FIBONACCI AND LUCAS NUMBERS THROUGH THE ACTION OF THE MODULAR GROUP ON REAL QUADRATIC FIELDS

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1. INTRODUCTION

The modular group $PSL(2,Z)$ is the group of all linear-fractional transformations $z \mapsto \frac{az+b}{cz+d}$, where a,b,c,d are integers satisfying $ad-bc=1$. Such transformations have matrix representations $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$. It is well-known that the group $PSL(2,Z)$ is generated by two elements $x : z \mapsto \frac{-1}{z}$ and $y : z \mapsto \frac{z-1}{z}$ which satisfy the relations $x^2 = y^3 = 1$. A natural action of $PSL(2,Z)$ on the rational projective line $Q(\sqrt{n})$, where n is a square-free positive integer, produces interesting results. An effective way of viewing this action is to use a graphical presentation known as a *coset diagram.*

2. COSET DIAGRAMS

Let S be a group generated by the elements $x_1, x_2, \ldots, x_k$ and acting on a set Ω. Then the elements of Ω may be represented by the vertices of a digraph, with edge of 'colour i' directed from vertex u to vertex v whenever $ux_i = v$.

$$u \longmapsto ux_i = v$$

The resulting diagram is a connnected graph whose vertices can be identified with the right cosets in S of the stabiliser N of any given point of Ω. Hence an edge of colour i joins the coset Ng to the coset Ngx_i, for each g in S, and the resulting diagram is called a *coset diagram.*

This is very similar to the notion of a Schreier coset graph whose vertices represent the cosets of any given subgroup in a finitely-generated group, and also to that of a Cayley graph whose vertices are the group elements themselves (see [1]), with trivial stabiliser.

As propounded by G. Higman, the coset diagrams defined for the actions of $PSL(2,Z)$ are special in a number of ways. First, they are defined for a particular group, namely $PSL(2,Z)$, which has a presentation in terms of two generators x and y. Since there are only two generators, it is possible to avoid using colours as well as the orientation of edges associated with the involution x. For y, which has order 3, there is a need to distinguish y from y^2. The 3-cycles of y are therefore represented by small triangles, with the convention that y permutes their vertices counterclockwise, while the fixed points of x and y, if any, are denoted by heavy dots (see Fig. 1).

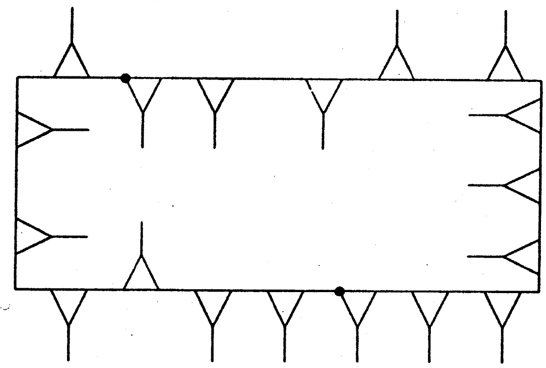

Figure 1

Thus the geometry of the figure makes the distinction between x-edges and y-edges obvious.

If $P = \{v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k\}$ is an alternating sequence of vertices and edges of a coset diagram for an action of $PSL(2, Z)$, then π is a *path* in the diagram if e_i joins v_{i-1} and v_i for each i, and $e_i \neq e_j$, where $i \neq j$. By a circuit we mean a *closed path* of edges and triangles. If $n_1, n_2, \ldots, n_{2k}$ is a sequence of $2k$ positive integers, then by a *circuit of type* $(n_1, n_2, \ldots, n_{2k})$, we mean the circuit in which n_1 triangles have one vertex outside the circuit and n_2 triangles have one vertex inside the circuit and so on. Such a circuit evolves an element of $PSL(2, Z)$ and fixes a particular vertex of a triangle lying on the circuit. It is important to mention here a result from [4].

