in (14) we obtain

$$g_e(n) = g_e(n-1) + g_e(n-2),\quad g_e(0) = g_e(1) = 1\ . \tag{17}$$

The numbers $g_e(n)$ are Fibonacci numbers. The Fibonacci numbers arise in other cases of restricted combinations. For example, if f(n) denotes the number of combinations from n, no two consecutive, and T(n) denotes the number of combinations from n with odd elements in odd position and even elements in even position, then [7, p. 17. problem 15].

$$g_e(n) = f(n-1) = T(n-1), \qquad n > 0 .$$

Also, by considering those combinations which do not contain n, those combinations which contain $n, n-1$ but not $n-2$, those containing $n, n-1, n-2, n-3$ but not $n-4, \cdots$, etc., we obtain for $n > k \geq w$, w even, the relation

$$g_e(n,k;w) = \sum_{r=0}^{w/2} g_e(n-2r-1, k-2r; w) . \tag{18}$$

In the case of odd parts, a relation comparable to (10) is not readily obtainable. However, for $n > k \geq w$, w odd, we have

$$g_0(n, k; w) = g_0(n-1, k; w) + \sum_{r=1}^{w+1/2} g_0(n-2r, k-2r+1; w) . \tag{19}$$

The first term on the right side counts those choices not containing n, the second term those choices containing n but not $n-1$, the third term those choices containing $n, n-1, n-2$ but not $n-3, \cdots$, etc., the last term those choices containing $n, n-1, \cdots, n-w+1$ but not $n-w$.

Denote by $g_0(n; w)$ the number of combinations from n with all parts odd and $\leq w$ (odd). Then, for $n > w$ (odd), summing (19) over k yields

$$g_0(n; w) = g_0(n-1; w) + \sum_{r=1}^{\overline{w+1}/2} g_0(n-2r; w) . \tag{20}$$

Taking w sufficiently large in (19), we have for $n > k$,

$$g_0(n,k) = g_0(n-1,k) + \sum_{r=1} g_0(n-2r, k-2r+1) \tag{21}$$

and

$$g_0(k,k) = \begin{cases} 0 & \text{if } k = 0, 2, 4, 6, \cdots \\ 1 & \text{if } k = 1, 3, 5, 7, \cdots \end{cases} .$$

For example,

$$g_0(2,1) = g_0(1,1) + g_0(0,0) = 1 + 1 = 2 .$$

Using (21), it is easily seen that

$$(22) \qquad g_0(n) = \begin{cases} g_0(n-1) + g_0(n-2) + g_0(n-4) + g_0(n-6) + \cdots + g_0(1) + 1 & \text{for } n \text{ odd and } n \geq 3, \\ g_0(n-1) + \sum_{r=1}^{n/2} g_0(n-2r) & \text{for } n \text{ even,} \end{cases}$$

with $g_0(0) = 1$ and $g_0(1) = 2$.

3. In a k-combination from $\{1, \cdots, n\}$ if we consider 1 and n as adjacent, then we obtain "circular" k-combinations from n. For example, the set $\{1, 2, 6, 8, 9, 12\}$ is a circular 6-combination from 12 consisting the 3 parts $\{12, 1, 2\}$, $\{6\}$, $\{8, 9\}$ of length 3, 1, and 2, respectively. Corresponding to the 6 symbols of List 2, we obtain 6 circular k-combination symbols denoted by h^i, $i = 1, \cdots, 6$. Then

$$(23) \qquad h^i = \frac{n}{n-k}\binom{n-k}{r}C^i, \qquad 0 < k < n .$$

This is easily established by noting the proof for

$$h(n, k; r, w) = \frac{n}{n-k}\binom{n-k}{r}C(k, r; w)$$

in [3]. The special case of $r = k$ and $i = 1$ in (23) gives

$$h(n,k; k) = \frac{n}{n-k}\binom{n-k}{k} ,$$

the number of k-combinations, no two consecutive, of $\{1, \cdots, n\}$ arranged in a circle, the lemma of Kaplansky [5]. The numbers

$$H(n) = \sum_{k=0}^{[n/2]} h(n,k;\, k) \quad ,$$

with $h(n, 0;\, 0) = 1$, have the relation

$$H(n) = H(n - 1) + H(n - 2)$$

for $n \geq 4$ with $H(2) = 3$ and $H(3) = 4$. [$H(n) = L_n$, the Lucas numbers.]

The relation between g_i and h_i is, of course, given by

$$g_i = \frac{(n - k)(n - k + 1)}{n(n - k + 1 - r)} h_i, \qquad i = 1, \cdots, 6, \qquad n - k \geq r .$$

4. In examining the proof of (3), it is clear that each of the numbers g_i and

$$\sum_{r=1} g_i, \qquad i = 1, \cdots, 6 ,$$

may be interpreted as the number of ways of putting loke objects into $n - k$ 1 unlike cells subject to corresponding conditions. Putting $n = m + k - 1$ we are then placing k like objects into m unlike cells with the corresponding conditions. For example, $g(m + k - 1, k;\, r)$ is the number of ways of doing this such that exactly r of the m cells are occupied while

$$B(m, k;\, w) = \sum_{r=1} g_e(m + k - 1, k;\, r, w)$$

is the number where any occupied cell contains an even number, not greater than w, of the like objects. Using (18),

$$B(m, k; w) = \sum_{r=0}^{w/2} B(m - 1, k - 2r; w) ,$$

w even. In particular,

$$g(m + k - 1, k; m) = \binom{k - 1}{m - 1}$$

and

$$\sum_{r=1} g(m + k - 1, k; r) = \sum_{r=1} \binom{m}{r}\binom{k - 1}{r - 1} = \binom{m + k - 1}{m}$$

are the well known occupancy formulae [Riordan, 7, p. 92 and p. 102, Problem 8], the first having none of the m cells empty and the second having no restriction on the distributions of the k objects. Also, the numbers

$$g(m + k - 1, k; r) \quad \text{and} \quad \sum_{r=1} g(m + k - 1, k; r, w)$$

are treated as occupancy problems by Riordan [7, pp. 102-104, Problems 9, 13, and 14].

The restricted combinations also have applications to certain ballot and random walk problems. For example, in an election between two candidates, the probability that a certain candidate leads after n votes but does not obtain more than $u > 0$ runs of votes nor a run of votes of length greater than w is equal to

$$\frac{\sum_{r=1}^{u} \sum_{k=\lceil n+2/2 \rceil} g(n, k; r, w)}{2^n} .$$

Finally, by noting that for $w > 1$,

$$\sum \binom{a_2}{a_1}\binom{a_3}{a_1}\cdots\binom{a_w}{a_{w-1}} = C(k, a_w; w) \quad , \tag{24}$$

the sum taken over all solutions $(a_1, \cdots, a_{w-1})$,

$$a_i \geq 0, \qquad \text{of} \qquad a_1 + a_2 + \cdots + a_{w-1} = k - a_w ,$$

many of the expressions in [1] are simplified. In particular for $w = k$ (24) becomes

$$\binom{a_2}{a_1}\binom{a_3}{a_2}\cdots\binom{a_k}{a_{k-1}} = C(k, a_k) = \binom{k-1}{a_k - 1} \quad . \tag{25}$$

Upon change of variables and some elementary manipulation, (25) becomes Theorem 16 of [4],

$$\binom{n-1}{r-1} = \sum_{\substack{b_1 + 2b_2 + 3b_3 + \cdots + nb_n = n \\ b_1 + b_2 + \cdots + b_n = r, \quad b_i \geq 0}} \frac{r!}{b_1!\, b_2! \cdots b_n!}$$

for all natural numbers n and r.

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WHY FIBONACCI SEQUENCE FOR PALM LEAF SPIRALS ?

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On account of their very large, prominently-stalked leaves, palms are ideal material to study phyllotaxis, which means the arrangement of leaves on the trunk. The leaves of palms are produced one after another, and their arrangement is termed alternate, which is the case with the majority of plants that display spiral phyllotaxis. That is, two consecutive leaves are placed on the stem at different heights with an angular deflection of less than 180°. Some other plants like most grasses subtend an angle of 180° between two consecutive leaves, and this arrangement is known as distichous of 1/2 phyllotaxis, as two leaves are produced in one complete rotation. Even a rare palm (Wallichia disticha) shows this peculiarity. In alternate phyllotaxis, some plants will have five leaves produced before completing two complete revolutions. This system is referred to as 2/5 phyllotaxis. Also, other plants may show 3/8, 5/13, 8/21, 13/34, 21/55, ···, or 2/3, 3/5, 5/8, 8/13, 13/21, 21/34, 34/55, and so on phyllotaxis. The numerators or the denominators of this series, when considered alone, form the successive stages of the Fibonacci sequence. It is known [1-5] that the Fibonacci phyllotaxis gives optimum illumination to the photosynthetic surface of plants since the leaves overlap least.

2. VARYING NUMBERS OF LEAF SPIRALS IN PADS

Different species of palms display different numbers of leaf spirals, and the numbers always match with Fibonacci numbers. For example, in the arecanut palm (Areca catechu) (Fig. 1), or the ornamental Ptychosperma macarthurii palm, only a single foliar spiral is discernible, while in the sugar palm (Arenga saccharifern) (Fig. 2), or Arenga pinnata, two spirals each are visible. In the palmyra palm (Borassus flabellifer) (Fig. 3), or Corypha elata, as well as a number of other species of palms, three clear spirals are visible. The coconut palm (Cacos nucifera) as well as Copernicia spirals (Fig. 4) have five spirals, while the African oil palm (Elaeis guineensis) (Fig. 5) bears eight spirals. The wild date palm (Phoenix sylvestris) and a

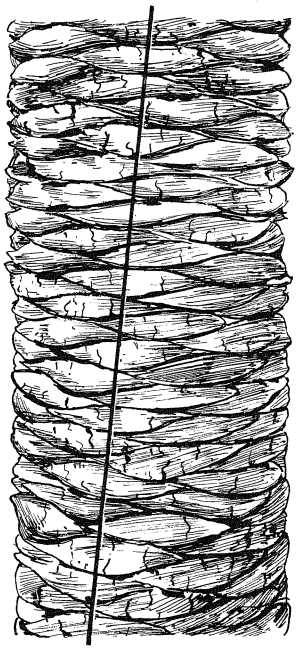
ARECA CATECHU(1)
Figure 1

ARENGA PINNATA(2)
Figure 2

BORASSUS FLABELLIFER (3)
Figure 3

COPERNICIA SP.(5)
Figure 4

ELAEIS GUINEENSIS(8)
Figure 5



PHOENIX CANARIENSIS(13)
Figure 6

few other species of palms also show eight spirals. On the stout trunks of the Canary Island palm (Phoenix canariensis) (Fig. 6), thirteen spirals can be observed. Also in some of these plams, twenty-one spirals can be made out. It is surprising that all the above-mentioned numbers (1, 2, 3, 5, 8, 13 and 21) happen to be Fibonacci numbers. Palms bearing 4, 6, 7, 9, 10, 11 or 12 obvious leaf spirals are not known.

3. FIBONACCI NUMBERS IN VERTICAL PALMS

In some exceptional palms, the arrangement of leaves does not show any spiral mechanism. Instead, the leaves are arranged one vertically above another along two or more rows. As mentioned, in Wallichia disticha, there are two vertical rows of leaves and the angle between any two consecutive leaves is 180°. In Madagascar's three-sided palm, Neodypsis decaryi, the leaves fall along three vertical rows, and the successive leaves are formed at a constant angular deflection of 120° (narrow angle). In the case of most individuals of Syagrus treubiana, five impressive vertical rows of persisting leaf bases can be seen. The angular deflection between any two successive leaves is 135°. Also, it is possible to observe on some trunks of the Canary Island palm, leaf scars that match with thirteen vertical rows, the angular deflection between their successive leaves will be 138.5°. If a palm showing twenty-one vertical rows of leaves can be detected, the angular deflection between two consecutive leaves will be 137.14°. Incidentally, the number of vertical rows of leaves in the above-mentioned palms also turn out to be all Fibonacci numbers. If the figures of the above angular deflections (180, 120, 144, 135, 138.5, 137.14, $\cdots$) are examined, one finds that the alternate numbers turn out to be more than 137.5° and the others less, and the difference between two numbers progressively gets narrower. This narrow angle makes with the remaining angle (222.5°) to complete one full circle, a proportion of 0.618, which is the golden proportion. This phenomenon exactly demonstrates one of the specific properties of the Fibonacci Sequence. That is, the proportion between any two consecutive Fibonacci numbers is alternately more (or less) than the golden proportion. If the values 137.14, 138.5, 135, 144, 120 and 180 are subtracted by the golden proportion angle of 137.5,

we get -0.36, 1.0, -2.5, 6.5, -17.5, 42.5, which when multiplied by 2 gives values approximating to alternate Fibonacci numbers.

The number of green leaves a palm bears at a time generally indicates the number of foliar spirals exhibited by the species. Palm species having fewer green leaves manifest smaller numbers of foliar spirals, and those bearing larger numbers of leaves show greater numbers of spirals. This situation can be easily explained by the help of the schematic representation of a palm crown shown in Fig. 7. There is also an indication that the mean number of gren leaves of a palm is more or less a Fibonacci number.

4, MAKING A PALM CROWN

The centralmost point in the schematic crown (Fig. 7) represents the serial view of a palm trunk, and the radial lines, its leaves. The outermost leaf which is the oldest, is numbered 1. Leaf No. 2 is drawn at an arbitrary angular deflection of 137.5° to the left of leaf No. 1. Since leaf No. 2 is nearer to leaf No. 1 by the left-hand side of an observer looking from the tip of leaf No. 1, this crown may be regarded as representing a left-spiralled palm. Similarly, leaf No. 3 is nearer to leaf No. 2 by the left, and the subsequent leaves are also similarly located. In another palm, leaf No. 2 can as well be nearer to leaf No. 1 by the right, in which case the diagram will represent a right-spiralled crown. The two types of individuals for any species of palms are distributed more or less equally in any locality, although for the coconut, there is an excess of one kind of individual in the Northern hemisphere, while the Southern hemisphere has more of the other kind, the hemispherical difference being significant statistically [6].

The younger leaves are represented progressively by shorter radial lines, and leaf No. 90 is the youngest visible leaf in this crown. The tips of all these leaves are connected by a line which forms a clockwise (left-handed) coil, and this will represent the only visible spiral in some palms such as the areca. In a palm showing two foliar spirals, one spiral will comprise leaves 1, 3, 5, 7, 9, and so on, while the second spiral will comprise all the even-numbered leaves. It is to be noticed that both the spirals move counter-clockwise. In a palm bearing three clearly visible spirals, the following leaves constitute the three spirals. Spiral one will connect

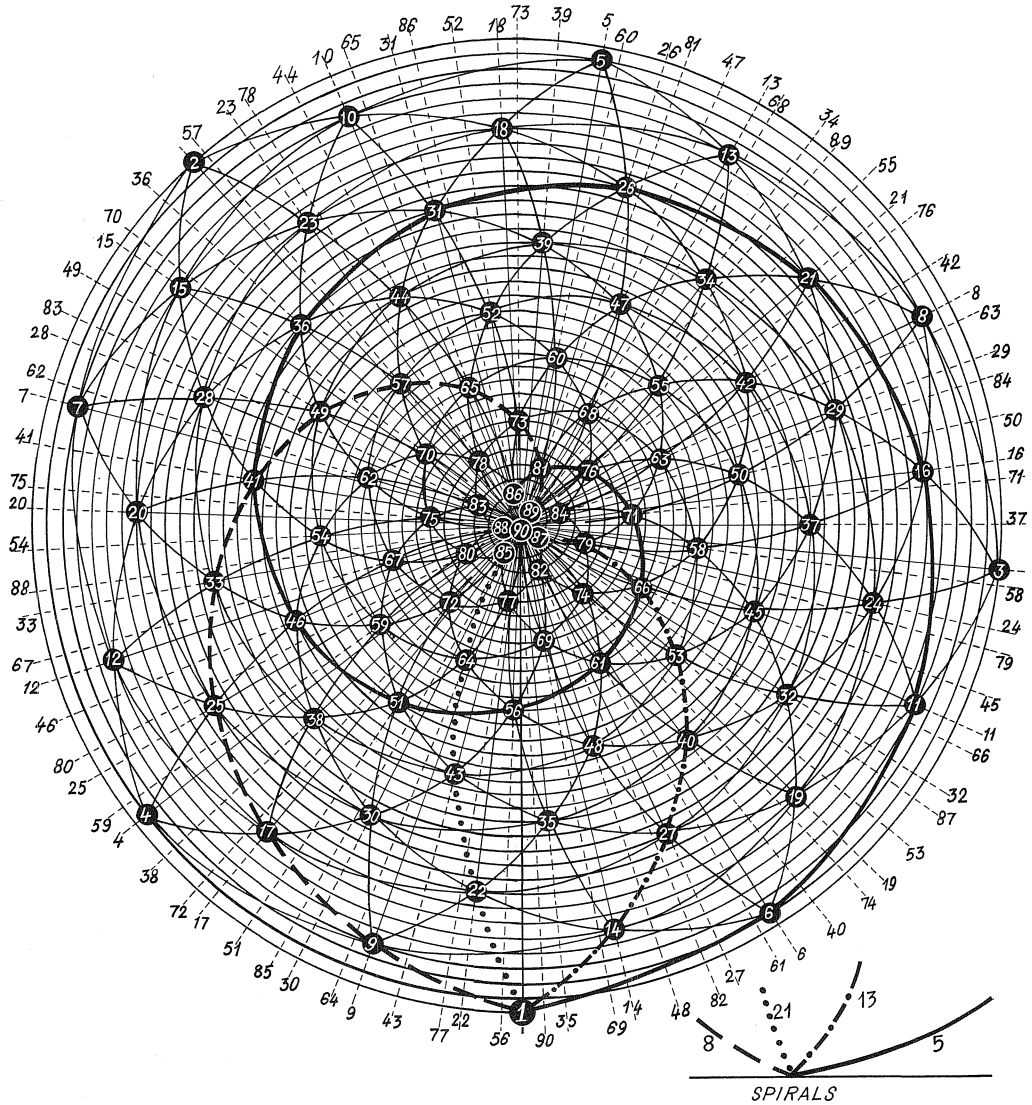


Figure 7

leaves 1, 4, 7, 10, 13, and so on. The second spiral will have leaves 2, 5, 8, 11, 14, and so on, while the third spiral will comprise leaves 3, 6, 9, 12, 15, and so on. All the three spirals run clockwise as opposed to the direction of the two spirals. No palm shows four clear spirals. This is in part due to the fact that leaves 1 and 5 which should form the two consecutive leaves of one of the four spirals, are located almost opposite each other. In a five-spiralled crown, leaves 1, 6, 11, 21, and so on, constitute one of the spirals, the other four starting with leaves 2, 3, 4, and 5, respectively. All the five spirals clearly move counter-clockwise which is opposite to the direction of the three spirals. In a palm with eight spirals, leaves 1, 9, 17, 25, 33, and so on, will form one of the spirals (shown in bold broken line) and the remaining spirals commence from leaves 2, 3, 4, 5, 6, 7 and 8. The eight spirals move opposite to the five spirals. In a like manner, if the diagram is to represent a thirteen-spiralled palm (one spiral comprising leaves 1, 14, 27, 40, 53, 66, etc.), the spirals will move opposite to the eight spirals, and similarly, the twenty-one spirals (shown in dots) move slantingly opposite to the thirteen spirals. Thus, in this diagram, the more obvious numbers of foliar spirals synchronize with Fibonacci numbers. Foliar spirals representing numbers 1, 3, 8, 21 move clockwise and the others counter-clockwise. This situation is in conformity with some properties of the Fibonacci Sequence.

In a palm crown having four or five leaves, only the single spiral is discernible, and two spirals may be clear if the leaf number goes up to seven or eight. Three spirals may be made out in a crown having ten to twelve leaves, and five spirals in a crown having about twenty leaves. Therefore, as the number of green leaves in a crown increases (a situation which normally necessitates a proportional increase in the girth of the trunk), higher orders of foliar spirals are displayed. Moreover, from a crown showing, say, eight spirals, those representing the two neighboring Fibonacci numbers (5 and 13) could also be made out. The leaf bases on the Elaeis guineensis trunk (Fig. 5) and the leaf scars on the stem of Phoenix canariensis (Fig. 6) bear testimony to this.

The angular deflection of 137.5° has been chosen arbitrarily because this would provide no two leaves in the diagram exactly superimposing each other till the 145th leaf. No palm is likely to have one hundred functional leaves

at a time, and so, this angular deflection gives the leaves scope for maximum exposure to sunlight.

