ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2025. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1356 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let $\{P_n\}_{n\geq 0}$ be the Padovan sequence, defined by $P_0 = P_1 = P_2 = 1$, and $P_n = P_{n-2} + P_{n-3}$ for $n \geq 3$. Evaluate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2P_n} L_{2P_{n+1}} L_{2P_{n+2}} L_{2P_{n+3}} L_{2P_{n+6}} L_{2P_{n+7}}}.$$

B-1357 Proposed by Toyesh Prakash Sharma (undergraduate), Agra College, Agra, India.

Show that

(i) $F_{2n+1} \exp\left(1 - \frac{F_n}{F_{n+1}}\right) \geq 2F_{n+1}^2$, where $n \geq 1$.

(ii) $(L_{2n} + L_{2n+2}) \exp\left(1 - \frac{L_n}{L_{n+1}}\right) \geq 2L_{n+1}^2$, where $n \geq 0$.

B-1358 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Let a and b be any two real numbers with $a < b$. Show that for all integers $n \geq 1$,

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{b^k - a^k}{k} = \sum_{k=1}^{n} (-1)^k \frac{(b-1)^k - (a-1)^k}{k}.$$

Proposer's Remark: Using this identity, one can show that, for examples,

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{2^k - 1}{k} = \sum_{k=1}^{n} \frac{(-1)^k}{k}, \qquad \text{and} \qquad \sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{F_{2k}}{k} = \sum_{k=1}^{n} \frac{(-1)^k F_k}{k}.$$

B-1359 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{nF_{2n}} = \sqrt{5} \lim_{r\to\infty} \log \frac{\alpha^{r^2}}{L_1 L_3 L_5 \cdots L_{2r-1}}.$$

B-1360 Proposed by Michel Bataille, Rouen, France.

Let $A_n = (a_{i,j})$ be the $n \times n$ matrix with entries $a_{i,i} = F_{2i-1}L_{2i}$, and

$$a_{i.j} = F_{2j}L_{2j-1}, \qquad i, j = 1, 2, \ldots, n, \quad i \neq j.$$

Prove that $5 \det(A_n) = 2^{n-1}(L_{4n+1} + 9 - 5n)$.

SOLUTIONS

Make the Sums Telescope!

B-1336 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany. (Vol. 61.4, November 2023)

Show the following identities:

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{L_{4n} + 18} = \frac{1}{8}, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{F_{4n}}{(L_{4n} + 18)^2} = \frac{9}{800}.$$

Solution by Hans J. H. Tuenter, Toronto, Canada.

Both sums are basically wrappers for telescoping sums. For the first part of the question, we start with

$$\sum_{n=0}^{a-1} \frac{1}{F_{2n-a}} = \sum_{n=0}^{\infty} \left(\frac{1}{F_{2n-a}} - \frac{1}{F_{2n+a}} \right) = \sum_{n=0}^{\infty} \frac{F_{2n+a} - F_{2n-a}}{F_{2n+a}F_{2n-a}},$$

where a is a positive integer and odd (to avoid a division by zero). When a is odd, the product formulas $5F_sF_t = L_{s+t} - (-1)^t L_{s-t}$ and $F_sL_t = F_{s+t} + (-1)^t F_{s-t}$ yield

$$5F_{2n+a}F_{2n-a} = L_{4n} + L_{2a}, \qquad \text{and} \qquad F_{2n}L_a = F_{2n+a} - F_{2n-a}.$$

A simple substitution in the right-hand side of the above identity and a little rearranging gives

$$\sum_{n=0}^{\infty} \frac{F_{2n}}{L_{4n}+L_{2a}} = \frac{1}{5L_a} \sum_{n=0}^{a-1} \frac{1}{F_{2n-a}}.$$

For $a = 3$, the right-hand side evaluates to $1/8$. As $L_6 = 18$, this gives the first summation identity that needed to be proved. Note that the Fibonacci numbers at negative indices are well defined and satisfy the reflection formula $F_{-n} = (-1)^{n+1}F_n$.