Proposition 2.1: *Every element of $PSL(2, Z)$, except the (group theoretic) conjugates of $x, y^{\pm 1}$ and $(xy)^n$, where $n > 0$ has real quadratic irrational numbers as fixed points.*

For the picture of a circuit as described above, consider for example the circuit of the type (3,2,1,2,3,2) (see Fig. 2).

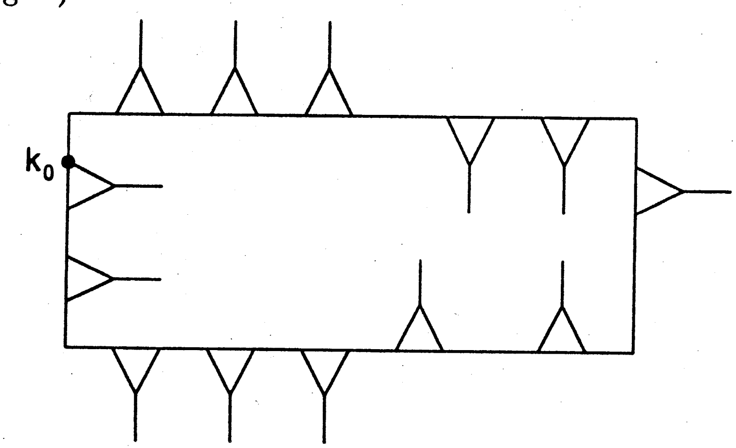

Figure 2

This circuit evolves the element $g = (xy)^3(xy^{-1})^2(xy)(xy^{-1})^2(xy)^3(xy^{-1})^2$ of $PSL(2, Z)$ which fixes the particular vertex k_0 as shown.

If k is the number of sets of triangles on the circuit, with one vertex outside the circuit, and k' is the number of sets of triangles on the circuit, with one vertex inside the circuit, then

$k = k'$, which means that the total number of sets of triangles in a circuit is $2k$. Observe that such sets of triangles with one vertex outside inside occur alternately. At this juncture, we take note of the following relevant result which was proved in [3].

Proposition 2.2: *For given positive integers $n_1, n_2, \ldots, n_{2k}$ there does not exist a circuit of the type $(n_1, n_2, \ldots, n_{2k'}, n_1, n_2, \ldots, n_{2k'}, \ldots, n_1, n_2, \ldots, n_{2k'})$, where k' divides k.*

3. THE ACTION OF $PSL(2,Z)$ ON $Q(\sqrt{n})$

Let $\alpha \in Q(\sqrt{n})$ be of the form $\frac{(a+\sqrt{n})}{c}$, n being a square-free positive integer, and the integers a, c and $\frac{(a^2-n)}{c}$ being relatively prime. Let $\overline{\alpha} = \frac{(a-\sqrt{n})}{c}$ be the algebraic conjugate of α. For some fixed square-free positive integer n, an element $\alpha = \frac{(a+\sqrt{n})}{c}$ and its algebraic conjugate $\overline{\alpha}$ may have different signs. In this case we call α an *ambiguous number*. If α and $\overline{\alpha}$ are both negative (positive), then we call α *totally negative (totally positive) number.*

Now, if k is one of the three vertices of a triangle in a coset diagram representing the action of $PSL(2,Z)$ on real quadratic irrational number fields, then the other two vertices will be ky and ky^{-1}. It is then not hard to see that one of k, ky and ky^{-1} will be a (totally) positive number and the other two will be ambiguous numbers.

Let us take note of the following relevant result which was proved in [3].

Theorem 3.1:

For every real quadratic irrational number in the orbit of α under the action of $PSL(2,Z)$ on real quadratic fields, the square-free positive integer n has the same value.

This result was then used to show (see Theorem 3 of [3]) that there is only a finite number of ambiguous numbers and, in particular there is only a finite number of such numbers in the orbit. It is also useful to note from [3] that in a coset diagram for the orbit of α under $PSL(2,Z)$, not only the ambiguous numbers form a circuit, but also this is the only circuit in the orbit of α.