From observations made on a few species of palms having large numbers of leaves, the smaller angle subtended by any two consecutive leaves has been found to be 137.5° which makes a 0.618 proportion with the larger angle of 222.5°. The proportion between any two consecutive Fibonacci numbers (excepting the few smaller ones) also turns out to be 0.618. However, it is difficult to explain why most palms conform to this rule of golden proportion in the provision of the angular deflection between consecutive leaves. Obviously, it is genetically controlled. In Nature, small variations are noticed in the deflection between leaves in some palms, and sometimes within a palm at different heights. The foliar spirals of a tree at different heights tend to move along varied curves which is caused by the varying lengths of the internodes between leaves as well as the changes in the thicknesses of the stem. However, the numbers of these spirals are constant for a species.

5. FURTHER PECULIARITIES

If the schematic crown (Fig. 7) is examined critically, one will find that one of the five spirals connecting leaves 1, 6, 11, 16, ···, and one of the eight spirals connecting leaves 1, 9, 18, 25, ··· meet first at leaf No. 41, i.e., after forty leaves, and again at regular intervals of forty leaves. Similarly, if spirals of stages No. 1 (connecting leaves 1, 2, 3, 4, ···) and 2 (connecting leaves 1, 3, 5, 7, ···) are considered, they will meet first at leaf No. 3; likewise, spirals stages Nos. 2 and 3 meet at leaf 7; spirals 3 and 5 meet at leaf 16; spirals 8 and 13 at leaf 105, and so on. Each of the numbers of the above-mentioned leaves minus the first one gives the product of the two spirals and they are always Fibonacci numbers. This situation is applicable not only between leaf spirals representing consecutive stages, but also to any two different spirals. For example, the spirals Nos. 3 and 8 meet first at leaf 25 (i.e., after 24 leaves), while spirals Nos. 3 and 13 which meet at leaf 40 (i.e., after 39 leaves). Therefore, these numbers are always the products of any set of two Fibonacci numbers.

The 1/3, 2/5, 3/8, 5/13, 8/21, ··· phyllotaxis mentioned earlier can be clearly made out from the drawing. After leaf No. 1, three leaves are

formed by the time the spiral has completed one full revolution (and a little more); five leaves are formed in two revolutions (a little less); eight leaves are formed after three revolutions; twenty-one leaves are formed after eight revolutions, and so on. It is also clear from the figure that the figures 1/3, 2/5, 8/21, ··· are not absolute figures since the specified numbers of leaves cover a little more or a little less distance than what they actually stand for. This is again in accordance with the mathematical properties of Fibonacci numbers.

I thank Mr. S. K. De, our Artist, for making the drawings.

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PELL IDENTITIES

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1. INTRODUCTION

Recent issues of this Journal have contained several interesting special results involving Pell numbers. Allowing for extension to the usual Pell numbers to negative subscripts, we define the Pell numbers by the Pell sequence $\{P_n\}$ thus:

$$(1) \qquad \{P_n\}: \begin{array}{cccccccccccc} \cdots & P_{-4} & P_{-3} & P_{-2} & P_{-1} & P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & \cdots \\ \cdots & -12 & 5 & -2 & 1 & 0 & 1 & 2 & 5 & 12 & 29 & \cdots \end{array}$$

in which

$$(2) \qquad P_0 = 0, \qquad P_1 = 1, \qquad P_{n+2} = 2P_{n+1} + P_n$$

and

$$(2') \qquad P_{-n} = (-1)^{n+1} P_n \ .$$

The purpose of this article is to urge a greater use of the properties of the generalized recurrence sequence $\{W_n(a,b;\ p,q)\}$, discussed by the author in a series of papers [2], [3], and [4]. The Pell sequence is but a special case of the generalized sequence.

2. THE SEQUENCE $\{W_n(a,b;\ p,q)\}$

Our generalized sequence $\{W_n(a,b;\ p,q)\}$ is defined [2] as

$$(3) \qquad \{W_n\}: \begin{array}{ccccccc} \cdots & W_{-1}, & W_0, & W_1, & W_2, & W_3, & W_4 \ \cdots \\ \cdots & \dfrac{pa-b}{q}, & a, & b, & pb-qa, & p^2b-pqa-qb, & \cdots \ \cdots \end{array}$$

in which

$$W_0 = a, \qquad W_1 = b, \qquad W_{n+2} = pW_{n+1} - qW_n , \tag{4}$$

where a, b, p, q are arbitrary integers at our disposal.

The Pell sequence is the special case for which

$$a = 0, \qquad b = 1, \qquad p = 2, \qquad q = -1, \tag{5}$$

i.e., $P_n = W_n(0, 1; 2 - 1)$.

From the general term W_n [2], namely,

$$W_n = \frac{b - \alpha\beta}{\alpha - \beta}\alpha^n + \frac{a\alpha - b}{\alpha - \beta}\beta^n , \tag{6}$$

where

$$\begin{cases} \alpha = (p + d)/2, \quad \beta = (p - d)/2 \\ d = (p^2 - 4q)^{1/2} \end{cases} , \tag{7}$$

We have, for the Pell sequence, using (5),

$$\begin{cases} d = 2^{3/2} \\ \alpha = 1 + \sqrt{2} \\ \beta = 1 - \sqrt{2} \end{cases} \tag{8}$$

so that, from (5), (6) and (8), the nth term of the Pell sequence is

$$P_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2^{3/2}} . \tag{9}$$

A generating function for $\{W_n\}$, namely [4],

(10) $$\frac{a + (b - pa)x}{1 - pz + qx^2} = \sum_{n=0}^{\infty} W_n x^n$$

becomes, using (5) for $\{P_n\}$,

(11) $$\frac{x}{1 - 2x - x^2} = \sum_{n=0}^{\infty} W_n x^n$$

(11') $$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} P_{n+1} x^n .$$

Associated with $\{W_n\}$ is [2] the characteristic number

(12) $$e = pab - qa^2 - b^2$$

with Pell value

(13) $$ep = -1$$

by (5).

Another special case of subsequent interest to us in (32) is the sequence $\{U_n(p,q)\}$ defined by

(14) $$U_0 = 0, \qquad U_1 = 1, \qquad U_{n+2} = pU_{n+1} - qU_n ,$$

i.e.,

$$U_n(p,q) = W_n(0, 1; p,q) ,$$

for which

(15) $$e_U = -1$$

and

$$U_{-n} = -q^{-n} U_n \ . \tag{16}$$

Result (16) was noted long ago by Lucas [6], p. 308, to whom much of the knowledge of sequences like $\{U_n(p,q)\}$ is due. Obviously, by (5) and (14),

$$P_n = U_n(2, -1) \ . \tag{17}$$

3. PELL IDENTITIES

Specific Pell identities to which we refer are:

$$P_k = \sum_{r=0}^{[(k-1)/2]} \binom{k}{2r+1} 2^r \tag{18}$$

$$P_{2k} = \sum_{r=1}^{k} \binom{k}{r} 2^r P_r \tag{19}$$

$$P_{2n+1} = P_n^2 + P_{n+1}^2 \tag{20}$$

$$P_{2n+1} + P_{2n} = 2P_{n+1}^2 - 2P_n^2 - (-1)^n \tag{21}$$

$$(-1)^n P_a P_b = P_{n+a} P_{n+b} - P_n P_{n+a+b} \ . \tag{22}$$

These identities occur as Problems B-161 [5], B-161 [5], B-136 [7], B-137 [7], and B-155 [8], respectively.

Identity (18) follows readily from formula (3.20) of [2]:

$$\begin{aligned} 2^n W_n = a \sum_{j=0}^{[n/2]} p^{n-2j} d^{2j} \binom{n}{2j} \\ + (2b - pa) \sum_{j=0}^{[(n-1)/2]} \binom{n}{2j+1} p^{n-2j-1} d^{2j} \end{aligned} \tag{23}$$

on using (5) and (8).

Identity (19) follows from formula (3.19) of [2]:

$$W_{2n} = (-q)^n \sum_{j=0}^{n} \binom{n}{j} \left(\frac{-p}{q}\right)^{n-j} W_{n-j} \tag{24}$$

on using (5) and recognizing that

$$\sum_{r=0}^{k} \binom{k}{r} 2^{k-r} P_{k-r} = \sum_{r=1}^{k} \binom{k}{r} 2^r P_r \ . \tag{25}$$

Employing the formula (3.14) of [2] and replacing U_n therein (and subsequently as required) by U_{n+1} in accordance with (14) to get

$$W_{n+r} = W_r U_{n+1} - qW_{r-1} U_n \ , \tag{26}$$

we put $r = n + 1$, and identity (20) follows immediately with the aid of (5) and (17).

Furthermore, (20) may simply be obtained from formula (4.5) of [2]:

$$W_{n+r} W_{n-r} = W_n^2 + e q^{n-r} U_r^2 \tag{27}$$

on choosing $r = n + 1$ and utilizing (1), (5), (13) and (17). ($P_{-1} = P_1 = 1$.)

An immediate consequence of (26) is, by (5) and (17), the result

$$P_{n+r} = P_r P_{n+1} + P_{r-1} P_n \ . \tag{28}$$

Setting $r = n$ in (28), we deduce that

$$P_{2n} = P_n (P_{n+1} + P_{n-1}) \ . \tag{29}$$

From (27), with $r = 1$ and using (5), (13) and (17) ($P_1 = 1$), we have

$$P_{n+1} P_{n-1} - P_n^2 = (-1)^n \ . \tag{30}$$

Now, to prove identity (21), merely add (20) and (29). Then

$$\begin{aligned} P_{2n+1} + P_{2n} &= P_{n+1}^2 + (P_{n+1}P_{n-1} - (-1)^n) + P_n(2P_{n+1} - 2P_n) \\ &= P_{n+1}^2 + P_{n+1}(P_{n+1} - 2P_n) - (-1)^n + P_n(2P_{n+1} - 2P_n) \\ &= 2(P_{n+1}^2 - P_n^2) - (-1)^n \end{aligned}$$

on using (2) twice, and (30).

Next, consider formula (4.18) of [2]:

$$W_{n-r}W_{n+r+t} - W_nW_{n+t} = eq^{n-r}U_rU_{r+t}. \tag{31}$$

Put $r = -a$, $b = r + t$, $t = a + b$ in (31). Using (2'), (5), (13), and (17), we observe that identity (22) evolves without difficulty.

4. CONCLUDING COMMENTS

I. Problem B-174, proposed by Zeitlin [10] from the solution to Problem B-155 [8], namely, to show that

$$U_{n+a}U_{n+b} - U_nU_{n+a+b} = q^nU_aU_b, \tag{32}$$

is proved for identity (22) from (31) on using (14), (15), and (16).

II. Discussing briefly the sequence $\{T_n\}$ for which

$$T_n = Ar^n + Bs^n, \tag{33}$$

where

$$r = \frac{1+\sqrt{5}}{2}, \qquad s = \frac{1-\sqrt{5}}{2} \tag{34}$$

and A, B depend on initial conditions, Bro. Brousseau [1] asks, and answers, the questions:

(i) Which sequences have a limiting ratio T_n/T_{n-1}?

(ii) Which sequences do not have a limiting ratio?

(iii) On what does the limiting ratio depend?

He finds that

$$\lim_{n\to\infty}\left[\frac{T_n}{T_{n-1}}\right] = r \ . \tag{35}$$

This accords with our more general result (3.1) of [2]:

$$\lim_{n\to\infty}\left[\frac{W_n}{W_{n-1}}\right] = \begin{matrix} \alpha & \text{if} & |\beta| \le 1 \\ \beta & \text{if} & |\alpha| \le 1 \end{matrix} \ , \tag{36}$$

where α, β are defined in (7). Result (36) probably answers Bro. Brousseau's queries (i), (ii), (iii) from a slightly different point of view.

Clearly, the particular sequence he quotes, namely, the one defined by

$$T_1 = 5, \quad T_2 = 9, \quad T_{n+2} = 3T_{n+1} - 4T_n \ , \tag{37}$$

i.e., our $\{W_n(5,9;\ 3,4)\}$, cannot converge to a real limit, since by (7),

$$\begin{cases} \alpha = (3 + i\sqrt{7})/2 \\ \beta = (3 - i\sqrt{7})/2 \end{cases} \tag{38}$$

which are both complex numbers.

III. Corresponding to the specifically stated Pell identities (18)-(22), and to the incidental Pell identities (28)-(30), one may write down identities for the

$$\begin{cases} \text{Fibonacci sequence} & \{F_n\} = \{W_n(0,\ 1;\ 1,\ -1)\} \\ \text{Lucas sequence} & \{L_n\} = \{W_n(2,\ 1;\ 1,\ -1)\} \\ \text{Generalized sequence} & \{H_n(s,r)\} = \{W_n(s,\ r;\ 1,\ -1)\} \end{cases} . \tag{39}$$

Readers are invited to explore these pleasant mathematical pastures. Reversing our previous procedure of using the $\{W_n\}$ sequence to obtain special

(Pell) identities, one could be motivated to discover generalized W_n identities commencing with only a simple recurrence-relation result.

Consider, for example, the relationship

$$F_n^2 + F_{n+3}^2 = 2(F_{n+1}^2 + F_{n+2}^2) . \tag{40}$$

an aesthetically attractive result known in embryonic form, at least, in 1929 when it was described in a philosophical article by D'Arcy Thompson [9] as "another of the many curious properties" of $\{F_n\}$. Readily, we have

$$L_n^2 + L_{n+3}^2 = 2(L_{n+1}^2 + L_{n+2}^2) \tag{41}$$

$$H_n^2 + H_{n+3}^2 = 2(H_{n+1}^2 + H_{n+2}^2) . \tag{42}$$

Not unexpectedly, the results (40)-(42) are alike simply because we have $p = 1$, $q = -1$ for each of the sequences concerned. But what, we ask, will happen in the case of the Pell sequence, for which $p = 2$, $q = -1$?

Proceeding to the generalized situation, we find

$$\begin{aligned} W_n^2 + W_{n+3}^2 = q^{-2}\,(p^2q^2 + 1)W_{n+2}^2 + (p^2 + q^4)W_{n+1}^2 \\ - 2p(q^3 + 1)W_{n+1}W_{n+2} . \end{aligned} \tag{43}$$

Pell's sequence reduces (43) to

$$P_n^2 + P_{n+3}^2 = 5(P_{n+1}^2 + P_{n+2}^2) . \tag{44}$$

IV. By now, the message of this article should be evident. Simply, it is this:

While the discovery of individual properties of a particular sequence, elegant though they may be, is a satisfying experience, I believe that a more fruitful mathematical enterprise is an investigation of the properties of the generalized sequence $\{W_n\}$. In this way, otherwise hidden relationships are brought to light. To this objective, I commend the reader.

[Continued on page 263.]

FIBONACCI SYSTEM IN AROIDS

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INTRODUCTION

The Aroids (family Araceae) are a group of attractive ornamental plants which include the very familiar Aglacnemas, Alocasias, Anthuriums, Arums, Caladiums, Colocasias, Dieffenbachias, Monsteras, Philodendrons, Scindapsuses and Spathyphyllums. The numerous species and varieties of some of them bearing leaves of different sizes, shapes and colors, and a most attractive fleshy spathe (cover) that surrounds the cylindrical inflorescence (flower bunch) spoken of as the spadix, are popular throughout the world. The spadix stands distinctive against the background of the hoodlike spathe. The fleshy cylindrical column of the spadix bears a multitude of stalkless flowers. Usually there are three kinds of flowers in a spadix — the females which occupy the lowest portion, the males the topmost region and the bisexual flowers located between the males and females. One would hardly believe that these flowers are packed in a mathematical pattern.

In most Aroids, clear spirals are discernible on the arrangement of the flowers. The numbers of these spirals generally synchronize with Fibonacci Numbers. But in some species they do not. Observations were made on a number of spadices each of 20 species of Aroids at the Royal Agri-Horticultural Society's garden at Calcutta in 1970, and the numbers of spirals in each of them recorded. In each inflorescence which follows the Fibonacci system positively and where the flowers are arranged in spirals, one can trace out the spirals running clockwise as well as counter-clockwise. The spirals in a spadix numerically always happen to be two consecutive Fibonacci numbers. According to the size of the inflorescence, the numbers of spirals generally vary, the thinner ones having smaller numbers. Moreover, in a species where the numbers of spirals are, say, 5 and 8, some individuals have the five spirals moving clockwise (and the eight spirals, counter-clockwise). In other individuals of the same species, the reverse is the situation, which is

like the left- and right-handedness reported for the coconut and other palms (Davis, 1971).

PRESENTATION OF DATA

A number of spadices from six Anthurium species were examined. In addition to recording the numbers of spirals veering to the left and to the right, the total length of the spadix and its maximum thickness were also measured. The spadix of Anthurium is slender, elongated and uniform bearing only bisexual flowers (Fig. 1). Data on Anthurium macrolobium are presented in Table 1.

Table 1
Data on 20 Spadices of Anthurium Macrolobium

No.	Spirals Left	Spirals Right	Length (cm)	Diameter (cm)
1	5	8	18.4	1.2
2	5	8	15.4	1.3
3	8	5	11.8	1.2
4	8	5	11.1	0.9
5	5	8	14.1	1.2
6	5	8	12.1	1.0
7	5	8	9.9	1.1
8	8	5	15.0	0.9
9	5	8	12.2	0.9
10	5	8	13.6	1.1
11	8	5	12.1	0.8
12	8	5	15.3	1.0
13	8	5	11.8	0.8
14	5	8	16.2	1.3
15	5	8	12.7	1.1
16	8	5	14.1	1.2
17	5	8	11.5	1.0
18	8	5	11.9	0.9
19	5	8	12.8	0.8
20	5	8	14.8	1.2
			266.8	20.9

L 5 and R 8 = 12
L 8 and R 5 = 8
Length of spadix = 13.34 cm
Maximum thickness = 1.05 cm

Fig. 1 Spadices of Five Species of Anthurium

Without an exception, all the spadices of Anthurium macrolobium bear floral spirals synchronizing the Fibonacci numbers 8 and 5, and they happen to be two consecutive stages in the sequence. The number of spadices having 5 left-veering spirals is higher than those with 8 left-veering spirals, but the difference is not statistically significant.

The summarized data on the spadices of Anthurium clarinervum are given below.

Data on Anthurium clarinervum

Left 8 and Right 13	=	5
Left 13 and Right 8	=	2
Left 8 and Right 5	=	1
Length of spadix	=	14.2 cm
Maximum thickness	=	0.6 cm

While a great majority of the spadices in the above species showed 13 and 8 spirals, one manifested the next lower numbers, i. e., 8 and 5.

In Anthurium ornatum, all the spadices examined showed 13 and 8 spirals as per data shown below.

Data on Anthurium ornatum

Left 8 and Right 13	=	8
Left 13 and Right 8	=	4
Others	=	Nil
Length of spadix	=	9.43 cm
Maximum thickness	=	0.89 cm

In Anthurium polyrrhizum and Anthurium andraeanum rubrum, the numbers of floral spirals are fixed at 13 and 8 and no exception was met with as the data in Table 2 show.

Table 2
Number of Spadices for Anthurium polyrrhizum and andraeanum rubrum

Spiral numbers	Number of Spadices	
	polyrrhizum	andraeanum rubrum
L 8 and R 13	6	6
L 13 and R 8	3	4
Others	Nil	Nil
Length of Spadix	7.43 cm	7.52 cm
Maximum Thickness	0.71 cm	0.73 cm

Anthurium crassinervum is a species having a larger spadix, and accordingly, it shows still higher numbers of floral spirals, i.e., 13 and 21 as per data shown below.

Data on Anthurium crassinervum

Left 13 and Right 21	=	8	
Left 21 and Right 13	=	6	
Left 11 and Right 18	=	1	(Lucas)
Length of spadix	=	15.04 cm	
Maximum thickness	=	1.39 cm	

The aberrant spadix bears spiral numbers short of 2 (left) and 3 (right) to match the rest. Incidentally, these are also Fibonacci numbers.

Schizocasia poteia, with an exception of one spadix, shows 13 and 8 floral spirals.

A Spathyphyllum spadix also conforms more or less to the Fibonacci system by displaying 8 and 5 floral spirals in a great majority of the spadices as per data given below.

Data on Spathyphyllum Spadices

Left 8 and Right 5	=	10
Left 5 and Right 8	=	3
Left 6 and Right 5	=	1
Left 6 and Right 7	=	1
Left 6 and Right 8	=	1
Length of spadix	=	7.66 cm
Maximum thickness	=	0.76 cm

Out of the total 16 spadices examined, three did not conform to the Fibonacci system, even though in two of them, the right-veering spirals numbering 5 and 8 show their affinity to the Fibonacci system.