For the second part of the question, we use telescoping sums (as before) and the first product formula to obtain

$$\sum_{n=0}^{a-1} \frac{1}{F_{2n-a}^2} = \sum_{n=0}^{\infty} \frac{F_{2n+a}^2 - F_{2n-a}^2}{F_{2n+a}^2 F_{2n-a}^2}, \qquad \text{and} \qquad F_{4n}F_{2a} = F_{2n+a}^2 - F_{2n-a}^2,$$

where a is again an odd, positive integer. This gives

$$\sum_{n=0}^{\infty} \frac{F_{4n}}{(L_{4n}+L_{2a})^2} = \frac{1}{25F_{2a}} \sum_{n=0}^{a-1} \frac{1}{F_{2n-a}^2}.$$

For $a = 3$, the right-hand side evaluates to $9/800$, and establishes the second summation identity that needed to be proved.

In closing, we note that, in a completely analogous manner, one can derive the identities

$$\sum_{n=0}^{\infty} \frac{L_{2n}}{L_{4n}-L_{2a}} = \frac{1}{L_a} \sum_{n=0}^{a-1} \frac{1}{L_{2n-a}}, \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{F_{4n}}{(L_{4n}-L_{2a})^2} = \frac{1}{5F_{2a}} \sum_{n=0}^{a-1} \frac{1}{L_{2n-a}^2},$$

where a is an odd, positive integer. The Lucas equivalents of the identities in the proposed problem are

$$\sum_{n=0}^{\infty} \frac{L_{2n}}{L_{4n}-18} = -\frac{1}{16}, \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{F_{4n}}{(L_{4n}-18)^2} = \frac{33}{640}.$$

Note that the Lucas numbers at negative indices are well defined and satisfy the reflection formula $L_{-n} = (-1)^n L_n$.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, I. V. Fedak, Dmitry Fleischman, Won Kyun Jeong, Hideyuki Ohtsuka, Ángel Plaza, Albert Stadler, Yunyong Zhang, and the proposer.

Extendable to Generalized Fibonacci Numbers

B-1337 Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.
(Vol. 61.4, November 2023)

Prove that

(i) $\dfrac{F_n^3}{F_{n-1}} + \dfrac{F_{n+2}^3}{F_n} - \dfrac{F_{n+1}^4}{F_{n-1}F_n} = 2F_{n+1}F_{n+2}$ for any integer $n \geq 3$,

(ii) $\dfrac{L_n^3}{L_{n-1}} + \dfrac{L_{n+2}^3}{L_n} - \dfrac{L_{n+1}^4}{L_{n-1}L_n} = 2L_{n+1}L_{n+2}$ for any integer $n \geq 1$.

Solution by Wei-Kai Lai, University of South Carolina Salkehatchie, Walterboro, SC.

We shall prove the general identity

$$\frac{G_n^3}{G_{n-1}} + \frac{G_{n+2}^3}{G_n} - \frac{G_{n+1}^4}{G_{n-1}G_n} = 2G_{n+1}G_{n+2}$$

for any nonzero sequence $\{G_i\}$ satisfying the recursive formula $G_i + G_{i+1} = G_{i+2}$. The identity is equivalent to

$$G_n^4 + G_{n-1}G_{n+2}^3 - G_{n+1}^4 = 2G_{n-1}G_nG_{n+1}G_{n+2},$$

which can also be rearranged, and factored into

$$G_{n-1}G_{n+2}(G_{n+2}^2 - 2G_nG_{n+1}) = (G_{n+1}^2 + G_n^2)(G_{n+1} + G_n)(G_{n+1} - G_n).$$

Therefore, we only need to show that

$$G_{n+2}^2 - 2G_nG_{n+1} = G_{n+1}^2 + G_n^2,$$

or equivalently,

$$G_2^2 = (G_{n+1} + G_n)^2.$$

The last equation is true according to the recursive formula, hence proving the new identity.

Since Lucas numbers are nonzero, the second identity in the original problem is true for any integer n. The first identity, however, is true only for integers $n \geq 2$, to avoid the denominator F_{n-1} being zero.