Thus, if we are given a real quadratic irrational number α we can find the circuit in the orbit of α under $PSL(2,Z)$. This also means that if we have two real quadratic irrational numbers α and β, then we can test to see whether or not they belong to the same orbit. For some values of n, there is more than one orbit containing numbers of discriminant n. For example, the circuits associated with the elements $(xy)^2(xy^{-1})^3xyxy^{-1}$ and $(xy)^{18}xy^{-1}$ of $PSL(2,Z)$ contain numbers of the same discriminant 99.

4. MATRIX REPRESENTATIONS OF ELEMENTS IN $PSL(2,Z)$

Consider the element $g \in PSL(2,Z)$ of the form $g = xyxy^{-1}xyxy^{-1}\ldots xyxy^{-1} = (xyxy^{-1})^m = (xyxy^2)^m$, observing that $(xyxy^{-1})^m$ is the m^{th} power of a commutator. Let $A(g)$ be the matrix (representation) of g. We shall show that $A(g)$ is a 2×2 symmetric matrix whose entries are Fibonacci numbers. A 2×2 symmetric matrix A, whose entries are Fibonacci numbers, is called a Fibonacci matrix [7] if its powers satisfy the recurrence (matrix) relation

$$A^n = A^{n-1} + A^{n-2}, \text{ where } n \in Z. \tag{4.1}$$

As we shall see shortly, the matrix $A(g)$, however, is not a Fibonacci matrix as it does not satisfy (4.1).

Recalling that $x : z \mapsto \frac{-1}{z} = \frac{1}{-z}$ and $y : z \mapsto \frac{z-1}{z}$, so that $xy : z \mapsto z+1$ and $xy^{-1} : z \mapsto \frac{z}{z+1}$, we have $A(xy) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A(y).A(x)$ and $A(xy^{-1}) = A(xy^2) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = A(y^2).A(x) = (A(xy))^T$. Hence, with B denoting the matrix $A(xyxy^{-1})$, we have $B = A(xyxy^{-1}) = A(xy^{-1}).A(xy) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_2 & F_3 \end{bmatrix}$, where F_k is the k^{th} Fibonacci number. It readily follows that $B^2 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} F_3 & F_4 \\ F_4 & F_5 \end{bmatrix}$, $B^3 = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} = \begin{bmatrix} F_5 & F_6 \\ F_6 & F_7 \end{bmatrix}$.

Using induction, with the help of the Fibonacci relations $F_n = F_{n-1} + F_{n-2}, n > 2$ and $F_0 = 0, F_1 = 1$, we can easily show that $B^m = \begin{bmatrix} F_{2m-1} & F_{2m} \\ F_{2m} & F_{2m+1} \end{bmatrix}, m \geq 1$. It is then immediate that the trace $tr(B^m)$ of B^m, is $tr(B^m) = F_{2m-1} + F_{2m+1} = L_{2m}$, where for integers $n \geq 0$, the Lucas numbers L_n satisfy the relations $L_n = L_{n-1} + L_{n-2}, n > 2$ and $L_0 = 2, L_1 = 1$.

The norm of B^m, that is of $A(g)$, which must be 1 by the defintions of x and y, is given by $\begin{vmatrix} F_{2m-1} & F_{2m} \\ F_{2m} & F_{2m+1} \end{vmatrix} = F_{2m-1}F_{2m+1} - F_{2m}^2 = (-1)^{2m} = 1$, as it is well-known that $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ (see [7]).