The four species of Dieffenbachia whose spadices were examined (Dieffenbachia picta viridis, Dieffenbachia picta, Dieffenbachia dagneus, and an unidentified species) have their female flowers, which are much larger and sparser, are borne only on one side of the flattened spadix, the opposite side being fused with the spathe. However, it was possible to make out 5 and 3 spirals out of the arrangement of these female flowers. Six of the 9 spadices

of Diffenbachia dagneus had five left-moving spirals and three right-moving ones. In the rest, a reverse order was noticed.

In the species of Dieffenbachia studied, only very few bisexual flowers were present, and this region of the column is considerably barren. The upper region consisting of the closely packed male flowers is quite prominent. Although regular spiral arrangement could be made out on these flowers, the numbers of spirals were not always Fibonacci numbers. Moreover, in many of them the numbers moving to the left and to the right were similar. The data on 18 spadices of Dieffenbachia picta viridis are shown below.

Data on Dieffenbachia picta viridis

Left 8 and Right 8	=	15
Left 8 and Right 7	=	2
Left 7 and Right 8	=	1
Length of spadix	=	12.39 cm

Similar data for Dieffenbachia picta given below relating to 17 spadices show a greater variation.

Data on Dieffenbachia picta

Left 7 and Right 7	=	1
Left 6 and Right 6	=	3
Left 3 and Right 4	=	1
Left 5 and Right 6	=	5
Left 6 and Right 5	=	6
Left 7 and Right 6	=	1

In some species like Aglaonema commutation and Arisaema ringens, a number of spadices each were examined. But it was very difficult to make out regular spirals in them.

In another unidentified species of Aglaonema, the following data were obtained from 13 spadices. Similar data on Syngonium spadices which closely resembles Aglaonema spadices are also shown in Table 3 with those for Aglaonema.

Table 3
Data for *Aglaonema* Spadices and *Syngonium* Spadices

Spiral numbers	*Aglaonema* Spadices	*Syngonium* Spadices
L 5 and R 5	9	5
L 5 and R 6	1	4
L 5 and R 7	Nil	1
L 5 and R 8	1	Nil
L 6 and R 5	1	2
L 6 and R 6	Nil	5
L 7 and R 5	1	Nil
L 7 and R 6	Nil	1
L 7 and R 7	Nil	2
Total	13	20
Length of spadix	2.96 cm	3.26 cm
Maximum thickness	0.7 cm	1.83 cm

The varying patterns in the number of floral spirals in an *Alocasia* spadix and in *Alocasia indica mettalica* are given in Table 4.

Table 4
Data on Two Alocasia Spadices

Alocasia Spadix	*Alocasia indica mettalica*
L 5 and R 5 = 1	L 9 and R 9 = 2
L 6 and R 6 = 4	L 11 and R 11 = 1
L 7 and R 7 = 4	L 9 and R 8 = 1
L 5 and R 6 = 1	L 9 and R 10 = 1
L 6 and R 7 = 2	L 10 and R 11 = 3
L 6 and R 8 = 1	L 11 and R 9 = 1
L 7 and R 6 = 3	L 12 and R 11 = 1
L 7 and R 8 = 1	Total 10
L 8 and R 7 = 1	Length of spadix = 17.65 cm
Total 18	Maximum thickness = 0.98 cm
Length of spadix = 3.08	
Maximum thickness = 1.76 cm	

A species of Caladium also showed irregularities by exhibiting 9 to 12 spirals, none of them synchronizing a Fibonacci number.

Given below are data on Philodendron spirals relating to 15 spadices.

Data on Philodendron spirals

L 12 and R 12	=	2
L 13 and R 13	=	7
L 15 and R 15	=	1
L 12 and R 13	=	1
L 14 and R 13	=	2
L 14 and R 15	=	1
L 15 and R 14	=	1
Length of spadix	=	12.88 cm
Maximum thickness	=	2.89 cm

In this species, a majority of the spadices possess equal numbers of floral spirals running clockwise and counter-clockwise. Also in many spadices, the number of spirals do not synchronize Fibonacci numbers.

DISCUSSION

To the list of pine cones (Brousseau, 1968), palms (Davis, 1971), sunflowers and the very many situations in plants arising out of alternate arrangement of leaves, may be included the Aroids so far as their affinity to the Fibonacci sequence is concerned.

Among the aroids, the several species of Anthurium show, without an exception, floral spirals whose numbers synchronize the Fibonacci numbers. This is due to the fact that any two consecutive flowers subtend between them an angular deflection which make with the remaining angle to complete one full revolution, the familiar golden ratio. At the tip of these spadices, the flowers end in smaller numbers of spirals. From the very last flower which can be easily made out in these cases, its nearness to the just preceding one can be made out. All the five spadices in Fig. 1 show this arrangement. Moreover, these compact flower-bunches although taper smoothly, there is no irregularity of any sort in any region. On the other hand, the five spadices shown in Fig. 2 are uneven, the last one (Monstera deliciosa) being an exception. By careful observation of the female flowers, spiral numbers which

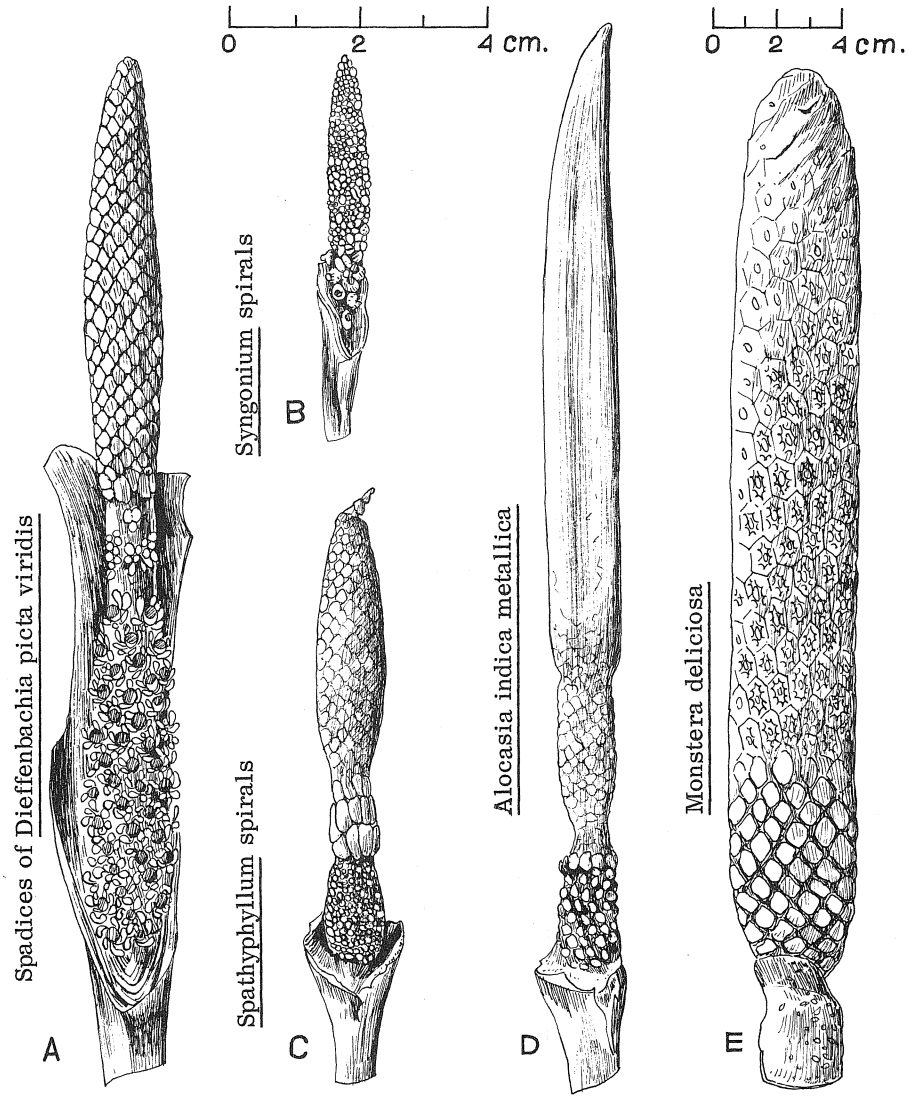

Fig. 2 Spadices of Dieffenbachia picta viridis

synchronize Fibonacci numbers can be made out. The bisexual flowers distributed in the narrow region are devoid of any obvious spirals. Moreover, at this region, the spadix remains considerably narrow. It is at this region, presumably, the angle between consecutive flowers undergo a change. As a result, the male flowers at the upper region either do not fall in regular spirals, or the spirals do not conform to Fibonacci numbers. In the same

species, different spadices show much differing numbers of spirals. A better understanding of the phyllotaxis in these species may be essential to study the cause of such a variation.

Pineapples as well as the male and female reproductive bodies (cones) of many species of Cycas show clear spirals in the arrangement of the individual fruits and generative leaves respectively, and these numbers are always Fibonacci numbers. In small pineapples (Fig. 3), 3 and 5 spirals are visible. As in Aroids and palms, the 3 spirals in a pineapple may veer clockwise or counter-clockwise. In a larger variety, there are 5 and 8 spirals, and in still larger pineapples, there are 8 and 13 spirals. In some exceptionally larger ones, even 13 and 21 spirals can be made out.

Among Cycas, too, there are species which show 3 and 5 spirals and the numbers in some other species may go up to 13 and 21 as in the male cone of Cycas circinalis seen in Fig. 4.

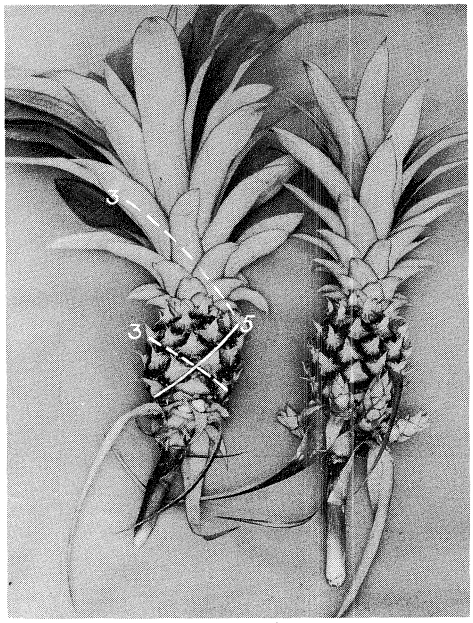

Fig. 3 Pineapples showing 3 and 5 spirals

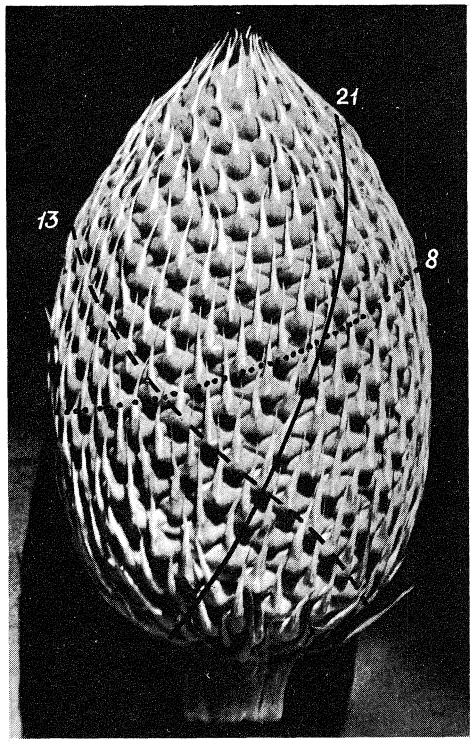

Fig. 4 Cones of Cycas circinalis showing 3 and 5 spirals

We thank Mr. S. K. De, artist of the Indian Statistical Institute, for the drawings.

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SOME PROPERTIES OF CERTAIN GENERALIZED FIBONACCI MATRICES

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INTRODUCTION

1. In this paper, we will derive a number of identities for the generalized Fibonacci sequence $\{H_n\}$ of Horadam [4] defined by the second-order recurrence relation

$$H_{n+2} = H_{n+1} + H_n \qquad \text{(n an integer, unrestricted)} \quad , \tag{1.1}$$

with initial values

$$H_0 = q \qquad \text{and} \qquad H_1 = p \quad , \tag{1.2}$$

by the use of generalized (square) Fibonacci matrices.

2. A generalized Fibonacci matrix is a matrix whose elements are generalized Fibonacci numbers.

3. The technique adopted is basically paralleling that due to Hoggatt and Bicknell [1], [2], and [3], where we establish numerous identities by examining the lambda functions or the characteristic equations of certain generalized Fibonacci matrices.

4. If we were to proceed as in Hoggatt and Bicknell [1] by selecting the 2-by-2 matrix defined by

$$A = \begin{bmatrix} p+q & p \\ p & q \end{bmatrix} \quad , \tag{4.1}$$

which becomes the Q matrix of [1] when $q = 0$ and $p = 1$, and where

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*Part of the substance of an M. Sc. thesis presented to the University of New England in 1968.

$|A| = -d$ where $d = p^2 - pq - q^2$ (which is the e of [4]), we would find that we would be unable to obtain a compact expression for the matrix A^n.

5. Instead, we commence our investigations by starting with the generalized Fibonacci matrix defined by

$$A_n = \begin{bmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{bmatrix} \tag{5.1}$$

where

$$\begin{aligned} |A_n| &= H_{n+1}H_{n-1} - H_n^2 \\ &= (-1)^n d \end{aligned} \tag{5.2}$$

Then the matrix A defined by (4.1) is a special case of A_n when $n = 1$. The matrix A_n becomes the matrix Q^n of [1] when $q = 0$ and $p = 1$. This approach is used throughout this paper where, by changing the powers of various characteristic equations to suffixes, we are able to develop numerous easily verified identities.

THE LAMBDA FUNCTION

6. We adopt the definition of the <u>lambda function</u> $\lambda(M)$ of the matrix M used by Hoggatt and Bicknell [1] where, if a_{ij} is the $i - j^{th}$ element in M, then

$$\lambda(M) = |a_{ij} + 1| - |a_{ij}| \tag{6.1}$$

7. Thus, for the Fibonacci matrix A_n defined by (5.1), we have

$$\begin{aligned} \lambda(A_n) &= \begin{vmatrix} H_{n+1} + 1 & H_n + 1 \\ H_n + 1 & H_{n-1} + 1 \end{vmatrix} - |A_n| \\ &= H_{n-3} \end{aligned} \tag{7.1}$$

on simplification.

Hence, from (7.1) and the easily verified identity (1) of [1], viz:

$$|a_{ij} + k| = |a_{ij}| + k\lambda(M) \quad , \tag{7.2}$$

we have

$$\begin{vmatrix} H_{n+1} + k & H_n + k \\ H_n + k & H_{n-1} + k \end{vmatrix} = (H_{n+1}H_{n-1} - H_n^2) + k(H_{n-1} + H_{n+1} - 2H_n) \tag{7.3}$$

$$= |A_n| + kH_{n-3}$$

8. For a 3-by-3 matrix, the associated lambda function may be found more conveniently by the application of a theorem of [1], where, for the matrix

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

$$\lambda(M) = \begin{vmatrix} 1 & b & c \\ 1 & e & f \\ 1 & h & j \end{vmatrix} + \begin{vmatrix} a & 1 & c \\ d & 1 & f \\ g & 1 & j \end{vmatrix} + \begin{vmatrix} a & b & 1 \\ d & e & 1 \\ g & h & 1 \end{vmatrix} \tag{8.1}$$

or

$$\lambda(M) = \begin{vmatrix} a + e - (b + d) & b + f - (c + e) \\ d + h - (g + e) & e + j - (h + f) \end{vmatrix} \quad . \tag{8.2}$$

For example, consider the generalized Fibonacci matrix E, where

$$E = \begin{bmatrix} H_{2p} & H_{2p+1} & H_m \\ H_{2p+1} & H_{2p+2} & H_m \\ H_{2p+2} & H_{2p+3} & H_m \end{bmatrix} \tag{8.3}$$

so that

$$
\begin{aligned}
|E| &= H_m[H_{2p+1}H_{2p+3} - H_{2p+2}^2 - H_{2p}H_{2p+3} + H_{2p+1}H_{2p+2} \\
&\qquad + H_{2p}H_{2p+2} - H_{2p+1}^2] \\
&= H_m[H_{2p+1}H_{2p+2} - H_{2p}H_{2p+3}] \qquad (8.4) \\
&= (-1)^{2(p+1)}\, d\, H_m \\
&= d\, H_m
\end{aligned}
$$

on using (12) of Horadam [4] where $n = 2p + 1$, $r = 0$, and $s = 1$.

One may evaluate $\lambda(E)$ by the use of (8.1) and a few simple column operations, whence

$$\lambda(E) = d \; . \tag{8.5}$$

The matrix E defined by (8.3) reduces to the matrix U of [1].

9. If we let $k = H_{m-1}$ in (7.2), we have

$$
\begin{aligned}
|E + H_{m-1}| &= |E| + H_{m-1} \cdot d \\
&= d H_m + d H_{m-1} \qquad (9.1) \\
&= d H_{m+1} \; .
\end{aligned}
$$

Similarly, if we put $k = H_n$ in (7.2), then we have

$$
\begin{aligned}
|A_n + H_n| &= \begin{vmatrix} H_{n+1} + H_n & 2H_n \\ 2H_n & H_{n-1} + H_n \end{vmatrix} \qquad (9.2) \\
&= |A_n| + H_n \lambda(A_n)
\end{aligned}
$$

so that, by (5.2) and (7.1),

$$\begin{vmatrix} H_{n+2} & 2H_n \\ 2H_n & H_{n+1} \end{vmatrix} = (-1)^n d + H_n H_{n-3} \tag{9.3}$$

from which we have

(9.4) $$4H_n^2 = H_{n+2}H_{n+1} - H_nH_{n-3} + (-1)^{n+1}d .$$

10. From Paragraphs 6 to 9, we can see that it is possible to derive many identities for the generalized Fibonacci sequence $\{H_n\}$ by the use of generalized Fibonacci matrices and the lambda function.

CHARACTERISTIC EQUATIONS

11. As a special case of the generalized Fibonacci matrix

(11.1) $$W_n = \begin{bmatrix} H_{n-1}^2 & H_{n-1}H_n & H_n^2 \\ 2H_{n-1}H_n & H_{n+1}^2 - H_{n-1}H_n & 2H_nH_{n+1} \\ H_n^2 & H_nH_{n+1} & H_{n+1}^2 \end{bmatrix},$$

when $n = 1$, we have the matrix W (say) where, on calculation, we have

$$W = W_1 = \begin{bmatrix} q^2 & pq & p^2 \\ 2pq & (p+q)^2 - pq & 2p(p+q) \\ p^2 & p(p+q) & (p+q)^2 \end{bmatrix}$$

whence

(11.2) $$|W| = -d^3$$

Since the Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation, namely,

$$|W - \lambda I| = \lambda^3 - h\lambda^2 - dh\lambda + d^3 = 0 ,$$

W satisfies the equation

(11.3) $$W^3 - hW^2 - dhW + d^3I = 0$$

where $h = 2p^2 + 3pq + 3q^2$.

Hence, from (11.3), we have, on multiplying throughout by W^n,

$$W^{n+3} - hW^{n+2} - dhW^{n+1} + d^3W^n = 0 . \tag{11.4}$$

Now, from the relations

$$\begin{cases} H_{n+3}^2 - 2H_{n+2}^2 - 2H_{n+1}^2 + H_n^2 = 0 \\ H_{n+3}H_{n+4} - 2H_{n+2}H_{n+3} - 2H_{n+1}H_{n+2} + H_nH_{n+1} = 0 \\ H_{n+4}^2 - H_{n+2}H_{n+3} - 2H_{n+3}^2 + 2H_{n+1}H_{n+2} - 2H_{n+2}^2 + 2H_nH_{n+1} \\ \qquad + H_{n+1}^2 - H_{n-1}H_n = 0 \end{cases} \tag{11.5}$$

and so on, we can form the matrices W_{n+3}, W_{n+2}, and W_{n+1}, which will satisfy the recurrence relation

$$W_{n+3} - 2W_{n+2} - 2W_{n+1} + W_n = 0 \tag{11.6}$$

adapted from Eq. (11.4) by analogy with the special case for the ordinary Fibonacci sequence $\{F_n\}$ for which $p = 1$, $q = 0$, $h = 2$, $d = 1$.