Editor's Notes:

1) The argument above works, provided $G_{n-1}, G_n \neq 0$. This leads us to investigate the zero-multiplicity of $\{G_n\}$; that is, the number of indices n for which $G_n = 0$. Here are some remarks that Tuenter included with his solution.

> For general recurrent sequences the zero-multiplicity is a difficult question. The American mathematician Morgan Ward (1901–1963) studied the multiplicity of general second-order linear recurrences, with arbitrary initial values, in the 1930s. A summary of Ward's conjectures, some of which were not resolved until the late 1970s, can be found in [1]. For the Tribonacci numbers it was shown, only a mere decade ago, that there are just four Tribonacci numbers that are zero [2]. In sharp contrast, for the generalized Fibonacci sequence, excluding the trivial case when both initial values are zero and the sequence consists of all zeros, the zero-multiplicity is easily determined. If $G_a = 0$, then it is not difficult to show that $G_n = G_{a+1}F_{n-a}$. Thus, the sequence $\{G_n\}$ has exactly one zero if and only if it is a scaled and/or shifted version of the Fibonacci sequence.

2) The proposers used a different approach in their solution. They first established the identity

$$\frac{x^3}{(x-y)(x-z)} + \frac{y^3}{(y-z)(y-x)} + \frac{z^3}{(z-x)(z-y)} = x + y + z,$$

and then setting $x = F_n$, $y = F_{n+1}$, and $z = F_{n+2}$, and their Lucas equivalents, respectively, to derive the two identities.

References

[1] R. Alter, *Remarks and results related to the Morgan Ward conjecture*, Congr. Numer., **14** (1975), 57–63.
[2] C. A. Gómez Ruiz and F. Luca, *The zero-multiplicity of third-order recurrences associated to the Tribonacci sequence*, Indag. Math. (N.S.), **25** (2014), no. 3, 579–587.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Charles K. Cook, Steve Edwards, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Ralph P. Grimaldi, Won Kyun Jeong, Hari Kishan, William Knuth and Connor Salch (both undergraduates) (jointly), Hideyuki Ohtsuka, Ángel Plaza, Patrick Rappa, Raphael Schumacher (graduate student), Albert Stadler, David Terr, Hans J. H. Tuenter, Daniel Văcaru, Yunyong Zhang, Nicusor Zlota, and the proposer.

They Come From the Same Inequality

B-1338 Proposed by Quang Hung Tran, HSGS School, Vietnam National University at Hanoi, Hanoi, Vietnam.
(Vol. 61.4, November 2023)

Prove that, for any integer $n \geq 0$,

(a) $\dfrac{1}{F_{n+1}} + \dfrac{1}{F_{n+2}} > \dfrac{16}{9L_{n+1} - 16F_n}$,

(b) $\dfrac{1}{L_{n+1}} + \dfrac{1}{L_{n+2}} > \dfrac{16}{45F_{n+1} - 16L_n}$.

Solution by Albert Stadler, Herrliberg, Switzerland.

We note that

$$\begin{aligned} 9L_{n+1} - 16F_n &= 9(F_n + F_{n+2}) - 16F_n \\ &= 9F_{n+2} - 7F_n \\ &= 9F_{n+2} - 7(F_{n+2} - F_{n+1}) \\ &= 2F_{n+2} + 7F_{n+1}, \end{aligned}$$

and

$$\begin{aligned} 45F_{n+1} - 16L_n &= 9(L_n + L_{n+2}) - 16L_n \\ &= 9L_{n+2} - 7L_n \\ &= 9L_{n+2} - 7(L_{n+2} - L_{n+1}) \\ &= 2L_{n+2} + 7L_{n+1}. \end{aligned}$$

Set $x = F_{n+1}$ and $y = F_{n+2}$ in part (a), and $x = L_{n+1}$ and $y = L_{n+2}$ in part (b). Both inequalities then read as

$$\frac{1}{x} + \frac{1}{y} > \frac{16}{7x + 2y},$$

which is equivalent to

$$7x^2 - 7xy + 2y^2 > 0.$$

This inequality holds true for $x, y \neq 0$, since

$$7x^2 - 7xy + 2y^2 = 7\left(x - \frac{y}{2}\right)^2 + \frac{y^2}{4} > 0,$$

thereby completing the proof.

