SEQUENCES $\{H_n\}$ FOR WHICH H_{n+1}/H_n APPROACHES THE GOLDEN RATIO

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ABSTRACT. The Golden Ratio Φ can be obtained as the limit n goes to $+\infty$ of the ratio H_{n+1}/H_n for an infinite number of sequences $\{H_n\}$.

1. Introduction

One of the properties of the Fibonacci sequence $\{F_n\}$ is

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi = \frac{1+\sqrt{5}}{2},$$

the Golden Ratio. It is well-known that $\Phi \approx 1.6180339$ has the unique property that Φ and Φ^{-1} have the same decimal part since $\Phi = 1 + \Phi^{-1}$.

Also, if $\{H_n\}$ satisfies the Fibonacci recursion with $H_1 = a$, $H_2 = b$, then $H_n = aF_{n-1} + bF_{n-2}$ and the ratio H_{n+1}/H_n also approaches Φ as a limit [1, 2, 3, 4]. This note demonstrates that an infinite number of other sequences have the property that the ratio of the $n+1$th to nth terms approaches the Golden Ratio.

2. Properties of the Sequences $\{H_n\}$

Theorem 2.1. *Let the sequence $\{H_n\}$ start with three arbitrary real numbers H_1, H_2, and H_3 such that $\Phi^2 H_3 - \Phi H_1 - H_2 \neq 0$. If*

$$H_n = \frac{H_{n+1} + H_{n-2}}{2}, \text{ for all } n \geq 3 \tag{2.1}$$

then $\{H_n\}$ has the property that

$$\lim_{n\to+\infty} \frac{H_{n+1}}{H_n} = \Phi = \frac{1+\sqrt{5}}{2} \approx 1.6180339.$$

Proof. Let us rewrite Equation (2.1) as follows:

$$H_{n+1} = 2H_n - H_{n-2}. \tag{2.2}$$

From Equation (2.2) and for $n \geq 3$, it is possible to calculate all the terms of the sequence:

$$\begin{aligned} H_4 &= 2H_3 - H_1 \\ H_5 &= 2H_4 - H_2 = 4H_3 - 2H_1 - H_2 \\ H_6 &= 2H_5 - H_3 = 7H_3 - 4H_1 - 2H_2. \end{aligned}$$

It is also possible to express H_{n+1} as a number only dependent on n and on the three initial numbers H_1, H_2, H_3:

$$H_{n+1} = \alpha_n H_3 - \alpha_{n-1} H_1 - \alpha_{n-2} H_2, \text{ for all } n \geq 3. \tag{2.3}$$

In the expression (2.3), $\{\alpha_n\}$ is the strictly increasing sequence of integers $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\alpha_4 = 4, \ldots$, with

$$\alpha_{n+1} = \alpha_n + \alpha_{n-1} + 1 \text{ with } n \geq 2. \tag{2.4}$$

By using Equations (2.3) and (2.4) we evaluate the ratio H_{n+1}/H_n as $n \to \infty$. Hence we have,

$$\lim_{n\to+\infty} \frac{H_{n+1}}{H_n} = \lim_{n\to+\infty} \frac{\alpha_n H_3 - \alpha_{n-1} H_1 - \alpha_{n-2} H_2}{\alpha_{n-1} H_3 - \alpha_{n-2} H_1 - \alpha_{n-3} H_2}.$$

Dividing both the numerator and the denominator by α_{n-1} we get

$$\lim_{n\to+\infty} \frac{H_{n+1}}{H_n} = \lim_{n\to+\infty} \frac{\frac{\alpha_n}{\alpha_{n-1}} H_3 - H_1 - \frac{\alpha_{n-2}}{\alpha_{n-1}} H_2}{H_3 - \frac{\alpha_{n-2}}{\alpha_{n-1}} H_1 - \frac{\alpha_{n-3}}{\alpha_{n-1}} H_2}.$$

Substituting

$$\lim_{n\to+\infty} \frac{\alpha_{n+1}}{\alpha_n} = \lim_{n\to+\infty} \frac{\alpha_n}{\alpha_{n-1}} = \lim_{n\to+\infty} \frac{\alpha_{n-1}}{\alpha_{n-2}} = \ldots = \lim_{n\to+\infty} \chi \tag{2.5}$$

we get,

$$\lim_{n\to+\infty} \frac{H_{n+1}}{H_n} = \lim_{n\to+\infty} \frac{\chi H_3 - H_1 - \chi^{-1} H_2}{H_3 - \chi^{-1} H_1 - \chi^{-2} H_2}.$$