As a special case of (11.6) for $n = 0$, we may re-write

$$W_3 - 2W_2 - 2W_1 + W_0 = 0 \tag{11.7}$$

in the equivalent form

$$W_3 + 3W_2 + 3W_1 + W_0 = 5W_2 + 5W_1 = 5(W_2 + W_1) , \tag{11.8}$$

from which, in general, it can be shown that

$$\binom{2n+1}{0}W_{2n+1} + \binom{2n+1}{1}W_{2n} + \cdots + \binom{2n+1}{2n+1}W_0 = 5^n(W_{n-1} - W_n). \tag{11.9}$$

On equating those elements in the first row and third column, and after using (9) of Horadam [4], we can deduce the result

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} = 5^n(H_{n+1}^2 + H_n^2) \tag{11.10}$$

$$= 5^n[(2p - q)H_{2n+1} - dF_{2n+1}] .$$

12. We can find a number of identities for the generalized Fibonacci sequence $\{H_n\}$ by proceeding as in Hoggatt and Bicknell [3] as follows.

Consider the generalized Fibonacci matrix defined by

$$J_n = \begin{bmatrix} H_{2n+2} & H_{2n} \\ -H_{2n} & -H_{2n-2} \end{bmatrix} \tag{12.1}$$

where, as a special case of (12.1), we have the matrix

$$J = J_1 = \begin{bmatrix} 3p + 2q & p + q \\ -p - q & -q \end{bmatrix}$$

for $n = 1$. Since J satisfies its own characteristic equation

$$J^2 - (3p + q)J + dI = 0 , \tag{12.2}$$

we can show that

$$(J + H_2 I)^2 = H_5 J + H_0 H_4 I . \tag{12.3}$$

This leads to the equations

$$J^m(J + H_2 I)^{2n} = J^m(H_5 J + H_0 H_4 I)^n \tag{12.4}$$

and

$$\sum_{k=0}^{2n} \binom{2n}{k} H_2^{2n-k} J^{k+m} = J^m(H_5 J + H_0 H_4 I)^n . \tag{12.5}$$

From the easily verified matrix equation

$$J_2 = 3J_1 + J_0 = 0 \tag{12.6}$$

obtained from observation of Eq. (12.2), we have the rearranged equation

$$J_2 + 2J_1 + J_0 = 5J_1 \quad . \tag{12.7}$$

In general, it can be shown that the J-matrices satisfy the equation

$$\binom{2n}{0}J_{2n} + \binom{2n}{1}J_{2n-1} + \cdots + \binom{2n}{2m}J_0 = 5^n J_n \quad , \tag{12.8}$$

whence

$$\sum_{k=0}^{2n} \binom{2n}{k} J_k = 5^n J_n \quad . \tag{12.9}$$

Hence, on equating those elements in the first row and second column, we have

$$\sum_{k=0}^{2n} \binom{2n}{k} H_{2k} = 5^n H_{2n} \quad . \tag{12.10}$$

13. If we now consider the same auxiliary matrix S as in [3], viz:

$$S = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} , \tag{13.1}$$

we have, on calculation:

$$J_n S = \begin{bmatrix} H_{2n+3} & H_{2n+1} \\ -H_{2n+1} & -H_{2n-1} \end{bmatrix} \tag{13.2}$$

By proceeding as in Paragraph 12, we can similarly establish the summation

$$\sum_{k=0}^{2n} \binom{2n}{k} J_k S = 5^n J_n S \ , \tag{13.3}$$

from which we deduce the result

$$\sum_{k=0}^{2n} \binom{2n}{k} H_{2k+1} = 5^n H_{2n+1} \ . \tag{13.4}$$

Similarly, we can generalize the equation

$$J_3 + 3J_2 + 3J_1 + J_0 = 5(J_2 + J_1) \tag{13.5}$$

so that

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} J_k = 5^n [J_{n+1} + J_n] \ , \tag{13.6}$$

from which we deduce that

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} H_{2k} = 5^n [H_{2n+2} + H_{2n}] \tag{13.7}$$

Again, we have the summation

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} J_k S = 5^n [J_{n+1} S + J_n S] \tag{13.8}$$

from which we have

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} H_{2k+1} = 5^n [H_{2n+3} + H_{2n+1}] \quad . \tag{13.9}$$

Finally, since we may re-write (12.6) in the form

$$J_2 - 2J_1 + J_0 = J_1 \quad , \tag{13.10}$$

we have, in general, that

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} J_k = J_n \quad , \tag{13.11}$$

so that, as before, we have the summation

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} H_{2k} = H_{2n} \quad . \tag{13.12}$$

Similarly, from the summation

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} J_k S = J_n S \tag{13.13}$$

we deduce the result

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} H_{2k+1} = H_{2n+1} \quad . \tag{13.14}$$

FURTHER SUMMATION IDENTITIES

14. As in [3], we can continue to establish further identities for the generalized Fibonacci sequence $\{H_n\}$ by letting

$$G_nS_0 = \begin{bmatrix} H_{4n+4} & H_{4n} \\ -H_{4n} & -H_{4n-4} \end{bmatrix} \quad \text{where } S_0 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$G_nS_1 = \begin{bmatrix} H_{4n+5} & H_{4n+1} \\ -H_{4n+1} & -H_{4n-3} \end{bmatrix} \qquad S_1 = \begin{bmatrix} 5 & 1 \\ -1 & -2 \end{bmatrix}$$

(14.1)

$$G_nS_2 = \begin{bmatrix} H_{4n+6} & H_{4n+2} \\ -H_{4n+2} & -H_{4n-2} \end{bmatrix} \qquad S_2 = \begin{bmatrix} 8 & 1 \\ -1 & 1 \end{bmatrix}$$

$$G_nS_3 = \begin{bmatrix} H_{4n+7} & H_{4n+3} \\ -H_{4n+3} & -H_{4n-1} \end{bmatrix} \qquad S_3 = \begin{bmatrix} 13 & 2 \\ -2 & -1 \end{bmatrix}$$

so that we have

$$G_n = \frac{1}{3}\begin{bmatrix} H_{4n+4} & H_{4n} \\ -H_{4n} & -H_{4n-4} \end{bmatrix} \tag{14.2}$$

As a special case of (14.2) we have, for $n = 1$, the matrix G which satisfies its characteristic equation $|G - \lambda I| = 0$, so that

$$G^2 - (7p + 4q)G + dI = 0 \ . \tag{14.3}$$

We can easily verify the matrix equation

$$G_2 - 7G_1 + G_0 = 0 \ , \tag{14.4}$$

so that, in general, we have

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} G_j = 5^n G_n . \tag{14.5}$$

Multiplying on the right by the auxiliary matrix S_s $(s = 0, 1, 2, 3)$ and equating the elements in the first row and second column gives

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} H_{4j+s} = 5^n H_{4n+s} . \tag{14.6}$$

Further, the matrix equation

$$G_3 - 3G_2 + 3G_1 - G_0 = 5(G_2 - G_1) , \tag{14.7}$$

may be generalized so that we have

$$\sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} G_j = 5^n [G_{n+1} - G_n] . \tag{14.8}$$

On postmultiplying by S_s, we have, therefore:

$$\sum_{j=0}^{2n+1} (-1)^{j+1} \binom{2n+1}{j} H_{4j+s} = 5^n [H_{4(n+1)+s} - H_{4n+s}] . \tag{14.9}$$

Again, Eq. (14.4) is equivalent to

$$G_2 + 2G_1 + G_0 = 3^2 G_1 , \tag{14.10}$$

which may be generalized to give

$$\sum_{j=0}^{2n} \binom{2n}{j} G_j = 3^{2n} G_n . \tag{14.11}$$

Postmultiplying by S_s leads to the identity

$$\sum_{j=0}^{2n} \binom{2n}{j} H_{4j+s} = 3^{2n} H_{4n+s} \quad . \tag{14.12}$$

Similarly, the matrix equation

$$G_3 + 3G_2 + 3G_1 + G_0 = 3^2(G_2 + G_1) \quad , \tag{14.13}$$

can be generalized, so that we have

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} G_j = 3^{2n}[G_{n+1} + G_n] \quad , \tag{14.14}$$

from which, on postmultiplying by S_s, we have the final identity

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} H_{4n+s} = 3^{2n}[H_{4(n+1)+s} + H_{4n+s}] \quad . \tag{14.15}$$

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ABOUT THE LINEAR SEQUENCE OF INTEGERS SUCH THAT EACH TERM IS THE SUM OF THE TWO PRECEDING

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1. The sequences of integers such that each term is equal to the sum of both preceding are infinite in number. Two of these have been especially investigated: the Fibonacci sequence, conceived at the beginning of the 13th Century by the mathematician Leonardo of Pisa, better known as Fibonacci, the Lucas sequence pointed out at the end of the last century by the French mathematician Lucas and named for him. Both sequences gave rise to many works which showed manifold properties of these sequences and conduced to strides in the numbers theory.

The present research work doesn't mean to go back on these questions, but it tends to make known how the use of the hyperbolic functions make much easier general feature works on the linear sequences defined at the beginning of the present paper, and from which Fibonacci and Lucas sequences are only special cases.[1] The author has recently had recourse to these functions in a very different field, that of mathematic geography, and he has been the first to show that their utilization simplified notably the determination of the conformal representations of the sphere or ellipsoide on the plane, that it lightened very much the algebraic expression of these representations and that it helped to state precisely the relationships existing between the different systems.

2. The sequences concerned are defined by the general relation:

$$z_n = z_{n-1} + z_{n-2} \quad , \tag{1}$$

in which z_n indicates the term of rank n.

[1]Lucas developed a very different generalization of both sequences. It will be reminded in paragraph 6.

Each sequence can therefore be characterized by two arbitrary integers which we call z_0 and z_1 and which don't seem, a priori, to be part of the sequence because they are not squaring with the definition (1); but, actually, they, too, enter into the sequence since it is possible to extend it without end, in the opposite direction, starting from the arbitrary terms z_0 and z_1.

3. The shape of the relation (1) between the successive terms of the sequences suggests immediately the use of circular or hyperbolic lines (functions) for expressing each term according to its place in the sequence. As it is a question of indefinitely increasing sequences, it is obviously suitable to have recourse to hyperbolic lines.

Let us write the relation (1) in the form:

$$z_{n+1} - z_{n-1} = z_n \ , \tag{2}$$

and designate by $m, \lambda,$ and ϕ, three constants to fix ulteriorly in terms of sequence's data. Let us set besides: either

$$z_{n+1} = m\,\mathrm{sh}\lambda\,(n + \phi + 1) \qquad \text{and} \qquad z_{n-1} = m\,\mathrm{sh}\lambda\,(n + \phi - 1)$$

or

$$z_{n+1} = m\,\mathrm{ch}\lambda\,(n + \phi + 1) \qquad \text{and} \qquad z_{n-1} = m\,\mathrm{ch}\lambda\,(n + \phi - 1).$$

Then the relation (2) conduces to:

$$z_n = 2m\,\mathrm{sh}\,\lambda\mathrm{ch}\lambda\,(n + \phi)$$

for the first case, or

$$z_n = 2m\,\mathrm{sh}\,\lambda\mathrm{sh}\,\lambda(n + \phi)$$

for the second case.

Let us define now the parameter λ by $\mathrm{sh}\,\lambda = 1/2$, from which it comes $\mathrm{ch}\lambda = \sqrt{5}/2$ and

$$\left(e = \frac{1 + \sqrt{5}}{2}\right) \text{ (golden number)}$$

Both expressions of z_n become simplified and it is obvious moreover that the terms of the sequence can be represented alternatively by hyperbolic sines and cosines

$$\text{(2 bis) } z_n = m \operatorname{ch}\lambda(n + \phi)$$

or

$$z_n = m \operatorname{sh}\lambda(n + \phi)$$

or, generally, speaking

$$z_n = m \frac{e^{\lambda(n+\phi)} \pm e^{-\lambda(n+\phi)}}{2} .$$

The parameters m and ϕ are easily obtained with the help of initial data z_0 and z_1, but it is obviously necessary to consider two cases according as one adopts for z_0, a hyperbolic sine or cosine, and the inverse for z_1. In the first case, the terms with an even index agree with hyperbolic sines, those with an odd index are represented by cosines. In the second case, the inverse occurs. To make a distinction between both cases, we shall write:

$$A = z_1 + z_0 e^{-\lambda} \qquad B = z_1 - z_0 e^{\lambda}$$

from what, taking the value of λ into consideration,

$$A - B = 2z_0 \operatorname{ch}\lambda \qquad A + B = 2z_1 - z_0 \qquad AB = z_1^2 - z_0 z_1 - z_0^2 .$$

Suppose, now, that one intends to adopt hyperbolic sines for the terms with an even index. It comes:

$$m \operatorname{sh}\lambda\phi = z_0 = \frac{A - B}{2 \operatorname{ch}\lambda} \qquad m \operatorname{ch}\lambda(\phi + 1) = z_1,$$

from what

$$m \operatorname{ch}\lambda\phi = \frac{z_1 - z_0 \operatorname{sh}\lambda}{\operatorname{ch}\lambda} = \frac{A - B}{2 \operatorname{ch}\lambda} ,$$

and therefore,

$$me^{\lambda\phi} = A/\operatorname{ch}\lambda \qquad me^{-\lambda\phi} = B/\operatorname{ch}\lambda \qquad m = \sqrt{AB}/\operatorname{ch}\lambda \qquad e^{\lambda\phi} = \sqrt{A/B}$$

B must so be positive, and we have consequently:

$$z_1 > z_0 e ,$$

either

$$z_1 > z_0 \frac{1 + \sqrt{5}}{2}$$

or

$$2z_1 - z_0 > z_0\sqrt{5} .$$

A parallel argument shows that if a hyperbolic cosine is adopted for the terms with an even index, $-B$ takes the place of B in the formulas of m and of $e^{\lambda\phi}$, and that consequently, B must be negative and

$$z_1 < z_0 \frac{1 + \sqrt{5}}{2} .$$

For example, the sequences defined by $z_0 = 3$ and $z_1 = 1$, or by $z_0 = 2$ and $z_1 = 3$ must be represented by

$$z_n = m \operatorname{ch}\lambda(n + \phi)$$

when n is even, whereas a hyperbolic sine is necessary for the sequence defined by $z_0 = 1$ and $z_1 = 2$.

Using the formulas of m and , we get the general expression

$$z_n = \frac{1}{2\,\mathrm{ch}\,\lambda}\,[Ae^{\lambda n} - B(-e^{-\lambda})^n] \tag{1}$$

Before going further in the study of the sequences, we deal first with the special case of ϕ integer; then, this parameter can be taken cipher, which is equivalent to shifting the number of the terms, the nth term receiving the index $n - 1$. The condition $\phi = 0$ produces $A = B$ if B is positive, $A = -B$ in the opposite case. Both cases correspond respectively to the Lucas and Fibonacci sequences.

The knowledge of both these sequences makes it much easier to set up formulas of the general sequence z. We add, besides, a special sequence G which also appears in the relations.

4. The Fibonacci Sequence. For this sequence, $A = B$, and consequently, $z_0 = 0$ and $z_1 = A$. Hence, for the general term,

$$z_n = \frac{z_1}{\mathrm{ch}\,\lambda}\,[e^{\lambda n} - (-e^{-\lambda})^n] \quad .$$

As no motive exists for keeping the same factor z_1 in all terms of the sequence, we can take $z_1 = 1$. Therefore, we have, with the symbol F instead of z:

$$F_n = \frac{\mathrm{sh}\lambda n}{\mathrm{ch}\,\lambda} \tag{3}$$

if n is even,

$$F_n = \frac{\mathrm{ch}\lambda n}{\mathrm{ch}\,\lambda}$$

if n is odd.

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[1]Substituting to the quantities A and B in this formula, their expressions in the terms of z_0 and z_1, one may obtain a relation which is no other than the relation (5), given further and then more directly obtained.

It would be possible to more quickly obtain these relations by departing from the usual definition $z_0 = 0$, $z_1 = 1$, and writing

$$m\ \mathrm{sh}\lambda\phi = 0 \qquad m\ \mathrm{ch}\lambda(\phi + 1) = 1 ,$$

relations giving $\phi = 0$ and $m = 1/\mathrm{ch}\lambda$.

As

$$e^{\lambda} = \frac{1 + \sqrt{5}}{2} \qquad -e^{-\lambda} = \frac{1 - \sqrt{5}}{2} ,$$

the expressions of the general term become:

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

or, more symmetrically,

$$F_n = \frac{\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n}{\frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2}}$$

and numerically,

$$F_n = \frac{(1,618 \cdots)^n - (-0,618 \cdots)^n}{2,236 \cdots} .$$

As shkx, with k integer, is always divisible by shx, and as chkx is divisible by chx when k is odd, the term F_{kn} is always divisible by F_n, which is also shown by the general formula. Specifically, the even terms have an index divisible by 3; the terms divisible by 3 have an index divisible by 4; the terms divisible by 5 have an index divisible by 5; and so on.

Likewise, when n becomes very great, which makes $\mathrm{th}\lambda n$ very near from the unity, the ratio of consecutive terms draws near to $\mathrm{ch}\lambda + \mathrm{sh}\lambda$, i.e.,

$$e^{\lambda} = \frac{1 + \sqrt{5}}{2} = 1,618 \cdots .$$

So the successive terms of the Fibonacci sequence are:

$n =$	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$F_n =$	0	1	1	2	3	5	8	13	21	34	55	89	144	$\cdots$
$\frac{\mathrm{sh}\lambda n}{\mathrm{ch}\,\lambda} =$	0		1		3		8		21		55		144	$\cdots$
$\frac{\mathrm{ch}\lambda n}{\mathrm{ch}\lambda} =$		1		2		5		13		34		89		$\cdots$.

5. The Lucas Sequence. We have seen that, for this sequence, $A = -B$, from which $z_0 = A/\mathrm{ch}\lambda$ and $z_1 = z_0/2$, and, for the general term, using the symbol L for the terms of the sequence, and taking $z_1 = 1$, as in the Fibonacci sequence, and for the same reason,

$$L_n = e^{\lambda n} + (-e^{-\lambda})^n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

and we have

$$L_n = 2\,\mathrm{ch}\lambda n \tag{4}$$

for n even, and

$$L_n = 2\,\mathrm{sh}\lambda n$$

for n odd. It would also be possible to get these expressions directly from the relations

$$m\,\mathrm{ch}\lambda\phi = z_0 = 2$$

and

$$m\ \mathrm{sh}\lambda\,(\phi + 1) = z_1 = 1$$

which give $\phi = 0$ and $m = 2$.

If one considers the product kn, the term L_{kn} is divisible by L_n when k is odd. Particularly, the terms having an index odd multiple of 3 are divisible by 4, whereas, as $\mathrm{ch}6\lambda$ is equal to 9, odd integer, the terms having for index a multiple of 6 and consequently for expression $2\mathrm{ch}6\lambda n$, are divisible by 2, and by no other power of this number, whatever the eveness of n may be.

The Lucas sequence, therefore, is as follows:

$n =$	0	1	2	3	4	5	6	7	8	9	10	11	12	...	
$L_n =$	2	1	3	4	7	11	18	29	47	76	123	199	312	...	322
$2\mathrm{ch}\ n =$	2		3		7		18		47		123		312	...	322
$2\mathrm{sh}\ n =$		1		4		11		29		76		199		...	.

6. The previous expressions of F_n and L_n in terms of

$$\frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad \frac{1 - \sqrt{5}}{2}$$

are connected with more general results set up by Edouard Lucas, who considers the functions U_n and V_n defined by

$$U_n = \frac{a^n - b^n}{a - b}$$

and

$$V_n = a^n + b^n \quad .$$

Lucas shows that $U_{2n} = U_n V_n$ (a similar formula is given further in paragraph 9) and that, on the other hand, he can write $U_n = 2 \sin n$ and $V_n = 2 \cos n$; for n real, the circular trigonometric lines fit, whereas for n imaginary, one must use hyperbolic functions.

It is also interesting to consider the quadratic equation having the roots a and b. In the special case where U_n and V_n agree, respectively, with F_n and L_n, this equation is $x^2 - x - 1 = 0$.

7. Connected Sequences. One can easily set up the relation:

$$z_n = z_0 F_{n-1} + z_1 F_n \tag{5}$$

permitting to deal with all sequences defined by relation (1) as soon as the Fibonacci sequence has been investigated.