Also solved by Thomas Achammer, Michel Bataille, Charles K. Cook and Michael R. Bacon (jointly), I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Won Kyun Jeong, Hideyuki Ohtsuka, Valentina Osorio Osorio (undergraduate), Ángel Plaza, Yunyong Zhang, and the proposer.

Floor After Floor

B-1339 Proposed by Michel Bataille, Rouen, France.
(Vol. 61.4, November 2023)

Let n be a positive integer. Prove that

$$\left\lfloor\sqrt{2\big(F_{n+1}^2 - F_{n+1}F_{n-1} + F_{n-1}^2\big)}\right\rfloor - \left\lfloor\sqrt{F_{n+1}F_{n-1}}\right\rfloor = F_n.$$

Solution by Won Kyun Jeong, Kyungpook National University, Daegu, Korea.

By Catalan's identity, $F_{n+1}F_{n-1} = F_n^2 + (-1)^n$. Then

$$F_{n+1}^2 - F_{n+1}F_{n-1} + F_{n-1}^2 = (F_{n+1} - F_{n-1})^2 + F_{n+1}F_{n-1} = 2F_n^2 + (-1)^n.$$

Hence, we obtain

$$\begin{aligned} \sqrt{2(F_{n+1}^2 - F_{n+1}F_{n-1} + F_{n-1}^2)} &= \sqrt{4F_n^2 + 2(-1)^n}, \\ \sqrt{F_{n+1}F_{n-1}} &= \sqrt{F_n^2 + (-1)^n}. \end{aligned}$$

If n is an even integer, then

$$2F_n < \sqrt{4F_n^2 + 2} < 2F_n + 1, \qquad \text{and} \qquad F_n < \sqrt{F_n^2 + 1} < F_n + 1.$$

If n is an odd integer, then

$$2F_n - 1 < \sqrt{4F_n^2 - 2} < 2F_n, \qquad \text{and} \qquad F_n - 1 < \sqrt{F_n^2 - 1} < F_n.$$

It follows that

$$\left\lfloor\sqrt{2(F_{n+1}^2 - F_{n+1}F_{n-1} + F_{n-1}^2)}\right\rfloor = \begin{cases} 2F_n, & \text{if } n \text{ is even;} \\ 2F_n - 1, & \text{if } n \text{ is odd;} \end{cases}$$

and

$$\left\lfloor\sqrt{F_{n+1}F_{n-1}}\right\rfloor = \begin{cases} F_n, & \text{if } n \text{ is even;} \\ F_n - 1, & \text{if } n \text{ is odd.} \end{cases}$$

The identity in the problem statement follows immediately.

Also solved by Thomas Achammer, Brian Bradie, I. V. Fedak, Dmitry Fleischman, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, David Terr, Yunyong Zhang, and the proposer.

A Binomial Sum of Fibonacci Polynomials

B-1340 Proposed by Hans J. H. Tuenter, Toronto, Canada.
(Vol. 61.4, November 2023)

Let a, b, and n be integers, with n nonnegative, and x any real or complex number. Evaluate

$$\sum_{i=0}^{n}\binom{n}{i}F_{a-1}^{n-i}(x)F_a^i(x)F_{b+i}(x),$$

where $F_n(x)$ are the Fibonacci polynomials, defined by the recurrence relation $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, with initial conditions $F_0(x) = 0$ and $F_1(x) = 1$. Note that the Fibonacci polynomials are defined at negative indices by extending the above recurrence relation and that they satisfy the relation $F_{-n}(x) = (-1)^{n+1}F_n(x)$.

Solution by Michel Bataille, Rouen, France.