We also observe that although the entries of $B^m = A(g)$ are Fibonacci numbers, neither B nor any of its powers B^m are Fibonacci matrices, since $B^{n-1} + B^{n-2} = \begin{bmatrix} F_{2n-3} & F_{2n-2} \\ F_{2n-2} & F_{2n-1} \end{bmatrix} + \begin{vmatrix} F_{2n-5} & F_{2n-4} \\ F_{2n-4} & F_{2n-3} \end{vmatrix} = \begin{vmatrix} L_{2n-4} & L_{2n-3} \\ L_{2n-3} & L_{2n-2} \end{vmatrix} \neq B^n$. To sum up, we have proved the following result.

Theorem 4.1:

Let α be a real quadratic irrational number and let $g = (xyxy^{-1})^m = (xyxy^2)^m \in PSL(2, Z)$ act on α, so that the orbit of α contains a circuit of type $\underbrace{(1, 1, \ldots, 1)}_{2m}$. Then the matrix $A(g)$ of g is

$$A(g) = \begin{bmatrix} F_{2m-1} & F_{2m} \\ F_{2m} & F_{2m+1} \end{bmatrix}, \quad (m \geq 1),$$

where F_k is the k^{th} Fibonacci number, with trace given by $tr(A(g)) = L_{2m}$, L_k being the k^{th} Lucas number.

Since we are dealing with circuits of coset diagrams, we may begin with xy^{-1}, so that we may consider the element $h = (xy^{-1}xy)^m = (xy^2xy)^m$ $(m \geq 1)$, instead of the element $g = (xyxy^{-1})^m = (xyxy^2)^m$, $(m \geq 1)$, of $PSL(2, Z)$, and obtain

$A(xy).A(xy^{-1}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_3 & F_2 \\ F_2 & F_1 \end{bmatrix}$. If we take A to be the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\left(= \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}\right)$, then we have $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_3 & F_2 \\ F_2 & F_1 \end{bmatrix}$, so that inductively $A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$. It is now a straightforward matter to verify that the matrices $A^n (n \in Z)$ are Fibonacci matrices. It follows that the matrix of $xy^{-1}xy$ is $A(xy^{-1}xy) = A(xy).A(xy^{-1}) = \begin{bmatrix} F_3 & F_2 \\ F_2 & F_1 \end{bmatrix} = A^2$, so that inductively, $A(h) = \begin{bmatrix} F_{2m+1} & F_{2m} \\ F_{2m} & F_{2m-1} \end{bmatrix} = A^{2m}$, a Fibonacci matrix, having the same trace and norm as $A(g)$. Observe that both matrices $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, that is, $A(xyxy^{-1})$ and $A(xy^{-1}xy)$, have the same eigenvalues, given by the roots λ_1 and λ_2 of $\lambda^2 - 3\lambda + 1 = 0$, where $\lambda_1 = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2} = 1 + \tau = \tau^2$, and $\lambda_2 = \frac{3-\sqrt{5}}{2} = 1 + \frac{1-\sqrt{5}}{2} = 1 + \sigma = \sigma^2$, τ being the golden-section number and σ, its (algebraic) conjugate. Since both of these matrices are 2×2 symmetric matrices that have two distinct eigenvalues, using known results of linear algebra, we can conclude that they are orthogonally diagonalizable, and hence that the eigenvalues of $A(g) = (A(xyxy^{-1}))^m$ (and also of $A(h) = (A(xy^{-1}xy))^m$ are $\lambda_1^m = \tau^{2m}$ and $\lambda_2^m = \sigma^{2m}$. Since it is well known that $\tau^n + \sigma^n = L_n$, we obtain once again the result $tr(A(g)) = L_{2m} = tr(A(h))$.

In the form $L_{2m} = \sum_{r=0}^{m} \frac{2m}{2m-r}\binom{2m-r}{r}$, the number L_{2m} is known (see [6]) to be the number of ways of selecting r objects no two of them adjacent, from $2m$ objects arranged circularly or, equivalently, the number of r-subsets of the 'circular set' (that is, cyclically-ordered set) $\{1, 2, \ldots, 2m\}$ that does not contain two consecutive integers (these subsets are also known as the r-subsets without unit separation of the circular set $\{1, 2, \ldots, 2m\}$ (see [2])).