We shall consider now that this relation (5) defined a function $G_n(z, F)$ of both sequences, and we shall spread it to any sequences y and z, writing:

$$G_n(y,z) = z_0 y_{n-1} + z_1 y_n \quad .$$

Through the relation (5), one shows without difficulty that $G_n(y,z) = G_n(z,y)$, and consequently,

$$G_n(y,z) = y_0 z_{n-1} + y_1 z_n \quad .$$

More generally, and if q is any integer, we find more:

$$G_n(y,z) = z_q y_{n-q-1} + z_{q+1} y_{n-q} \quad .$$

One can also show that $G_n = G_{n-1} + G_{n-2}$, and therefore that the sequence G is a linear sequence of the family (1) concerned and has ing terms:

$$G_0 = y_1 z_0 + z_1 y_0 - y_0 z_0$$

and

$$G_1 = y_0z_0 + y_1z_1 .$$

With the symbols of paragraph 3, we can show that, on the other hand,

$$A(y,z) = A(y)A(z) \qquad m(y,z) = m(y)m(z)\mathrm{ch}\lambda$$

$$B(y,z) = B(y)B(z) \qquad \phi(y,s) = \phi(y) + \phi(z)$$

hence,

$$G_n(y,z) = m(y)m(z)\ \mathrm{ch}\lambda\ \mathrm{sh}\lambda[n + \phi(y) + \phi(z)] \tag{5 bis}$$

or

$$G_n(y,z) = m(y)m(z)\ \mathrm{ch}\lambda\ \mathrm{ch}\lambda[n + \phi(y) + \phi(z)] ,$$

accordingly as to whether G_1 is superior to

$$G_0\ \frac{1 + \sqrt{5}}{2}$$

We have first, $G_n(z,F) = z_n$. The sequence $G_n(z,L)$ affords a special interest because

$$G_n(z,L) = L_0z_{n-1} + L_1z_n = 2z_{n-1} + z_n = z_{n-1} + z_{n+1}$$

which gives, in particular:

$$L_n = G_n(L,F) = F_{n-1} + F_{n+1} . \tag{6}$$

When the sequences y and z are the same, one may obtain, using $G_n(z)$ instead of $G_n(z,z)$,

$$G_n(z) = z_0z_{n-1} + z_1z_n = z_qz_{n-q-1} + z_{q+1}z_{n-q} .$$

In this sequence,

$$G_0(z) = z_0(2z_1 - z_0)$$

$$G_1(z) = z_0^2 + z_1^2$$

$$G_2(z) = z_1(2z_0 + z_1) \ .$$

We find also:

(6 bis) $$G_n(z) = m^2 ch\lambda\, ch\lambda\, (n + 2\phi) \quad \text{or} \quad m^2 ch\lambda\, sh\lambda\, (n + 2\phi) \ ,$$

according to the value of the ratio G_1/G_0.

Consequently, through these relations,

(7) $$\begin{aligned} G_n(F) &= F_n \\ G_n(L) &= 5F_n \\ z_n G_n(z, L) &= G_{2n}(z) \ . \end{aligned}$$

Likewise, the sequence z_n can be connected to the same sequence z_{n+p} shifted by an integer P. As $m(z) = m(z_{+p})$ and

$$(z, z_{+p}) = (z) + (z_{+p}) \ ,$$

it comes

(7 bis) $$\begin{aligned} G_n(z, z_{+p}) &= m^2 ch\lambda ch\lambda (n + p + 2\phi) \quad \text{if } n \text{ and } p \text{ have different eveness} \\ G_n(z, z_{+p}) &= m^2 ch\lambda sh\lambda (n + p + 2\phi) \quad \text{if } n \text{ and } p \text{ have the same eveness} \end{aligned}$$

Obviously, the terms of the connected sequences G must be, like those of the other linear sequences, alternatively a hyperbolic sine and a hyperbolic cosine.

8. Sundry Relations. Having resort to formulas interconnecting hyperbolic lines, we can set up many relations between the terms of the linear sequences of type (1).

(a) Formulas of addition and subtraction. One finds

$$z_{n+p} + z_{n-p} = \begin{cases} z_n L_p & \text{if } p \text{ is even} \\ (z_{n-1} + z_{n+1})F_p & \text{if } p \text{ is odd} \end{cases}$$

$$z_{n+p} - z_{n-p} = \begin{cases} (z_{n-1} + z_{n+1})F_p & \text{if } p \text{ is even} \\ z_n L_p & \text{if } p \text{ is odd} \end{cases} .$$

These relations can be condensed into the following form:

$$\begin{cases} z_{n+p} + (-1)^p z_{n-p} = z_n L_p \\ z_{n+p} - (-1)^p z_{n-p} = (z_{n+1} + z_{n-1})F_p . \end{cases}$$

One can write them more symmetrically:

$$(8) \quad \begin{cases} z_{n+p} + (-1)^p z_{n-p} = L_p G_n(z, F) \\ z_{n+p} - (-1)^p z_{n-p} = F_p G_n(z, L) \end{cases} \text{ or another way } \begin{cases} = G_p(L, F) G_n(z, F) \\ = G_p(F, F) G_n(z, L) \end{cases} .$$

Each of the above mentioned sums and differences concerns both terms z_{n+p} and z_{n-p} of which the indices are separated from $2p$ which is an even integer.

When the difference, which we call a, between the indices $q + a$ and q of the considered terms is odd, i.e., when we try to compute the sum $z_{q+a} + z_a$ or the difference $z_{q+a} - z_a$, the problem is much more difficult because the terms are expressed, one by a hyperbolic cosine, the other by a sine, and there is no general formula for the addition or subtraction of both lines. Then, it is possible, to make the investigation easier, to pass through the Fibonacci sequence by introducing the following auxiliary linear sequences, of which the number is unlimited and which are only interesting when a is odd. We use the letters x and y to denominate these sequences:

$$x_q(a) = F_{q+a} + F_q$$
$$y_q(a) = F_{q+a} - F_q .$$

Particularly:

$$x_q(1) = F_{q+2} \qquad y_q(1) = F_{q-1}$$

$$x_q(3) = 2F_{q+2} \qquad y_q(3) = 2F_{q+1}$$

$$x_q(5) = 5F_{q+1} + 4F_q \qquad y_q(5) = 5F_{q+1} + 2F_q \quad .$$

Generally speaking, we have

$$F_{q+a} = F_{a-1}F_q + F_aF_{q+1}$$

Hence, with the help of (5),

$$z_{q+a} - z_q = z_0x_{q-1}(a) + z_1x_q(a) = G_q[z,x_q(a)] .$$

In the same way,

$$z_{q+a} - z_q = G_q[z,y_q(a)]$$

(b) Sums or differences of Squares. Using the sums and differences just set up, one finds:

$$(9) \qquad \begin{aligned} z_p^2 - z_q^2 &= G_{p+q}(z)F_{p-q} \quad \text{if } p \text{ and } q \text{ have the same eveness} \\ z_p^2 + z_q^2 &= G_{p+q}(z)F_{p-q} \quad \text{if } p \text{ and } q \text{ have different eveness.} \end{aligned}$$

and, by condensing these relations:

$$z_p^2 - (-1)^{p+q}z_q^2 = G_{p+q}(z)F_{p-q} \quad .$$

The difference of the squares, when $p - q$ is odd, can be written:

$$z_p^2 - z_q^2 = G_q[z,x_q(p - q)]G_q[z,y_q(p - q)]$$

but this way does not lend itself to practical applications. Likewise, for the sum of the squares when $p - q$ is even.

(c) Sums of the terms of Linear Sequences. One easily finds by recurrence the following relation which is suitable to all linear sequences defined by formula (1):

$$Z_{p,q} = \sum_{i=p}^{i=q} z_i = z_{q+2} - z_{p+1} \quad .$$

We have, therefore, in the case of the first $n + 1$ terms, from $p = 0$ to $q = n$:

$$Z_n = z_{n+2} - z_1$$

In addition to this general method, there are, for two special cases, other methods making possible, for instance, to get checking of the computation:

In one of the cases, the number $n + 1$ of the implicated terms is a multiple of 4 and one gets

$$Z_n = \frac{F_{n+1}}{2}\left(\frac{z_{n+5}}{2} + \frac{z_{n+1}}{2}\right) = \frac{F_{n+1}}{2}\,\frac{G_{n+3}}{2}\,(z, L)$$

The second special case, which looks more interesting, concerns a number of terms which are multiples of 2 and of no other power of 2. In this case, $n - 1$ is a multiple of 4 and we have, consequently,

$$Z_n = \frac{L_{n+1}}{2}\,\frac{z_{n+3}}{2} \quad ,$$

i.e., the sum of the $n + 1$ implicated terms is equal to the product of the $(n+5)/2^{th}$ term of the sequence (index $(n+3)/2$), by the $(n+3)/2^{th}$ term of the Lucas sequence (index $(n + 1)/2$). There is, therefore, equality between the sum of the first six term $(n = 5)$ and the product of the fifth term by 4:

the sum of the first 10 terms ($n = 9$) and the product of the 7th term by 11, the sum of the first 14 terms ($n = 13$) and the product of the 9th term by 29, the sum of the first 18 terms ($n = 17$) and the product of the 11th term by 76.

and so on.

(d) Sums of the two-by-two terms. Let us add first the $(n/2) + 1$ terms with an even index, from 0 to n. We find

$$S_n = (z_0 - z_1) + z_{n+1} = z_{n+1} - z_{-1} .$$

For the $(n' + 1)/2$ terms with an odd index n', from z_1 to $z_{n'}$, we get likewise

$$S_{n'} = z_{n'+1} - z_0 .$$

One can easily check the accuracy of both expressions of S_n and $S_{n'}$ by taking $n' = n - 1$. Adding both sums, one must find again the sum Z_n of the $n + 1$ terms of the sequence z, from z_0 to z_n.

We have indeed, on the one hand for the number of terms

$$\left(\frac{n}{2} + 1\right) + \frac{n' + 1}{2} = n + 1 ,$$

and, on the other hand, for the formulas of the sums

$$S_{n'} = z_n - z_0$$

and

$$S_n + S_{n'} = z_{n+1} - z_{-1} + z_n - z_0 = z_{n+2} - z_1 = Z_n .$$

Moreover, man can try to get the sums of the two-by-two terms between the indices p and q, both even or both odd. The number of these terms is $(p - q)/2$, and their sum is the difference between the sums S_q and S_p made from the beginning of the sequence to q and to p. As p and q have

the same eveness, one obtains

$$S_q - S_p = z_{q+1} - z_{p+1}$$

from which it comes, by using the formulas (8):

$$(10)\quad \begin{cases} S_q - S_p = \dfrac{F_{q-p}}{2}\,\dfrac{G_{q+p}}{2} + 1^{(z,L)} & \text{if } \dfrac{q-p}{2} \text{ is even} \\[2ex] S_q - S_p = \dfrac{L_{q-p}}{2}\,\dfrac{G_{q+p}}{2} + 1^{(z,F)} & \text{if } \dfrac{q-p}{2} \text{ is odd} \end{cases}$$

9. Application to the Fibonacci and Lucas Sequences. The relations, which we shall get by application of the formula of the previous paragraphs, could be obtained by using the formulas (3) and (4), which give the terms of both the Fibonacci and Lucas sequences in the shape of hyperbolic lines of the index n. We think, nevertheless, more into the spirit of the present paper to consider both sequences as special cases of very great simplicity.

We have previously seen that in (6), $L_n = F_{n-1} + F_{n+1}$. Substituting F to z in the formulas (7), one finds $F_nL_n = F_{2n}$.

(a) Formulas of addition and subtraction. The Formulas (8) give:

$$F_{n+p} + (-1)^pF_{n-p} = F_nL_p \qquad L_{n+p} + (-1)^pL_{n-p} = L_nL_p$$

$$F_{n-p} - (-1)^pF_{n-p} = F_pL_n \qquad L_{n+p} - (-1)^pL_{n-p} = 5F_nF_p\ .$$

In particular,

$$\begin{cases} F_{n+1} + F_{n-1} = L_n \\ F_{n+1} - F_{n-1} = F_n \end{cases} \qquad \begin{cases} L_{n+1} + L_{n-1} = 5F_n \\ L_{n+1} - L_{n-1} = L_n\ , \end{cases}$$

and consequently,

$$F^2_{n+1} - F^2_{n-1} = F_nL_n = F_{2n} \qquad L^2_{n+1} - L^2_{n-1} = 5F_nL_n = 5F_{2n}\ .$$

(b) Sums or differences of squares. One finds with the help of formulas (9):

$$F_p^2 - (-1)^{p+q}F_q^2 = F_{p+q}F_{p-q} \qquad L_p^2 - (-1)^{p+q}L_q^2 = 5F_{p+q}F_{p-q} ,$$

from which we deduce, among others

$$F_n^2 + F_{n+1}^2 = F_{2n+1} \qquad L_n^2 + L_{n+1}^2 = 5\,F_{2n+1}$$

$$F_n^2 + (-1)^n = F_{n+1}F_{n-1} \qquad L_n^2 + (-1)^n = 5\,F_{n+1}F_{n-1} .$$

(c) Sums of the terms of each sequence. Let us nominate by Φ_n and Λ_n the sums of the first $n+1$ terms, from the index 0 to the index 1. Using the formulas of the paragraph 7c and taking $z_1 = 1$, we get:

$$\Phi_n = F_{n+2} - 1 \qquad \Lambda_n = L_{n+2} - 1 .$$

For $n+1$ multiple of 4, it becomes

$$\Phi_n = \frac{F_{n+1}}{2}\,\frac{F_{n+5}}{2} + \frac{F_{n+1}}{2} \qquad \Lambda_n = \frac{5F_{n+1}}{2}\,\frac{F_{n+3}}{2}$$

For $n-1$ multiple of 4, one finds

$$\Phi_n = \frac{L_{n+1}}{2}\,\frac{F_{n+1}}{2} \qquad \Lambda_n = \frac{L_{n+1}}{2}\,\frac{L_{n+3}}{2}$$

(d) Sums of the two-by-two terms. The sum of the first $(n/2)+1$ terms of even index is

$$S_n(F) = F_{n+1} - 1 \qquad S_n(L) = L_{n+1} + 1 .$$

That of the first $(n'+1)/2$ terms of odd index is:

$$S_{n'}(F) = F_{n'+1} \qquad S_{n'}(L) = L_{n'+1} - 2 .$$

For the two-by-two sums between the indices p and q, of the same eveness, we find, using the formulas (10):

$$S_q(F) - S_p(F) = \frac{L_{q-p}}{2}\frac{L_{q+p}}{2} + 1 \qquad S_q(L) - S_p(L) = \frac{L_{q-p}}{2}\frac{L_{q+p}}{2} + 1$$

when $(q - p)/2$ is odd. When this quantity is even, we have

$$S_q(F) - S_p(F) = \frac{F_{q-p}}{2}\frac{L_{q+p}}{2} + 1 \qquad S_q(L) - S_p(L) = \frac{5F_{q-p}}{2}\frac{F_{q+p}}{2} + 1$$

(e) Other relations between the terms of the sequences F and L. Cancelling out the hyperbolic lines between the expressions (3) and (4), we obtain the following relations, in which we can note again the prominent part taken by the factor 5 which is equal to 4 ch^2 .

$$L_n^2 = 5F_n^2 + 4(-1)^n \qquad L_{2n} = 5F_n^2 + 2(-1)^n$$

$$L_n^2 = F_n^2 + 4F_{n-1}F_{n+1} \qquad 2L_{2n} = L_n^2 + 5F_n^2$$

According to the first of these relations, we see that no one term of the Lucas sequence can be a multiple of 5, and that L_n draws nearer to $F_n\sqrt{5}$, when n grows indefinitely. One also finds

$$L_{2n} = L_n^2 - 2(-1)^n$$

10. Research of a Linear Sequence. The matter here is to research if a given number can be a term of given rank in a linear sequence. In other words, the values of z and n are given, and those of z_0 and z_1 are unknown. The relation

$$z_n = z_0F_{n-1} + z_1F_n \tag{5}$$

contains the solution of the problem. It is a simple equation which must be solved by integers, which is always possible. On the other hand, if z_0 and

z_1 are a solution, there is an infinity of other solutions defined by $z_0 + kF_n$ and $z_1 - kF_{n-1}$, where k is any integer, positive or negative.

As an example, let us search the sequences in qhich $z_7 = 81$. The equation of the problem is

$$8z_0 + 13z_1 = 81 .$$

Few trials show that z_0 and z_1 can be respectively taken equal to 2 and 5. Consequently, the solutions are:

$$\begin{array}{lcccccccc} z_0 & = & \cdots & -24 & -11 & 2 & 15 & 28 & \cdots \\ z_1 & = & \cdots & 21 & 13 & 5 & -3 & -11 & \cdots . \end{array}$$

The differences between the terms of two such sequences defined by the values k' and k'' of k are equal to the product by $k' - k''$ of the terms of a Fibonacci sequence.

11. We can generalize the notion of linear sequence if we admit that the parameter n can vary in a continuous way, without being limited to integers, so that z_n is a continuous fraction $z(n)$ of the parameter n and can consequently take irrational values. This expedient can be used to simplify the records, but it is not of practical value for the applications.

With the notation of paragraph 3, we can write the formulas (2bis), according to the case:

$$(2^{\text{ter}}) \qquad z_n = \sqrt{|AB|}F_{n+\phi} \qquad \text{or} \qquad z_n = \frac{\sqrt{|AB|}}{2\,\text{ch}\,\lambda} L_{n+\phi}$$

The quantities z and n are well integers, but it is not the case for the functions $F_{n+\phi}$ and $L_{n+\phi}$, like for the parameters λ, ϕ, and m (or $\sqrt{|AB|}$).

Thus, any linear sequence can be reduced to a generalized Lucas or Fibonacci sequence by use of an irrational factor.

In the special case of the connected sequences (paragraph 7), the formulas (5bis), (6bis) and (7bis) can be modified like the formula (2bis) and therefore simplified. One replaces to this end:

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RELATIONS BETWEEN A SEQUENCE OF FIBONACCI TYPE AND THE SEQUENCE OF ITS PARTIAL SUMS

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Let $\{F_n\}$ be a Fibonacci-type sequence, where

$$F_{n+2} = F_n + F_{n+1}, \quad F_1 = a, \quad F_2 = b .$$

Let $\{S_n\}$ be the sequence obtained from the given sequence by taking the partial sums of its terms, that is,

$$S_n = \sum_{i=1}^{n} F_i .$$

Then there is an unexpected relation between the F-sequence and the S-sequence, namely that S_{4r+2} is a multiple of F_{2r+1}. In fact,

$$S_{4r-2} = c_{2r-1} F_{2r+1} ,$$

where $\{c_n\}$ is itself a Fibonacci-type sequence with $c_1 = 1$ and $c_2 = 3$ (the Lucas sequence).

Proof. If α and β are the roots of the equation

$$t^2 - t - 1 = 0 ,$$

we may show, as usual, that

$$F_n = \frac{a(\alpha^{n-2} - \beta^{n-2}) + b(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta}$$

We may then obtain S_n by summing geometric progressions, which gives

$$S_n = \frac{a(\alpha^n - \beta^n) + b[\alpha(\alpha^n - 1) - \beta(\beta^n - 1)]}{\alpha - \beta} .$$

Hence,

$$F_{2r+1} = \frac{a(\alpha^{2r-1} - \beta^{2r-1}) + b(\alpha^{2r} - \beta^{2r})}{\alpha - \beta}$$

and

$$S_{4r-2} = \frac{a(\alpha^{4r-2} - \beta^{4r-2}) + b[\alpha(\alpha^{4r-2} - 1) - \beta(\beta^{4r-2} - 1)]}{\alpha - \beta} .$$

Since

$$\begin{aligned}(\alpha^{2r} - \beta^{2r})(\alpha^{2r-1} + \beta^{2r-1}) &= \alpha^{4r-1} - \beta^{4r-1} + (\alpha\beta)^{2r-1}(\alpha - \beta) \\ &= \alpha^{4r-1} - \beta^{4r-1} - (\alpha - \beta) ,\end{aligned}$$

we have

$$S_{4r-2} = c_{2r-1} F_{2r+1} ,$$

where

$$c_{2r-1} = \alpha^{2r-1} + \beta^{2r-1} .$$

It follows that $\{c_n\}$ is itself a Fibonacci-type sequence, with

$$c_1 = \alpha + \beta = 1$$

and

$$c_2 = \alpha^2 + \beta^2 = 3 .$$

There are two other results of interest. First, we have a somewhat similar relation between S_{4r} and F_{2r+2}, namely

$$S_{4r} = c_{2r}F_{2r+2} - 2b .$$

This can easily be proved in the same way as the earlier result.

Second, it follows from the earlier results that

$$S_n = F_{n+2} - b ,$$

and hence that

$$S_{n+2} = S_n + S_{n+1} + b .$$

[Continued from page 295.]

— for (5bis), the product AB by the product of the values of A and B relative to the sequences y and z, and, on the other hand, ϕ by the sum of the values of ϕ for these sequences.