We know that for all $m \in \mathbb{Z}$,

$$F_m(x) = \frac{\alpha^m(x) - \beta^m(x)}{\alpha(x) - \beta(x)},$$

where $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$, and $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$ are the roots of the characteristic equation $q^2 - xq - 1 = 0$. Using this Binet formula, it is easy to show that

$$\begin{aligned} F_{m-1}(x) + \alpha(x)F_m(x) &= \alpha^m(x), \\ F_{m-1}(x) + \beta(x)F_m(x) &= \beta^m(x). \end{aligned}$$

Now, let

$$\begin{aligned} A(x) &= \sum_{i=0}^{n}\binom{n}{i}F_{a-1}^{n-i}(x)F_a^i(x)\alpha^i(x), \\ B(x) &= \sum_{i=0}^{n}\binom{n}{i}F_{a-1}^{n-i}(x)F_a^i(x)\beta^i(x). \end{aligned}$$

From the binomial theorem, we have

$$A(x) = (F_{a-1}(x) + \alpha(x)F_a(x))^n = \alpha^{an}(x).$$

and, similarly, $B(x) = \beta^{an}(x)$. It follows that

$$\begin{aligned} \sum_{i=0}^{n}\binom{n}{i}F_{a-1}^{n-i}(x)F_a^i(x)F_{b+i}(x) &= \frac{\alpha^b(x)A(x) - \beta^b(x)B(x)}{\alpha(x) - \beta(x)} \\ &= \frac{\alpha^b(x)\alpha^{an}(x) - \beta^b(x)\beta^{an}(x)}{\alpha(x) - \beta(x)} \\ &= \frac{\alpha^{an+b}(x) - \beta^{an+b}(x)}{\alpha(x) - \beta(x)} \\ &= F_{an+b}(x). \end{aligned}$$

Editor's Notes: Both the proposer and Frontczak mentioned that the result can be extended to the Lucas polynomials:

$$\sum_{i=0}^{n} \binom{n}{i} F_{a-1}^{n-i}(x) F_a^i(x) L_{b+i}(x) = L_{an+b}(x).$$

Frontczak also noted that, for an odd integer m,

$$\sum_{i=0}^{n} \binom{n}{i} F_{m(a-1)}^{n-i}(x) F_{ma}^i(x) F_{m(b+i)}(x) = F_m^n(x) F_{m(an+b)}(x),$$

with a similar result holding for the Lucas polynomials.

Proposer's Historical Notes:

1) Problem B-1340 is a generalization of Advanced Problem H-13 [1], which asked the readers to prove that

$$F_n = \sum_{i=0}^{r} \binom{r}{i} F_{a-1}^{r-i} F_a^i F_{n+i-ar}.$$

One of the proposers, Verner E. Hoggatt, Jr. (1921–1980), was also a founding member of the Fibonacci Association and a long-time editor of *The Fibonacci Quarterly*.

2) The Dutch mathematician Jan Cornelis Kluyver (1860–1932) proposed the same problem as H-13 in 1928 [2], in a periodical dedicated to mathematical problems and their solutions, published by the Dutch Mathematical Society. The identity in B-1340 is a special case of a more general and complex identity for an arbitrary, linear recurrent sequence that was posed (also in 1928) as a problem by another Dutch mathematician, Johannes Gaultherus van der Corput (1890–1975) [3]. Kluyver was van der Corput's thesis advisor at Leiden University.

Just goes to show that, with "Things Fibonacci," one can never be totally certain that a result is new.

References

[1] H. W. Gould and V. E. Hoggatt, Jr., *Problem H-13*, The Fibonacci Quarterly, **1.2** (1963), 54.
[2] J. C. Kluyver, *Vraagstuk CXLII*, Wiskundige Opgaven met de Oplossingen, **14** (1928), no. 3, 282–284, 296–297.
[3] J. G. van der Corput, *Vraagstuk CXXI*, Wiskundige Opgaven met de Oplossingen, **14** (1928), no. 3, 248–251.

Also solved by I. V. Fedak, Sergio Falcón and Ángel Plaza (jointly), Dmitry Fleischman, Robert Frontczak, Albert Stadler, Yunyong Zhang, and the proposer.