Using this known fact and a rather involved combinatorial argument while looking at the ways the trace of the matrix,

$$A(g) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{4.4}$$

can be computed from the product on the right side of (4.4), without having to determine the matrix $A(g)$ itself, we can prove the following result.

Theorem 4.2:

Let $W = \{1, 2, \ldots, 2m\}$ be a cyclically-ordered set of positive integers and let the orbit of α contain a circuit of type $\underbrace{(1, 1, \ldots, 1)}_{2m}$. If P is the collection of non-empty subsets of W obtained by striking out any number of adjacent pairs of elements of W, then

$$tr(A(g)) = 2 + |P|, \tag{4.5}$$

where as usual $|P|$ is the cardinality of P.

We observe that $|P| = \sum_{r=1}^{m} L_{2r-1}$, since it can be shown by induction that $L_{2m} = 2 + \sum_{r=1}^{m} L_{2r-1}$.

5. THE CASE WHEN $(xy^{-1}xy)^m$ OR $(xyxy^{-1})^m$ FIXES ELEMENTS OF THE FIELD $Q(\sqrt{n})$

Here, we shall require that, for all integers $m \geq 1$, the transformation $h = (xy^{-1}xy)^m = (xy^2xy)^m$, or $g = (xyxy^{-1})^m = (xyxy^2)^m$, fix elements of $Q(\sqrt{n})$. We shall show that $n = 5$ in this case.

Let $\alpha \in Q(\sqrt{n})$. If h is the fix α, then $\frac{F_{2m+1}\alpha+F_{2m}}{F_{2m}\alpha+F_{2m-1}} = \alpha$, so that $F_{2m}\alpha^2 + (F_{2m-1} - F_{2m+1})\alpha - F_{2m} = 0$, that is $F_{2m}(\alpha^2 - \alpha - 1) = 0$, for all integers ≥ 1, whence $\alpha = \frac{1 \pm \sqrt{5}}{2}$. This implies that $n = 5$, and the elements fixed by h are $\tau = \frac{1+\sqrt{5}}{2}$ and $\sigma = \frac{1-\sqrt{5}}{2} = \overline{\tau}$. On the other hand, if g is to fix $\alpha \in Q(\sqrt{n})$, then $\frac{F_{2m-1}\alpha+F_{2m}}{F_{2m}\alpha+F_{2m+1}} = \alpha$, leads to $F_{2m}(\alpha^2 + \alpha - 1) = 0$, for all integers ≥ 1, whence $\alpha = \frac{-1\pm\sqrt{5}}{2}$. This again implies that $n = 5$, and the elements fixed by g are $\tau - 1 = \tau^{-1} = \frac{-1+\sqrt{5}}{2}$ and $\sigma - 1 = \sigma^{-1} = \frac{-1-\sqrt{5}}{2} = \overline{\tau} - 1 = (\overline{\tau})^{-1} = \overline{\tau^{-1}}$.

It follows that when the elements x and y of $PSL(2, Z)$ act on $Q(\sqrt{n})$, under the condition that for all integers $m \geq 1, h = (xy^{-1}xy)^m$, or $g = (xyxy^{-1})^m$, fixes the elements of $Q(\sqrt{n})$, then $n = 5$ and the circuit corresponding to h or g reduces to the circuit corresponding to $xy^{-1}xy$ or to $xyxy^{-1}$, and hence contains only two pairs of ambiguous numbers, namely, $\tau = \frac{1+\sqrt{5}}{2}$, $\sigma = \frac{1-\sqrt{5}}{2}$, and $\tau^{-1} = \frac{-1+\sqrt{5}}{2}, \sigma^{-1} = \frac{-1-\sqrt{5}}{2}$ (see Fig. 3).