— for (6bis) $\sqrt{AB}$ by AB and ϕ by 2ϕ ,

— for (7bis) $\sqrt{AB}$ by AB and ϕ by $p + 2\phi$.

12. The author thinks he has shown, by the present study which does not maintain to be exhaustive, how much the use of hyperbolic lines to express the terms of the linear sequences of the type (1) is favorable by the simplicity which it introduces in the calculations bearing on these sequences, by the fact also that it suggests relations, which makes it easier to set up. These advantages are specially clear in the case of the Fibonacci and Lucas sequences, for which it is possible to re-establish quickly the well known formulas concerning them.

RECIPROCALS OF GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

The purpose of this paper is to find expressions for

$$\sum_{n=1}^{\infty} H_{2n}^{-1}, \qquad \sum_{n=1}^{\infty} H_n^{-t} z^n, \qquad H_n^{-1} \quad \text{and} \quad H_{n+1}^{-t},$$

where $\{H_n\}$ is the generalized Fibonacci sequence defined by Horadam [6] as follows:

$$(1.1) \qquad H_n = H_{n-1} + H_{n-2} \quad (n \geq 3), \qquad H_1 = p, \qquad H_2 = p + q,$$

where p, q are arbitrary integers, and

$$(1.2) \qquad H_n = (2\sqrt{5})^{-1} (\ell a^n - m b^n)$$

with $\ell = 2(p - qb)$, $m = 2(p - qa)$ and where a, b are the roots of $x^2 - x - 1 = 0$.

The required expressions will be obtained as results (2.1), (2.2), (2.3), and (3.6), respectively. They will be seen to involve Lambert series and Bernoulli-type polynomials.

Let

$$(1.3) \qquad H = \frac{p - qb}{p - qa}.$$

We define the Lambert series

*Part of the substance of a thesis presented for the Bachelor of Letters degree to the University of New England in 1968.

$$L_1(x) = \sum_{r=1}^{\infty} H^{-r/2} \frac{x^r}{1 - x^r} \tag{1.4}$$

and

$$L_2(x) = \sum_{r=1}^{\infty} H^{-r} \frac{x^r}{1 - x^r} . \tag{1.5}$$

Details of some of the properties of the Lambert series may be found in Hardy and Wright [5] and Landau [7].

We also need to introduce a new expression

$$\sum_{r=0}^{\infty} B_r^{(t)'}(x) \frac{n^r}{r!} = \frac{n^t e^{nx}}{(e^n - H)^t} , \tag{1.6}$$

in which the $B_r^{(t)'}(x)$ is analogous to the general Bernoulli polynomials of higher order which have been discussed by Gould [3].

A Bernoulli polynomial $B_r(x)$ is defined by means of

$$\sum_{r=0}^{\infty} B_r(x) \frac{n^r}{r!} = \frac{n e^{nx}}{e^n - 1} . \tag{1.7}$$

Some of their properties are developed by Carlitz [2], Hardy and Wright [5], and Gould [3] and [4] who relates the Bernoulli and Euler numbers.

In fact, the $B_r^{(t)'}(x)$ are generalized Bernoulli polynomials and satisfy the recurrence relation

$$B_r^{(t)'}(x + 1) - H B_r^{(t)'}(x) - n B_r^{(t-1)'}(x) = 0 . \tag{1.8}$$

The proof of (1.8) is as follows.

$$\sum_{r=0}^{\infty} \{B_r^{(t)'}(x+1) - H B_r^{(t)'}(x)\} \frac{n^r}{r!}$$

$$= \frac{n^t e^{nx} e^n}{(e^n - H)^t} - \frac{H n^t e^{nx}}{(e^n - H)^t}$$

$$= n \frac{n^{t-1} e^{nx}}{(e^n - H)^{t-1}} = n \sum_{n=0}^{\infty} B_r^{(t-1)'}(x) \frac{n^r}{r!}$$

We shall also use a special case of $B_r^{(t)'}(x)$, when $r = 1$, defined by

$$\sum_{r=0}^{\infty} B'_r(x) \frac{n^r}{r!} = \frac{n e^{nx}}{e^n - H} . \tag{1.9}$$

The $B'_r(x)$ also satisfy a recurrence relation

$$B'_r(x+1) - H B'_r(x) = r x^{r-1} .$$

This recurrence relation follows since

$$\sum_{r=0}^{\infty} \{B'_r(x+1) - H B'_r(x)\} \frac{n^r}{r!}$$

$$= \frac{n e^{nx} e^n}{e^n - H} - \frac{H n e^{nx}}{e^n - H}$$

$$= n e^{nx} = n \sum_{r=0}^{\infty} \frac{(nx)^r}{r!} .$$

2. CALCULATION OF THE RECIPROCALS

$$\sum_{n=1}^{\infty} H_{2n}^{-1} = 2\sqrt{5}\sum_{n=1}^{\infty} \frac{1}{\ell a^{2n} - mb^{2n}}$$

$$= 2\sqrt{\frac{5}{\ell m}}\sum_{n=1}^{\infty} \frac{\sqrt{\frac{m}{\ell}}\, b^{2n}}{1 - \frac{m}{\ell} b^{4n}}$$

$$= 2\sqrt{\frac{5}{\ell m}}\sum_{n=1}^{\infty} \left\{ \frac{\frac{1}{\sqrt{H}}\, b^{2n}}{1 - \frac{1}{\sqrt{H}} b^{2n}} - \frac{\frac{1}{H} b^{4n}}{1 - \frac{1}{H} b^{4n}} \right\}$$

$$= 2\sqrt{\frac{5}{\ell m}}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty} \left\{ H - \frac{r}{2} b^{2nr} - H^{-r} b^{4nr} \right\}$$

$$= 2\sqrt{\frac{5}{\ell m}}\sum_{r=1}^{\infty} \left\{ H^{-\frac{r}{2}} \frac{b^{2r}}{1 - b^{2r}} - H^{-r} \frac{b^{4r}}{1 - b^{4r}} \right\} .$$

Thus

$$\sum_{n=1}^{\infty} H_{2n}^{-1} = 2\sqrt{\frac{5}{\ell m}} \left(L_1\left(\frac{3 - \sqrt{5}}{2}\right) - L_2\left(\frac{7 - 3\sqrt{5}}{2}\right)\right) \tag{2.1}$$

That is, the required expression is seen to involve Lambert series defined in (1.4) and (1.5).

Write

$$H_n^{-t} = \left(\frac{2\sqrt{5}}{-m}\right)^t \cdot \frac{1}{a^{nt}} \cdot \frac{1}{(C^n - H)^t} ,$$

where $C = b/a$.

Then

$$H_n^{-t} = \left(\frac{2\sqrt{5}}{-m}\right)^t \frac{1}{(C^x a^t)^n} \frac{C^{nx}}{(C^n - H)^t}$$

$$= \left(\frac{2\sqrt{5}}{-m}\right)^t \frac{1}{(C^x a^t)^n} \frac{e^{x(n\log C)}}{(e^{n\log C} - H)^t}$$

$$= \left(\frac{2\sqrt{5}}{-m}\right)^t \frac{1}{(n\log C)^t (C^x a^t)^n} \frac{z^t e^{xz}}{(e^n - H)^t}$$

where $z = n \log C$. Thus

$$H_n^{-t} = \left(\frac{2\sqrt{5}}{-m}\right)^t \frac{1}{(n\log C)^t (C^x a^t)^n} \sum_{r=0}^{\infty} B_r^{(t)'}(x) \frac{(n\log C)^r}{r!}$$

(α)

$$= \left(\frac{2\sqrt{5}}{-m}\right)^t \frac{1}{(C^x a^t)^n} \sum_{r=0}^{\infty} B_r^{(t)'}(x) \frac{(\log C)^{r-t}}{r!} n^{r-t} .$$

From this, the generating function for powers of the reciprocals can be set up. This is

$$(2.2) \quad \sum_{n=1}^{\infty} H_n^{-t} z^n = \left(\frac{2\sqrt{5}}{-m}\right)^t \sum_{r=0}^{\infty} B_r^{(t)'}(x) \frac{(\log C)^{r-t}}{r!} \cdot \sum_{n=1}^{\infty} n^{r-t} \left(\frac{z}{a^{t-x} b^x}\right)^n .$$

Thus, the required expression involves the generalized Bernoulli polynomials of higher order (1.6).

As a special case of (α) with $t = 1$, it follows that

$$(2.3) \qquad H_n^{-1} = \frac{-2\sqrt{5}}{m(a^{1-x} b^x)^n} \sum_{r=0}^{\infty} B_r'(x) \frac{(\log C)^{r-1}}{r!} n^{r-1} .$$

As expected from (2.2), our expression involves the Bernoulli polynomials (1.9).

Following Gould [3], let

$$H(x) = \sum_{n=1}^{\infty} H_n^{-1} x^n . \tag{2.4}$$

Then

$$\ell H(ax) - mH(bx) = \sum_{n=1}^{\infty} H_n^{-1} \left(\frac{\ell a^n - mb^n}{2\sqrt{5}} \right) 2\sqrt{5}\, x^n$$

$$= \sum_{n=1}^{\infty} 2\sqrt{5}\, x^n .$$

Thus

$$\ell H(ax) - m H(bx) = \frac{2\sqrt{5}x}{1 - x} , \tag{2.5}$$

which is a succinct expression involving

$$\sum_{n=1}^{\infty} H_n^{-1} x^n .$$

3. THE OPERATOR E

We introduce an operator E, defined by

$$EH_n = H_{n+1} . \tag{3.1}$$

Thus

$$H_{n+2} - H_{n+1} - H_n = 0$$

becomes

$$(E^2 - E - 1)H_n = 0$$

or

$$(E - a)(E - b)H_n = 0 . \tag{3.2}$$

Let

$$G_n = (E - b)H_n = H_{n+1} - bH_n .$$

Then from (3.2),

$$(E - a)G_n = 0 \qquad \text{or} \qquad G_{n+1} = aG_n , \tag{3.3}$$

and so

$$G_1 = H_2 - bH_1 = ap + q . \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$G_n = a^{n-1}(ap + q) . \tag{3.5}$$

Now

$$H_{n+1} = bH_n + G_n ,$$

and so

$$H_{n+1}^{-t} = b^{-t} H_n^{-t} \left(1 + \frac{G_n}{b H_n}\right)^t$$

$$= b^{-t} H_n^{-t} \sum_{r=0}^{\infty} (-1)^r \frac{t(t+1)\cdots(r+r-1)}{r!} \left(\frac{G_n}{b H_n}\right)^r$$

$$= b^{-t} H_n^{-t} \sum_{r=0}^{\infty} \frac{(-1)^r (t)_r a^{nr-r}}{r!\, b^r H_n^r} (ap+q)^r \quad ,$$

where

$$(t)_r = t(t+1)(t+2)\cdots(t+r-1) \quad [1] .$$

Thus

$$H_{n+1}^{-t} = \sum_{r=0}^{\infty} \frac{(-1)^r (t)_r}{r!} \sum_{s=0}^{\infty} \frac{r!}{s!\, r-s!} \frac{p^{r-s} q^s}{a^{s-nr} b^{r+t}} H_n^{-t-r}$$

and so

$$H_{n+1}^{-t} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r (t)_r}{s!\, r-s!} \frac{p^{r-s} q^s}{a^{s-nr} b^{r+t}} H_n^{-t-r} . \tag{3.6}$$

See also (α).

We have thus established expressions for the reciprocals stated at the beginning of this article.

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A GENERALIZED PYTHAGOREAN THEOREM

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1. INTRODUCTION

The results in this paper arose from the efforts of the first-named author to adapt Horadam's Fibonacci number triples [3] to generate direction of numbers in solid geometry for a multivariable calculus course. This effort was unsuccessful in that the equation obtained was true for quadratic diophantine equations in general, but it did not use any properties of the Fibonacci sequence. However, it did give rise to some results for higher order sequences.

Horadam [2], [3], [4] has studied the properties of a generalized Fibonacci sequence defined by

$$H_{n+2} = H_{n+1} + H_n, \qquad (n \geq 1), \tag{1}$$

with $H_1 = p$, $H_2 = p + q$. One of the properties he found was that

$$(H_n H_{n+3}) + (2H_{n+1} H_{n+2})^2 = (2H_{n+1} H_{n+2} + H_n^2)^2, \tag{2}$$

Which connects generalized Fibonacci numbers with Pythagorean triples.

In the next section of this paper, an analogous result is obtained for generalized "Tribonacci" numbers. The theorem is then extended to general linear difference equations of order r with unit coefficients.

2. TRIBONACCI NUMBER TRIPLES

The general Tribonacci series (see Feinberg [1]) is defined by

$$U_{n+3} = U_{n+2} + U_{n+1} + U_n, \qquad (n \geq 1) \tag{3}$$

with initial values U_1, U_2, U_3.

Theorem 1.

$$(4)\qquad (U_n U_{n+4})^2 + (2(U_{n+1} + U_{n+2})U_{n+3})^2 = (U_n^2 + 2(U_{n+1} + U_{n+2})U_{n+3})^2 .$$

Proof.

$$\begin{aligned} U_n^2 &= (U_{n+3} - (U_{n+1} + U_{n+2}))^2 \\ &= U_{n+3}^2 + (U_{n+1} + U_{n+2})^2 - 2(U_{n+1} + U_{n+2})U_{n+3} \end{aligned}$$

and so

$$U_n^2 + 2(U_{n+1} + U_{n+2})U_{n+3} = U_{n+3}^2 + (U_{n+1} + U_{n+2})^2 .$$

This gives

$$\begin{aligned} &(U_n^2 + 2(U_{n+1} + U_{n+2})U_{n+3})^2 \\ (5)\qquad &= U_{n+3}^4 + (U_{n+1} + U_{n+2})^4 + 2(U_{n+1} + U_{n+2})U_{n+3})^2 \\ &= \left\{U_{n+3}^2 - (U_{n+1} + U_{n+2})^2\right\}^2 + (2(U_{n+1} + U_{n+2})U_{n+3})^2 . \end{aligned}$$

Now

$$\begin{aligned} &\left\{U_{n+3}^2 - (U_{n+1} + U_{n+2})^2\right\}^2 \\ (6)\qquad &= (U_{n+3} - (U_{n+1} + U_{n+2}))^2 + (U_{n+3} + (U_{n+1} + U_{n+2}))^2 \\ &= U_n^2 U_{n+4}^2 . \end{aligned}$$

Substitution of (6) in (5) gives the result (4).

Theorem 2. All Pythagorean triples are Fibonacci triples.

Proof. Put $U_1 = x - y$, $U_2 = y$, $U_3 = 0$. Then

$$U_4 = x \quad \text{and} \quad U_5 = x + y .$$

For $n = 1$, Eq. (4) becomes

$$(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2 .$$

For example, when $x = 5$ and $y = 2$, we get the triple 20, 21, 29.

3. GENERALIZED PYTHAGOREAN THEOREM

Comparison of (4) with (2) suggests that for a general recurring sequence $\{V_n\}$ of order r where

$$V_{n+r} = \sum_{i=0}^{r-1} V_{n+i} , \qquad (n \geq 1) , \tag{7}$$

with initial values $V_1, V_2, \cdots, V_r$, there is a Pythagorean theorem of the form

Theorem 3.

$$\begin{aligned}(V_n V_{n+r+1})^2 + (2V_{n+r}(V_{n+r} - V_n))^2 \\ = (V_n^2 + 2V_{n+r}(V_{n+r} - V_n))^2 .\end{aligned} \tag{8}$$

For example, when $r = 2$, we get

$$(V_n V_{n+3})^2 + (2V_{n+2}V_{n+1})^2 = (V_n^2 + 2V_{n+2}V_{n+1})^2 ,$$

which agrees with (2). When $r = 3$, we get

$$\begin{aligned}(V_n V_{n+4})^2 + (2V_{n+3}(V_{n+2} + V_{n+1}))^2 \\ = (V_n^2 + 2(V_{n+1} + V_{n+2})V_{n+3})^2 ,\end{aligned}$$

which agrees with (4).

Lemma 1.

$$2V_{n+r} - V_{n+r+1} = V_n . \tag{9}$$

Proof.

$$\begin{aligned} &2V_{n+r} - V_{n+r+1} \\ &\quad = V_{n+r} + V_{n+r} - V_{n+r+1} \\ &\quad = (V_{n+r-1} + V_{n+r-2} + \cdots + V_{n+1} + V_n) + V_{n+r} \\ &\qquad - (V_{n+r} + V_{n+r-1} + V_{n+r-2} + \cdots + V_{n+1}) \\ &\quad = V_n \ . \end{aligned}$$

Lemma 2.

$$2V_{n+r} + V_{n+r+1} = 4V_{n+r} - V_n \ . \tag{10}$$

Proof. This reduces immediately to

$$2V_{n+r} - V_{n+r+1} = V_n \ ,$$

which has just been proved.

Proof of Theorem 3. From Lemmas 1 and 2, we have

$$(2V_{n+r} - V_{n+r+1})(2V_{n+r} + V_{n+r+1}) = V_n(4V_{n+r} - V_n) \ ,$$

which becomes

$$4V_{n+r}^2 - V_{n+r+1}^2 = V_n(4V_{n+r} - V_n) \ .$$

This can be rearranged as

$$V_{n+r+1}^2 = V_n^2 + 4V_{n+r}(V_{n+r} - V_n) \ . \tag{11}$$

On multiplication by V_n^2 and addition of $(2V_{n+r}(V_{n+r} - V_n))^2$ to each side of (11), the result in (8) follows.

For example, when $r = 4$, we get a "tetranacci" series [1] and (7) becomes

$$V_{n+r} = \sum_{i=0}^{3} V_{n+i} \ ,$$

with $V_1 = V_2 = V_3 = V_4 = 1$, say. Then $V_5 = 4$, $V_6 = 7$. At the same time, (8) becomes

$$(12)\qquad \begin{aligned}&(V_n^2 + 2(V_{n+1} + V_{n+2} + V_{n+3})V_{n+4})^2\\ &= (V_n V_{n+5})^2 + (2(V_{n+1} + V_{n+2} + V_{n+3})V_{n+4})^2 .\end{aligned}$$

$n = 1$ gives the Pythagorean triple 7, 24, 25.

If we call the type of triangle in (8) a recurrence triple, we get

<u>Theorem 4.</u> All Pythagorean triples are recurrence triples.

<u>Proof.</u> Put

$$V_1 = x - y, \quad V_2 = y, \quad V_3 = V_4 = 0$$

in (7). Then $V_5 = x$, $V_6 = x + y$, and for $n = 1$, Eq. (8) becomes

$$(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2 .$$

4. CONCLUDING COMMENTS

The results in Theorem 3 can be used to produce various properties for recurrence relations of different orders. For instance, when $r = 2$ and $n = m - 1$, we get

$$(13)\qquad 4(H_{m+1}H_{m-1} - H_{m+1}^2) = H_{m-1}^2 - H_{m+2}^2 ,$$

which, in conjunction with Eq. (11) of [2], gives

$$(14)\qquad 4(H_m^2 - H_{m+1}^2 + (-1)^m e) = H_{m-1}^2 - H_{m+2}^2 ,$$

where $e = p^2 - pq - q^2$.

For a third-order relation, the property analogous to (13) is

$$(15)\qquad 4(U_{m+2}U_{m-1} - U_{m+2}^2) = U_{m-1}^2 - U_{m+3}^2 .$$

This may provide a convenient method for the development of properties of third and higher order recurrence relations, which have been studied in a

number of papers in the Fibonacci Quarterly in recent years. For earlier studies, Morgan Ward [5] provides a useful reference.

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ON AN INITIAL-VALUE PROBLEM FOR LINEAR PARTIAL DIFFERENCE EQUATIONS *

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SUMMARY

Sufficient conditions are given for the existence and unity of the solution of an initial-value problem with linear partial difference equations. From this, in particular, assertions about the existence of compatibility conditions between initial values can be derived in case, by the formulation of a problem (perhaps a discretization of a partial differential equation) or by the method of solution, more than the required initial values goes into the calculation. With the aid of a two-dimensional operational calculus, certain applications are investigated.

INTRODUCTION

In the classical work [1] of A. A. Markoff, there is an existence and uniqueness theorem for partial difference equations of the form

$$x_{m+1,n+1} - a_{mn}x_{m,n+1} = b_{mn}x_{m+k,n} \,, \tag{1}$$

$$(m,n \geq 0, \text{ integral, } k \text{ fixed natural number}) \,,$$

for a desired complex-valued function $x = x_{mn}$ with given initial values x_{mo} $(m \geq k)$ and x_{on} $(n \geq 1)$. The proof is conducted by investigation of a system of infinitely many ordinary difference equations equivalent to (1). Here, in Theorem 1, an essentially more general initial-value problem for linear partial difference equations of arbitrary order will be treated by which the ideas of Ch. Jordan [2] on the subject are made precise.