Note that Equation (2.5) is true because $\{\alpha_n\}$ is a strictly increasing sequence of integers. Finally, multiplying and dividing by χ^2, we obtain:

$$\lim_{n\to+\infty} \frac{H_{n+1}}{H_n} = \lim_{n\to+\infty} \chi \frac{\chi^2 H_3 - \chi H_1 - H_2}{\chi^2 H_3 - \chi H_1 - H_2} = \lim_{n\to+\infty} \chi \tag{2.6}$$

where we observe that the last simplification is valid only if $\chi^2 H_3 - \chi H_1 - H_2 \neq 0$. Also, it is worth pointing out that the following relationships hold:

$$\begin{aligned}
\lim_{n\to+\infty} \chi &= \lim_{n\to+\infty} \frac{\alpha_{n+1}}{\alpha_n} \\
&= \lim_{n\to+\infty} \frac{\alpha_n + \alpha_{n-1} + 1}{\alpha_n} \\
&= \lim_{n\to+\infty} 1 + \frac{\alpha_{n-1}}{\alpha_n} + \frac{1}{\alpha_n} \\
&= \lim_{n\to+\infty} 1 + \frac{\alpha_{n-1}}{\alpha_{n-1} + \alpha_{n-2} + 1} \\
&= \lim_{n\to+\infty} 1 + \frac{1}{1 + \frac{\alpha_{n-2}}{\alpha_{n-1}} + \frac{1}{\alpha_{n-1}}} \\
&= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\cdots}}}} = \Phi.
\end{aligned}$$

This result appears fully consistent with the preliminary assumption of the theorem here reported: $\Phi^2 H_3 - \Phi H_1 - H_2 \neq 0$. $\square$

Corollary 2.2. *Consider three arbitrary real numbers H_1, H_2 and H_3 with the following constraint:*

$$H_3 = H_1 + H_2 + k \text{ with } k \in \mathbb{R} \tag{2.7}$$

Then the numeric sequences are built according to the following formulas:

$$H_n = \frac{H_{n+1} + H_{n-2}}{2} \tag{2.8}$$

and

$$H_{n+1} = H_n + H_{n-1} + k \tag{2.9}$$

are coincident.

Proof. If we consider Equation (2.8) for $n = 3$ and apply the relationship (2.7) in order to get H_4 such that the average of H_4 and H_1 equals H_3 we can write:

$$\begin{aligned} H_4 &= 2H_3 - H_1 \\ &= 2H_1 + 2H_2 + 2k - H_1 \\ &= H_2 + (H_1 + H_2 + k) + k \\ &= H_2 + H_3 + k \end{aligned} \tag{2.10}$$

Applying the iterative process to Equation (2.10) we get Equation (2.9):

$$H_{n+1} = H_n + H_{n-1} + k.$$

This general expression converges toward the Fibonacci sequence once k, H_1 and H_2 are respectively chosen as 0, 0 and 1!

Given a k-value without any restriction apart from the one expressed as

$$\Phi^2(H_1 + H_2 + k) - \Phi H_1 - H_2 \neq 0$$

and the initial values H_1 and H_2, we can obtain an infinite number of sequences for which

$$\lim_{n\to+\infty} \frac{H_{n+1}}{H_n} = \Phi = \frac{1+\sqrt{5}}{2} \approx 1.6180339.$$

$\square$

Examples. Let us consider $k = 3$, $H_1 = 1$ and $H_2 = 2$. Applying Equation (2.9) we get:

1 2 6 11 20 34 57 94 154 251 408 662 1073 1738 ...

and, as it is easy to recognize, the ratio of H_{n+1} to H_n approaches Φ.

As a second example, let $k = 0.6$, $H_1 = 0.2$, and $H_2 = 5$. In this case the sequence is:

0.2 5 5.8 11.4 17.8 29.8 48.2 78.6 127.4 206.6 334.6 541.8 877 ...

and again, the ratio of H_{n+1} to H_n approaches Φ.

3. Conclusions

We have found and proved a general relationship which determines the existence of infinite sequences $\{H_n\}$ for which the ratio H_{n+1}/H_n approaches the Golden Ratio as n goes to ∞. The Fibonacci sequence appears as a particular case of this general relationship.

References

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