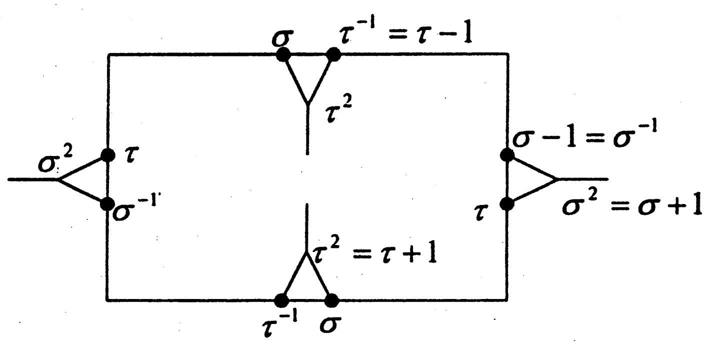

Figure 3.

This circuit, or its homomorphic image, has the following form (Figure 4).

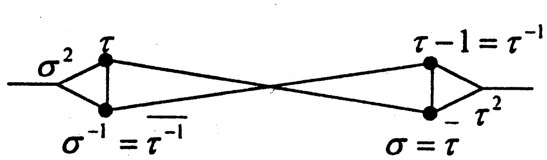

Figure 4.

From the circuit of Fig. 4, it is apparent that, instead of using the elements h and g of $PSL(2,Z)$ to fix elements of $Q(\sqrt{n})$ we can obtain the same end result by letting the field automorphism $s: z \mapsto \overline{z}$, and the inversion $t: z \mapsto \frac{1}{z}$ acting on $Q(\sqrt{5})$. Since $s^2 = t^2 = (st)^2 = 1$, we see that the Klein 4-group V_4 acts on $Q(\sqrt{5})$ much in the same way as the modular group $PSL(2,Z)$ acts on $Q(\sqrt{n})$ through its elements h and g, with h, or g, fixing elements of $Q(\sqrt{n})$. In other words, for integers $n_1, n_2, \ldots, n_m$, when, $n_1 = n_2 = \cdots = n_m = 1$, the orbit of τ under V_4 contains circuits of the type $\underbrace{(1,1,\ldots,1)}_{m}$ each with only two pairs of ambiguous numbers, namely, $\tau, \tau s = \overline{\tau}$ where $\tau = \frac{1+\sqrt{5}}{2}$ and $\tau t = \tau^{-1}, (\tau t)s = \overline{\tau^{-1}} = (\overline{\tau})^{-1} = (\tau s)t$.

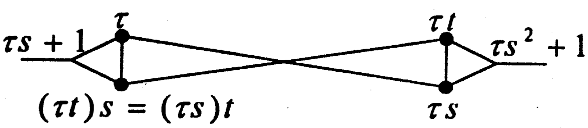

Figure 5.

By way of conclusion, let us summarize the preceding discussion in the following proposition.

Proposition 5.1:
Let $\alpha \in Q(\sqrt{n})$, and $h = (xy^{-1}xy)^m$ and $g = (xyxy^{-1})^m$ be elements of the modular group $PSL(2,Z)$ acting on α, so that the orbit of α contains circuits of the type $\underbrace{(1,1,\ldots,1)}_{2m}$. If h (or g) is to fix α, then $\alpha = \tau$, σ (or $\alpha = \tau^{-1}, \sigma^{-1}$), where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden section number and, σ its algebraic conjugate, so that $n = 5$ and the reduced circuit in the coset diagram for the action of h, or g, on $Q(\sqrt{5})$ contains only two pairs of ambiguous numbers, namely, τ, σ, and τ^{-1}, σ^{-1}, all being fixed points of the transformation(s). Moreover, this reduced circuit is the same as the circuit of type $\underbrace{(1,1,\ldots,1)}_{m}$ in the orbit of $\tau \in Q(\sqrt{5})$ under the action of the Klein 4-group: $V_4 = \langle s,t : s^2 = t^2 = (st)^2 = 1 \rangle$.

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