The applications in the second part of the work show that the two-dimensional discrete operational calculus developed in [3] is appropriate to give in certain cases the solution, determined uniquely according to Theorem 1, in closed form and the possibly necessary compatibility conditions between the initial values explicitly.

*Translated by P. F. Byrd, San Jose State College, San Jose, California.

EXISTENCE AND UNIQUENESS THEOREMS

We consider the linear partial difference equation

$$(2) \qquad D(x) = \sum_{k,j=1,0}^{k,\ell} a_{ij} x_{m+i,n+j} = b_{mn} \qquad (m,n \geq 0, \text{ integral}),$$

of order (k,ℓ) with given complex-valued functions

$$a_{ij} = a_{ij}(m,n), \; b_{mn}.$$

Let $k \geq 1$, $\ell \geq 1$, and for at least one i or j the coefficients a_{io}, a_{oj}, a_{i1}, a_{kj} should not vanish.

The question arises which of the initial values

$$x_{mj} \qquad (j = 0, 1, \cdots, \ell - 1)$$
$$x_{in} \qquad (i = 0, 1, \cdots, k - 1)$$

should be prescribed so that the function x_{mn} is uniquely determined by (2) for all remaining $m,n \geq 0$. An answer to this is given by the following:

<u>Theorem 1.</u> The difference equation (2) of order (k,ℓ) possesses exactly one solution if, for all $m,n \geq 0$,

$$\text{(a)} \quad a_{kj} \neq 0 \text{ for } j = \ell_k \leq 1 \text{ and for } j = \ell_o \leq \ell_k, \qquad a_{kj} = 0 \text{ for } j > \ell_k$$

holds, and the initial values

$$(3) \qquad \begin{aligned} x_{mj} &= \alpha_m^j \qquad (j = 0, 1, \cdots, \ell_k;\; j \neq \ell_0;\; m \geq 0), \\ x_{in} &= \beta_n^i \qquad (i = 0, 1, \cdots, k - 1;\; n \geq 0) \\ &\qquad \alpha_i^j = \beta_j^i \end{aligned}$$

are prescribed, or if

$$\text{(b)} \qquad \begin{aligned} a_{i\ell} &\neq 0 \text{ for } i = k_1 \leq k \quad \text{and for } i = k_0 \leq k_1 \\ a_{i\ell} &= 0 \quad \text{for } i > k_1 \end{aligned}$$

holds and the initial values

$$(4)\qquad \begin{aligned} x_{mj} &= \alpha_m^j \quad (j = 0, 1, \cdots, \ell - 1;\ m \geq 0)\ , \\ x_{in} &= \beta_n^i \quad (i = 0, 1, \cdots, k_\ell;\ i \neq k_0;\ n \geq 0) \ \text{ with } \alpha_i^j = \beta_j^i \end{aligned}$$

are prescribed. For $\ell_k = 0$ (in the case (a)) or $k_\ell = 0$ (case (b)) the first equation of (3) or the second of (4), respectively, drops out.

Proof. We consider case (a) and solve equation (2) for $x_{m+k,n+\ell_0}$ which is possible because $a_{k,\ell_0} \neq 0$. For $m = n = 0$, there results, after inserting the initial values (3),

$$x_{k,\ell_0} = 1/a_{k,\ell_0}\left(b_{00} - \sum_{i,j=0,0}^{k-1,\ell} a_{ij}\beta_j^i - \sum_{\substack{j=0\\ j\neq\ell_0}}^{\ell_k} a_{kj}\alpha_k^j\right).$$

For $\ell_k = 0$ and thus $\ell_0 = 0$, the sum $\sum a_{kj}\alpha_k^j$ drops out in agreement with the concluding remark of the theorem. Since $a_{k,\ell_k} \neq 0$, the equation

$$(5)\qquad x_{m+k,n+\ell_k} = 1/a_{k,\ell_k}\left(b_{mm} - \sum_{i,j=0.0}^{k-1,\ell} a_{ij}x_{m+j,n+j} \sum_{j=0}^{\ell_k-1} a_{kj}x_{m+k,n+j}\right)$$

follows from (2), and there results with (3) the function values x_{kn} $(n > \ell_k)$. If the function values up to x_{k,ℓ_k+p-1} $(p \geq 2)$ are determined, then it follows for $2 \leq p \leq \ell_k$ that

$$x_{k,\ell_k+p} = 1/a_{k,\ell_k}\left(b_{op} - \sum_{i,j=0,0}^{k-1,\ell} a_{ij}\beta_{j+p}^i - \sum_{i=\ell_k-p+1}^{\ell_k-1} a_{kj}x_{k,j+p} - \sum_{j=0}^{\ell_k-p} a_{kj}\alpha_k^{j+p}\right)$$

For $p > \ell_k$, the last sum drops out and the lower limit of the second sum is set to zero. The elements x_{mn} result analogously for the rows $m > k$ (the function x_{mn} being regarded as an infinite matrix) by use of the elements standing at hand in the immediate upper k rows, which are given either by (3) or are determined by (3) and (5).

For the case (b), one notes that (2) can be solved respectively for $x_{m+k_0,n+\ell}$ or $x_{m+k_1,n+\ell}$ because $a_{k_0,\ell} \neq 0$ or $a_{k_1,\ell} \neq 0$. In an analogous way as with (a) the function values x_{mn} $(m = k_0,\ m > k_1,\ n \geq 1)$ are determined column-wise.

The proof of uniqueness of solution is trivial. If there were two solutions $x \neq y$ in the case (a) and if $x_{m_0,n_0} \neq y_{m_0,n_0}$ for $m_0 \geq k$ and $n_0 \geq \ell_k$, while $x_{mn} = y_{mn}$ for $m < m_0$ and $m = m_0$, $n < n_0$, then there immediately results from (2), for $m = m_0 - k$, $n = n_0 - \ell_k$ because $a_{k,\ell_k} \neq 0$, a contradiction. In case (b), the same holds for $n_0 = \ell_0 < \ell_k$.

In applications, the case when $\ell_0 = \ell_k = 1$ and $k_0 = k_1$ often occurs; then it follows that $k_i = k$ and the distinction between cases is cancelled. The solution x_{mn} of (2) is then uniquely determined by the specification of $k + \ell$ initial functions, namely by the first k rows and the first columns. (See example 1^0, 2^0.) Also, if $\ell_k < \ell$ or $k_1 < k$, occasionally $k + \ell$ initial values x_{mn} $(j = 0, \cdots, \ell - 1)$, x_{in} $(i = 0, \cdots, k - 1)$ are considered as prescribed. Compatibility conditions between these must then exist so that in the case (a) the $\ell - \ell_k$ functions x_{mj} $(j = \ell_0, \ell_k + 1, \cdots, \ell - 1)$ and in case (b) the $k - k_1$ functions x_{in} $(i = k_0, k_1 + 1, \cdots, k - 1)$ are already respectively determined by the remaining $k + \ell_k$ or $\ell + k_1$ functions. (See example 3^0, 4^0, 5^0.)

APPLICATIONS

In the treatment of the following applications, we make use of the operational calculus developed in [3]. It is shown there that the set of complex-valued functions $x = x_{mn}$ of integral variables m,n with vanishing function values for $m < M$ and all n for $n < N(m)$, $m \geq M$ (for each function an integer M exists and a function $N(m)$) forms a field by means of ordinary addition and of two-dimensional Cauchy product as multiplication. The subset D of functions with $M = 0$ and $N(m) = 0$ is an integral domain. For functions $x \in D$, the difference theorem

$$
\begin{aligned}
x_{m+k,n+1} &= p^k q^\ell x_{mn} \\
&\quad - q^\ell \sum_{i=0}^{k-1} p^{k-i} x_{in} - p^k \sum_{j=0}^{\ell-1} q^{\ell-j} x_{mj} + \sum_{i,j=0,0}^{k-1,\ell-1} p^{k-i} q^{\ell-j} x_{ij} ,
\end{aligned} \tag{6}
$$

holds, where x_{mj}, x_{in} and x_{ij} can be understood as functions from D which at least for $n = 0$ or $m = 0$ and $m = n = 0$ possess nonvanishing function values; p, q are displacement functions from Q, with k, ℓ being natural numbers.

1^0. The equation

$$x_{m+2,n+2} - x_{m+1,n+2} - x_{m+2,n+1} - x_{m,n+2} + 3x_{m+1,n+1} - x_{m+2,n} = 0$$
$$(m, n \geq 0)$$

related to Fibonacci numbers was treated in [4] and [5]. Its solution according to Theorem 1 is uniquely determined because

$$\ell_k = \ell = k_1 = k = 2 ,$$

if the $k + 1 = 4$ initial values x_{m0}, x_{m1}, x_{0n}, x_{1n} (so far as $k_0 = \ell_0 = 2$ is chosen) are prescribed independently of one another. This solution was represented in [5] in closed form.

2^0. The equation

$$x_{m+1,n+1} = x_{m+1,n} + \frac{2m + n + 3}{2m + 2} x_{m,n+1} \qquad (m, n \geq 0)$$

possesses the solutions[1]

$$x = x_{mn} = \sum_{i=0}^{\infty} \binom{m+i}{m} \binom{2m+n+1}{2m+2i+1} \quad \text{and} \quad y = y_{mn} = 2^n \binom{m+n}{m} .$$

Here,

$$\ell_k = \ell = k_1 = k = 1 .$$

Thus if one chooses $\ell_0 = k_0 = 1$, then it follows from Theorem 1 that the equation is uniquely solvable if the initial functions x_{mo}, x_{on} are prescribed. Since $x_{mo} = y_{mo} = 1$ $(m \geq 0)$ and $x_{on} = y_{on} = 2^n$ $(n \geq 0)$, it immediately follows that $x \equiv y$, and thus

[1]According to a written communication from A. Kotzauer (treated there by complete induction).

$$\sum_{i=0}^{\infty} \binom{m+i}{m}\binom{2m+n+1}{2m+2i+1} = 2^n \binom{m+n}{m} \qquad m,n = 0,1,\cdots .$$

3^0. The equation

$$x_{m+3,n} + x_{m,n+2} = 0 \qquad (m,n \geq 0) \tag{7}$$

of order (3.2) possesses, on account of $\ell_k = 0 < 2 = \ell$, $k_\ell = 0 < 3 = k$, exactly one solution from D if either in the case (a) the three initial functions $x_{in} = \beta_n^i$ $(i = 0, 1, 2)$ according to (3), or in the case (b) the two functions $x_{mj} = \alpha_m^j$ $(j = 0,1)$ are prescribed according to (4). With application of the difference theorem (6) there appears, however, $k + \ell = 5$ initial functions in the operational representation of equation (7):

$$x = x_{mn} = \frac{y}{p^3}\left(p^3\beta_n^0 + p^2\beta_n^1 + p\beta_n^2 + q^2\alpha_m^0 + q\alpha_m^1\right) . \tag{8}$$

With it,

$$y = \frac{p^3}{p^3 + q^3} = \begin{cases} (-1)^{m/3} & \text{for } n = 2m/3, \ n = 0,3,\cdots \\ 0 & \text{otherwise} \end{cases}$$

The required compatibility conditions between the initial functions are, as result from (8) for $n = 0$ or $n = 1$ after easy calculation in the field Q,

$$\alpha_m^j = (-1)^{[m/3]}\beta_{j+2\,[m/3]}^{\epsilon_m} \quad (j = 0,1;\ m \geq 0) \text{ with } \epsilon_m = \begin{cases} 0 \text{ for } m \equiv 0(3) \\ 1 \text{ for } m \equiv 1(3) \\ 2 \text{ for } m \equiv 2(3) \end{cases} \tag{9}$$

or, after β_n^i is solved,

$$\beta_n^i = (-1)^{[n/2]}\alpha_{i+3\ n/2}^{\delta_n} \quad (i = 0,1,2;\ n \geq 0) \text{ with } \delta_n = \begin{cases} 0 \text{ for } n \equiv 0\ (2) \\ 1 \text{ for } n \equiv 1\ (2) \end{cases} . \tag{10}$$

If one combines the conditions (9) with the representation (8), there results the solution of equation (7) determined according to case (a) of Theorem 1 in D, namely,

(11) $$x_{mn} = (-1)^{[m/3]}\beta^{\epsilon_m}_{2[m/3]+n} \qquad (m,n \geq 0),$$

while in case (b), the solution can be represented with the aid of (10) in dependence of initial functions $x_{mj} = \alpha^j_m$ $(j = 0, 1)$, in the form

(12) $$x_{mn} = (-1)^{[n/2]}\alpha^{\delta_n}_{m+3[n/2]} \qquad (m,n \geq 0)$$

4°. As an example of a discretized partial differential equation, let us consider the difference equation

(13) $$z_{m+2,n+1} - z_{m+1,n+2} - z_{m+1,n} + z_{m,n+1} = 0 \qquad (m,n \geq 0)$$

of order (2.2) appropriate for the wave equation $z_{xx} = z_{tt}$. Because $\ell_k = k_1 = 1$, the solution of (13) according to Theorem 1 is uniquely secured if three initial functions are prescribed, in the case (a) z_{on}, z_{1n}, z_{mo}, and in the case (b), z_{mo}, z_{m1}, z_{on}. For k_0, ℓ_0, only the possibility $k_0 = \ell_0 = 1$ exists. A compatibility condition between the four initial functions z_{mj} $(j = 0, 1)$, z_{in} $(i = 0, 1)$ is thus necessary. One obtains in [6] further evidence and the proof of existence of a solution from D only after application of an operational calculus to equation (13) where the initial functions are specially selected. We again use the difference law (6) with which, for arbitrary initial values $z_{mj} = \alpha^j_m$ $(j = 0, 1)$, $z_{in} = \beta^i_n$ $(i = 0, 1)$, $\alpha^j_i = \beta^i_j$ $(i,j = 0,1)$, there results the operational representation

(14) $$z = pq/(pq - 1)(\beta^{0'}_n + \alpha^0_m) + uy(\beta^{1'}_n - \alpha^{1'}_m - v\beta^0_n + u\alpha^0_m)$$

(u,v in Q inverse to p,q) with

$$\beta^{i'}_n = \begin{cases} 0_i & \text{for } n = 0 \\ \beta^i_n & \text{for } n > 0 \end{cases}, \quad (i = 0,1), \qquad \alpha^{1'}_m = \begin{cases} 0 & \text{for } m = 0 \\ \alpha^1_m & \text{for } m > 0 \end{cases},$$

and

$$y = \frac{p^2q}{(p - q)(pq - 1)} = \begin{cases} m + n + 1 & \text{for } |n| \leq m,\ m \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \notin D$$

From this, there follows, after easy calculation in Q, upon use of

$$pq/(pq - 1) = \delta_{mn} \in D$$

(δ_{mn} Kronecker delta) for $n = 0$, the required compatibility condition

$$\alpha^1_{m+1} - \beta^1_{m+1} = \alpha^0_m - \beta^0_m, \qquad m \geq 0 . \tag{15}$$

If one specializes the initial functions according to [6], namely,

$$z_{m0} = z_{m1} = \alpha^0_m, \qquad z_{on} = 0, \qquad z_{in} = \beta^1_n , \tag{16}$$

then (15) transforms to the condition

$$\alpha^0_m - \alpha^0_{m-1} = \beta^1_m \qquad (m \geq 1) , \tag{17}$$

which is equivalent to the equation

$$\alpha^0_n = \sum_1^n \beta^1_i \qquad (n \geq 1) ,$$

given in [6].

With the compatibility condition (15), the solution of (13) can be represented in dependence on three initial functions. In the case of (a), these are α^0_m, β^i_n $(i = 0, 1)$ and there results

$$z = \delta_{mn}(\beta^{0'}_n + \alpha^0_m) + uy(\beta^{1'}_n - \beta^1_m + \beta^0_{m-1} - \beta^0_{n-1}) .$$

If one carrys out the multiplication (in Q), one obtains finally the solution $z \in D$ in the form

$$\left(\sum_1^0 a_i = 0 \quad \text{set} \right)$$

$$(18)\qquad z_{mn} = \sum_{i=1}^{\min(m,n)} (\beta^1_{m+n+1-2i} - \beta^0_{m+n-2i}) + \begin{cases} \beta^0_{m-n} & \text{for } 0 \le m \le n, \\ \alpha^0_{mon} & \text{for } 0 \le n \le m . \end{cases}$$

For the special initial functions (16), equation (18) yields

$$z_{mn} = \sum_{i=1}^{\mathrm{Min}(m,n)} \beta^1_{m+n+1-2i} + \begin{cases} 0 & \text{for } 0 \le m \le n , \\ \alpha^0_{m-n} & \text{for } 0 \le n \le m \end{cases}$$

and one easily recognizes with the aid of the special compatibility condition (17) that this function is in agreement with that given in [6].

5^0. The linear difference equation of order (1,1) with constant coefficients

$$(19)\qquad ax_{m+1,n} - bx_{m,n+1} - cx_{mn} = 0 \qquad (m,n \ge 0;\ a,b \ne 0)$$

leads to the operator representation

$$(20)\qquad x = \frac{ap}{ap - bq - c}\left(\beta - \frac{b}{a}\, uq\alpha\right)$$

with initial functions $x_{m0} = \alpha^0_m = a$, $x_{on} = \beta^0_n = \beta$. On account of the vanishing of the coefficients $x_{m+1;n+1}$, there exists, according to Theorem 1, a compatibility condition between and B. This results from (20), since, for $n = 0$,

$$y = \frac{ap}{ap - bq - c} = \begin{cases} \binom{m}{-n}\left(\frac{c}{a}\right)^m\left(\frac{c}{b}\right)^n & \text{for } -m \le n \le 0 , \\ 0 & \text{otherwise} \end{cases}$$

and $(qy\alpha)_{m0} = 0$, in the form

$$(21)\qquad \alpha_m = \left(\frac{c}{a}\right)^m \sum_{i=0}^{m} \binom{m}{i}\left(\frac{b}{c}\right)^i \beta_i \qquad (m \ge 0) .$$

In the case (a) of Theorem 1 (x_{on} prescribed), the solution of (19) can be represented, with the aid of the compatibility condition (21), as a function of β alone, namely

$$x_{mn} = \left(\frac{c}{a}\right)^m \sum_{i=0}^{m} \binom{m}{i} \left(\frac{b}{c}\right)^i \beta_{n+1} \in D \ , \tag{22}$$

which results, after easy calculation[1].

If one eliminates x and α in (20) with the aid of (21) and (22), there results the operator relation

$$a^{-m} \sum_{i=0}^{m} \binom{m}{i} b^i c^{m-i} \beta_{n+i} = \frac{1}{ap-bq-c} \left(ap\beta_n - bqa^{-m} \sum_{i=0}^{m} \binom{m}{i} b^{-i} c^{m-i} \beta_i \right) \qquad (m,n \geq 0)$$

which for $\beta^n = d^n$ (d = constant) changes to

$$\left(\frac{c + bd}{a}\right)^m d^n = \frac{1}{ap - bq - c} \left(apd^n - bq \left(\frac{c + bd}{a}\right)^m \right) \qquad (m,n \geq 0) \ ,$$

---

[1]For $-m \leq n \leq -1$

$$x_{mn} = \left(\frac{c}{a}\right)^m \left(\frac{c}{b}\right)^n \sum_{i=0}^{m+n} c^{-i} \left(\binom{m}{-n+i} b^i \beta_i - \binom{m-1-i}{-n-1} a^i \alpha_i \right) ,$$

and from (21) and

$$\sum_{i=0}^{p} \binom{p+q-i}{p-i} \binom{r+i}{i} = \binom{p+q+r+1}{r} \qquad (p,q,r \geq 0)$$

it follows that $x_{mn} = 0$.

and, for $a = b$, $c = 0$, to

$$(23)\quad \beta_{mn} = \frac{1}{p - q}\,(p\beta_n - q\beta_m) \qquad (m,n \geq 0;\ \beta_{mn} = \beta_{m+n} \in D).$$

A formula analogous to (23) is known in the operational calculus for functions of two continuous variables (see perhaps [7]; p,q difference operators) in the theory of two-dimensional Laplace transformation (see [8]).

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ON THE GENERATION OF FIBONACCI NUMBERS AND THE "POLYVIBRATING" EXTENSION OF THESE NUMBERS

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SUMMARY

The authors, while emphasizing the role of Fibonacci numbers in the elaboration of the method of sequential search for the optimum, exhibit new generations of these numbers, taking as the point of departure either a certain ordinary, second-order differential equation, or a minimization problem of a certain functional. Then they present, while continuing their studies concerning polyvibrating systems and generalized polyvibratings, a "polyvibrating" extension of Fibonacci numbers.

1. Developments concerning the method of sequential search for the determination of the optimum, having at its basis Fibonacci numbers, have motivated in the recent times the work which refers itself so much to methods of finding the optimum in domains of several dimensions, as well as to different extensions of Fibonacci numbers. It is in the framework of this admirable progress which we helped in founding seven years ago and to the regular publication since then of a periodical specializing in this area, namely THE FIBONACCI QUARTERLY.

In the following, the authors give new generations of Fibonacci numbers, taking as the point of departure either a certain second-order ordinary differential equation, or a problem of minimizing a certain functional. They finally present, while continuing their research concerning polyvibrating systems and generalized polyvibratings, a "polyvibrating" extension of Fibonacci numbers.

2. Given

$$(a_1x + a_2)y'' + (a_3x + a_4)y' + (a_5x + a_6)y = 0 \quad , \tag{2.1}$$

the second-order differential equation of Laplace. The recurrence relation between the coefficients K_i of one of the equations solutions is written in the form of a power series expansion, which is given by

$$\frac{a_2}{n+2} K_{n+1} = -\frac{a_1 n + a_4 - a_1}{n+1} K_n - \frac{n a_3 + a_6 - a_3}{n} K_{n-1} - a_5 K_{n-2} \tag{2.2}$$

and the first terms of the set of coefficients determined by (2.1) are

$$\begin{aligned} &1;\ 1;\ -\frac{3}{a_2}\left(\frac{a_4}{2} + a_6\right);\ \frac{4}{a_2^2}\left[(a_1 + a_2)\left(\frac{1}{2} a_4 + a_6\right) - \frac{3}{2}(a_3 + a_6) - 3a_5\right]; \\ &-\frac{20}{a_2^3}(2a_1 + a_4)\left[(a_1 + a_4)\left(\frac{1}{2} a_4 + a_6\right) - \frac{3}{2}(a_3 + a_6) - 3a_5\right] + \\ &\qquad \frac{5(2a_3 + a_6)}{a_2^2} - \frac{5a_5}{a_2};\ \cdots \ . \end{aligned} \tag{2.3}$$

The direct calculation leads to the following theorem:

<u>Theorem 1.</u> Equation (2.1) reduces for

$$a_1 = 1, \quad a_2 = 0, \quad a_3 = -1, \quad a_4 = 2, \quad a_5 = -1, \quad a_6 = -1 \tag{2.4}$$

to the differential equation whose solution, in the form of a power series expansion, have for coefficients the Fibonacci numbers.

3. Following the fact that the equation of Laplace (2.1) results in the problem of minimizing the functional

$$\begin{aligned} [y(x)] = \int_a^b \Bigg[&-\frac{a_3 x^2 + 2a_4 x - 2a_1 x}{2}\left(yy'' + \frac{1}{2} y'^2\right) \\ &+ \frac{a_5 x + a_6}{2} y^2 - \frac{a_1 x + a_2 y'^2}{2} \Bigg] dx \end{aligned} \tag{3.1}$$

in the set of functions satisfying the conditions

$$y(a) = y(b) = 0 \ , \tag{3.2}$$

results in the following theorem.

Theorem 2. The minimization of the functional in the set of

$$T[y(x)] = \int_a^b \left[- \frac{-x^2 + 2x}{2} \ yy'' + \frac{y'^2}{2} - \frac{x+1}{2} y^2 - \frac{x}{2} y'^2 \ dx \right] \tag{3.3}$$

functions $y(x)$ satisfying (3.2) leads to the second-order ordinary differential equation

$$xy'' + (2 - x)y' - (x + 1)y = 0 \ , \tag{3.4}$$

of which the successive coefficients of the solution expressed as a power series represent the sequence of Fibonacci numbers.

4. Now consider the polyvibrating extension of the sequence of Fibonacci numbers taking the point of departure of polyvibrating systems, of which the prototype is given by the eigenvalue problem

$$D[A(x)Du + \lambda B(x)u] + \lambda[B(x)Du + C(x)u] = 0, \quad u\Big|_{FrR} = 0 \ , \tag{4.1}$$
$$x = (x_1, x_2, x_3, \cdots, x_m) \ ,$$

whose novelty consists in taking the rectangular domain

$$R = \{a_i \le x_i \le b_i; \quad i = 1, 2, 3, \cdots, m\}$$

which has m dimensions and the symbol D denoting the polyvibrating derivative, or (better) the total derivative in the sense of M. M. Picone, namely

$$Du \equiv \frac{\partial^m u}{\partial x_1 \partial x_2 \cdots \partial x_m} \tag{4.2}$$

or (better) be the variational problem

$$G[f(x)] = \int_R A(x)[Df(x)]^2 dx, \qquad (x = (x_1, x_2, \cdots, x_m)), \tag{4.3}$$

$$R = \{a_i \le x_i \le b_i\}, \qquad (i = 1, 2, \cdots, m)$$

and

$$H[f(x)] = \int_R [2B(x)f(x)Df(x) + C(x)f^2(x)]dx = \pm 1, \quad f(x)\Big|_{FrR} = 0 . \tag{4.4}$$

In the case where one considers the polyvibrating equation of Laplace

$$(a_1\theta + a_2)D^2u + (a_3\theta + a_4)Du + (a_5\theta + a_6)u = 0, \quad \theta = \prod_{i=1}^{m} x_i , \tag{4.5}$$

the recurrence relation generating the coefficients in the form of a power series solution of the product $\theta = x_1, x_2, \cdots, x_m$ of the equation (4.5) is

$$\begin{aligned} a_2(n+1)^m n^m k_{n+1} &= -\left[a_1 n^m [n-1]^m + a_4 n^m\right](n+2)k_n - (n+1)(n+2) \\ &\cdot [a_3(n-1)^m + a_6]k_{n-1} - a_5 n(n+1)(n+2)k_{n-2} , \end{aligned} \tag{4.6}$$

while the relation generating the polyvibrating extension of Fibonacci numbers (the hypothesis (2.4) concerning the coefficients of equation (4.5) is given by

$$[n^m(n = 1)^m + 2n^m]k_n = (n+1)[(n-1)^m + 1]k_{n-1} + n(n+1)k_{n-2} \tag{4.7}$$

and the first terms of corresponding sequence are

$$1;\ 2^{1-m};\ \frac{4}{3^m(2^m+2)}\left[\frac{2^m+1}{2^m-1}+3\right];\ \cdots \quad . \tag{4.8}$$

REMARKS

1. It would be interesting to give a geometric interpretation for the coefficients of the sequence (2.3) and of its polyvibrating extension (4.8).

2. The application to the variational problem (3.2), (3.3) and to its polyvibrating extension (4.3) and (4.4) of the method of dynamic programming or by other present methods of optimization could perhaps clear up the (why) of the fundamental role which is performed by the sequence of Fibonacci numbers and the corresponding differential equation in sequential search, which is used with such success in the theory of supplies of all kinds, automatic sample control genetics, separation processes of separation of several phases and still others.

3. We refer finally for algorithmic details and other results concerning this class of ideas to the paper in Bulletin of Polytechnic Institute of Jassey. To be found there among other ideas, are the extensions of the Fibonacci numbers and the many relations that connect them, corresponding to generalized polyvibrating systems, that is to the systems of the form (2.1), (2.4), where the ordinary differential operator d/dx and the independent variable x are systematically and respectively replaced by the generalized polyvibrating operator 16 - 17.

$$D^*u = \frac{\partial^{n_1+n_2+\cdots+n_m}}{\partial x^{n_1}\,\partial x_2^{n_2}\cdots\partial x_m^{n_m}}$$

and by the product of independent variables

$$\theta^* = \prod_{i=1}^{m} x_i^{n_i} \quad . \tag{5.2}$$

CONTINUED ON INSIDE BACK COVER

ON PARTLY ORDERED PARTITIONS OF A POSITIVE INTEGER

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1. INTRODUCTION

The following problem is discussed. Let

$$V_1 = (n, \underbrace{0, \cdots, 0}_{n-1}),$$

where n is a finite positive integer. From V_1 are generated

$$V_{i+1} = (n - i, i, \underbrace{0, \cdots, 0}_{n-2}), \qquad 1 \leq i < n .$$

From V_2 are generated

$$V_{n+j} = (n - 1 - j, 1, j, \underbrace{0, \cdots, 0}_{n-3}), \qquad 1 \leq j < n - 1 ,$$

and so on, until the entire list of non-null vectors V_i has been considered.

Suppose the first k $(0 \leq k \leq n)$ components from left to right in each vector V_i are fixed, with $k = 0$ meaning that none is fixed, and the remaining components are arranged from left to right in descending order of magnitude. The positive integers in each vector V_i form a partition of n and on arranging the components as above, we obtain what we define as <u>partly ordered partitions</u> of the integer n.

Let $\phi_k(n)$ denote the number of distinct non-null vectors V_i in the system generated above in which the first k components are kept fixed. The primary object of this paper is to derive a recurrence relation for $\phi_k(n)$. Several other interesting results are obtained.

2. IMMEDIATE RESULTS

Let $p(n)$ denote the number of distinct partitions of the positive integer n. Several values of $p(n)$ can be found in [1], page 35.

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*This paper was written while the author was on an N. R. C. postdoctoral fellowship at the University of Waterloo.

Let V_i' be the vector obtained from V_i $(i = 1, 2, \cdots)$ by removing all zero components of V_i and let $[V]$, $[V']$ denote the set of non-null vectors V_i, V_i', respectively. There is a one-one correspondence between V_i and V_i' and hence between $[V]$, $[V']$. We have,

Theorem 1. $\phi_0(n) = p(n)$.

Proof. The components of V_i' constitute a partition of n. Suppose the components of each vector in $[V']$ are arranged from left to right in descending order of magnitude. Then each V_j' $(j \neq i)$ which has the same components as V_i' after rearrangement, hence the distinct vectors in $[V']$ are those vectors V_i' whose components are distinct partitions of n, hence

$$\phi_0(n) = p(n) .$$

Theorem 2. $\phi_k(n) = 2^{n-1}$, $k = n$ or $n - 1$, $(n \geq 1)$.

Proof. We show first that $\phi_{n-1}(n) = \phi_n(n)$.

$$V_i' = (1, \underbrace{1, \cdots}_{n}, 1)$$

is the only vector in $[V']$ which has more than $n - 1$ components, hence keeping $n - 1$ components fixed in $[V']$ is equivalent to keeping all n components fixed; that is,

$$\phi_{n-1}(n) = \phi_n(n) .$$

Now the system $[V']$ contains all the compositions of the integer n, hence by a result of [2, page 124], $\phi_n(n) = 2^{n-1}$.

This proves the theorem.

We come now to the more significant results.

3. MAIN RESULTS

Theorem 3. $\phi_k(n) = \phi_k(n-1) + \phi_{k-1}(n - 1)$, $(k \geq 1)$.

Proof. $\phi_k(n)$ is obtained from $\phi_{k-1}(r)$, $1 \leq r \leq n - 1$, in the following way:

Let $[U]$ be the system of distinct non-null vectors generated for a particular value of r $(1 < r \leq n - 1)$ in which the first $k - 1$ components in

each vector are fixed and the other components are arranged in descending order. Let

$$U = (u_1, u_2, \cdots, u_r) \qquad [U] .$$

Define

$$U' = (n - r, u_1, u_2, \cdots, u_r) .$$

There is a one-one correspondence between U, U' and as U runs through the vectors in $[U]$ we obtain a system of distinct non-null vectors in which the non-zero components sum to n and the first k components are fixed. As r runs through all integral values from 1 to $n - 1$ we obtain collectively all the distinct non-null vectors in $\phi_k(n)$ except

$$V = (n, \underbrace{0, 0, \cdots}_{n-1}, 0) ,$$

hence,

$$\begin{aligned}
\phi_k(n) &= 1 + \sum_{r=1}^{n-1} \phi_{k-1}(r) , \\
&= \left(1 + \sum_{r=1}^{n+2} \phi_{k-1}(r)\right) + \phi_{k-1}(n - 1) , \\
&= \phi_k(n - 1) + \phi_{k-1}(n - 1) .
\end{aligned}$$

Using this result and the values for $\phi_0(n)$ which are to be taken as initial values we obtain Table 1 for $1 \le n \le 10$. We take $\phi_0(0) = 0$, and for $k > n$ and finite we may also put $\phi_k(n) = \phi_n(n)$ since this simply entails expanding the vectors in $[V]$ by adding a further $k - n$ zero components on the right in each vector. These values of $\phi_k(n)$ fall below the leading diagonal in the table and are omitted.

We note also that the binomial coefficients also satisfy a similar recurrence relation.

Table 1

n	0	1	2	3	4	5	6	7	8	9	10
ϕ_0	0	1	2	3	5	7	11	15	22	30	42
ϕ_1		1	2	4	7	12	18	30	45	67	97
ϕ_2			2	4	8	15	27	46	76	121	188
ϕ_3				4	8	16	31	58	104	180	301
ϕ_4					8	16	32	63	121	225	405
ϕ_5						16	32	64	127	248	473
ϕ_6							32	64	128	255	503
ϕ_7								64	128	256	511
ϕ_8									128	256	512

Here ϕ_i stands for $\phi_i(n)$ $(0 \le i \le 8)$.

<u>Corollary 1.</u> $\phi_{n-2}(n) = 2^{n-1}$, $(n \ge 2)$.

<u>Proof.</u> By Theorem 3,

$$\sum_{s=0}^{n-3} (\phi_{n-2-s}(n - s) - \phi_{n-3-s}(n - s - 1)) = \sum_{s=0}^{n-3} \phi_{n-2-s}(n - s - 1) ,$$

that is,

$$\phi_{n-2}(n) - \phi_0(2) = \sum_{s=1}^{n-2} 2^s ,$$

by Theorem 2, hence,

$$\phi_{n-2}(n) = 2(2^{n-2} - 1) + \phi_0(2) ,$$

$$= 2^{n-1} .$$

The following result can also be obtained by using similar difference methods.

Corollary 2. $\phi_{n-3}(n) = 2^{n-1} - 1, \quad n \geq 3.$

Before we state a general expression for $\phi_{n-j}(n)$, $3 \leq j \leq n-1$, we prove the following lemmas.

Lemma 1.

$$\sum_{r=0}^{n-j-1} \binom{j-3+r}{r} = \binom{n-3}{n-j-1}, \quad 3 \leq j \leq n-1, \quad n \geq 4.$$

Proof.

$$\sum_{r=0}^{n-j-1} \binom{n-3+r}{r} = \sum_{r=1}^{n-j-1} \left[\binom{j-2+r}{r} - \binom{j-3+r}{r-1}\right] + 1$$

$$= \binom{n-j}{n-j-1} = 1 + 1$$

$$= \binom{n-3}{n-j-1} .$$

Lemma 2.

$$\sum_{r=0}^{q-2} \binom{p+r}{r} 2^{q-r} = \sum_{r=0}^{q-3} \binom{p+r+1}{r} 2^{q-r-1} + 4\binom{p+q-1}{q-2}, \quad q \geq 2.$$

Proof.

$$\sum_{r=0}^{q-2} \binom{p+r}{r} 2^{q-r} = \binom{p+1}{0} 2^{q-1} + \left[\binom{p+1}{0} + \binom{p+1}{1}\right] 2^{q-1}$$

$$+ \sum_{r=2}^{q-2} \binom{p+r}{r} 2^{q-r} ,$$

$$= \binom{p+1}{0} 2^{q-1} + \binom{p+2}{1} 2^{q-2} + \left[\binom{p+2}{1} + \binom{p+2}{2}\right] 2^{q-2}$$

$$+ \sum_{r=3}^{q-2} \binom{p+r}{r} 2^{q-r} ,$$

$$\vdots$$

$$= \sum_{r=0}^{q-3} \binom{p+r+1}{r} 2^{q-r-1} + 4\binom{p+q-1}{q-2} .$$

<u>Theorem 4.</u>

$$\phi_{n-j}(n) = \sum_{r=0}^{n-j-1} \binom{j-3+r}{r} 2^{n-j-r+1} + \sum_{r=3}^{j} \binom{n-r-1}{j-r} \phi_0(r) ,$$

$$3 \le j \le n-1, \quad n \ge 4 .$$

<u>Proof.</u> When $j = 3$, the right-hand side is

$$\sum_{r=0}^{n-4} 2^{n-r-2} + \phi_0(3)$$

$$= 2^{n-1} - 4 + 3$$

$$= 2^{n-1} - 1 .$$

By Corollary 2 above, theorem is true for $j = 3$. Assuming it is true for j, we have, by Theorem 3,

$$\sum_{s=0}^{n-j-2} (\phi_{n-j-s-1}(n-s) - \phi_{n-j-s-2}(n-s-1)) = \sum_{s=0}^{n-j-2} \phi_{n-j-s-1}(n-s-1) ,$$

$$= \sum_{s=0}^{n-j-2} \left(\sum_{r=0}^{n-j-s-2} \binom{j-3+r}{r} 2^{n-j-r-s} + \sum_{r=3}^{j} \binom{n-r-s-2}{j-r} \phi_0(r) \right) ,$$

$$= \sum_{r=0}^{n-j-2} \binom{j-3+r}{r} 2^{n-j-r} \sum_{s=0}^{n-j-r-2} 2^{-s} + \sum_{r=3}^{j} \phi_0(r) \sum_{s=0}^{n-j-2} \binom{n-r-s-2}{j-r},$$

$$= \sum_{r=0}^{n-j-2} \binom{j-3+r}{r} (2^{n-j-r+1} - 4) + \sum_{r=3}^{j} \binom{n-1-r}{j-r+1} \phi_0(r) \quad , \text{ by Lemma 1,}$$

$$= \sum_{r=0}^{n-j-1} \binom{j-3+r}{r} 2^{n-j-r+1} - 4\left\{\binom{n-4}{n-j-1} + \sum_{r=0}^{n-j-2} \binom{j-3+r}{r}\right\}$$

$$+ \sum_{r=3}^{j} \binom{n-r-1}{j-r+1} \phi_0(r) ,$$

$$= \sum_{r=0}^{n-j-2} \binom{j-2+r}{r} 2^{n-j-r} + 4\binom{n-3}{n-j-1} - 4\left\{\binom{n-4}{n-j-1} + \binom{n-4}{n-j-2}\right\}$$

$$+ \sum_{r=3}^{j} \binom{n-r-1}{j-r+1} \phi_0(r) ,$$

by Lemmas 1 and 2,

$$= \sum_{r=0}^{n-j-2} \binom{j-2+r}{r} 2^{n-j-r} + \sum_{r=3}^{j} \binom{n-r-1}{j-r+1} \phi_0(r) .$$

Hence,

$$\phi_{n-j-1}(n) = \sum_{r=0}^{n-j-2} \binom{j-2+r}{r} 2^{n-j-r} + \sum_{r=3}^{j} \binom{n-r-1}{j-r+1} \phi_0(r) + \phi_0(r+1) ,$$

$$= \sum_{r=0}^{n-j-2} \binom{j-2+r}{r} 2^{n-j-r} + \sum_{r=3}^{j+1} \binom{n-r-1}{j-r+1} \phi_0(r) .$$

Thus, if true for j, also true for $j + 1$. This proves the theorem.

This proves the theorem.

Further reductions on the result of Lemma 2 give the following:

Theorem 5.

$$\sum_{r=0}^{q-2} \binom{p+r}{r} 2^{q-r} = 4 \sum_{r=0}^{q-2} \binom{p+q-1}{r} .$$

Theorem 4 can now be stated in the following way:

Lemma 3.

$$\phi_{n-j}(n) = 4 \sum_{r=0}^{n-j-1} \binom{n-3}{r} + \sum_{r=3}^{j} \binom{n-r-1}{j-r} \phi_0(r) .$$

Two special cases which are easily obtained from Lemma 3 are stated in

Theorem 6.

$$\phi_{\frac{n-1}{2}}(n) = 2^{n-2} + 2\binom{n-3}{\frac{n-3}{2}} + \sum_{r=3}^{\frac{n+1}{2}} \binom{n-r-1}{\frac{n+1}{2}-r} \phi_0(r), \quad n \text{ odd} \qquad (\geq 5) ,$$

$$\phi_{\frac{n-2}{2}}(n) = 2^{n-2} + \sum_{r=3}^{\frac{n+2}{2}} \binom{n-r-1}{\frac{n+2}{2}-r} \phi_0(r), \quad n \text{ even} \qquad (\geq 4) .$$

The author is indebted to R. N. Burns of the University of Waterloo for his many helpful suggestions.

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