ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2006. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1006 Proposed by Paul S. Bruckman, Canada

For $n \geq 1$, let $\{A_n\}$ and $\{B_n\}$ be two sequences of positive integers denoting the lengths of the legs of a Pythagorean triangle such that $B_n = 2A_n - 2(-1)^n$. Determine $\{A_n\}$ and $\{B_n\}$ and obtain recurrence relations for these sequences.

B-1007 Proposed by Andrew Cusumano, Great Neck, New York

Prove or disprove:

$$\frac{1}{1} + \frac{\left[(1^2+1)^N - (1^2)^N\right]}{(1\cdot 1)^N} - \frac{\left[(2^2)^N - (2^2-1)^N\right]}{(1\cdot 2)^N} + \frac{\left[(3^2+1)^N - (3^2)^N\right]}{(2\cdot 3)^N}$$

$$- \frac{\left[(5^2)^N - (5^2-1)^N\right]}{(3\cdot 5)^N} + \frac{\left[(8^2+1)^N - (8^2)^N\right]}{(5\cdot 8)^N} - \frac{\left[(13^2)^N - (13^2-1)^N\right]}{(8\cdot 13)^N} + \cdots = \alpha^N.$$

B-1008 Proposed by the Problem Editor

Find all (a, b, c, d) that satisfy the system

$$\begin{aligned} a+b+c+d &= 0 \\ ab+ac+ad+bc+bd+cd &= -3 \\ abc+abd+acd+bcd &= 0 \\ abcd &= 1. \end{aligned}$$

B-1009 Proposed by José Luis Díaz-Barrero, Universitat Politècnia, de Catalunya, Barcelona, Spain

Let n be a positive integer. Prove that

$$4 + 2\sum_{k=1}^{n} \left\{ \frac{F_{k+1}}{\log\left(1 + \frac{F_{k+1}}{F_k}\right)} \right\} < F_{n+1} + 3F_{n+2}.$$

B-886 Proposed by Peter J. Ferraro, Roselle Park, NJ

For $n \geq 9$, show that

$$\lfloor \sqrt[4]{F_n} \rfloor = \lfloor \sqrt[4]{F_{n-4}} + \sqrt[4]{F_{n-8}} \rfloor.$$

Comment by the editor. This problem was first proposed in Volume 37, No. 4, November 1999. Since no solution has been submitted, the problem remains open.

SOLUTIONS

A Magic Square!

B-991 Proposed by Peter Jeuck, Hewitt, NJ (Vol. 43, no. 1, February 2005)

Consider a 3×3 magic square of the following form.

F_n	F_{n+2}	F_{n+3}
b_1	b_2	b_3
c_1	c_2	c_3

Prove or disprove: The integer b_1 must be a Lucas number.

Solution by Charles K. Cook, University of South Carolina Sumter, Sumter, SC

To be magic every row, column and diagonal must add to the sum of row 1, namely, $F_n + F_{n+2} + F_{n+3} = F_{n+2} - F_{n+1} + F_{n+2} + F_{n+2} + F_{n+1} = 3F_{n+2}$.

Solving the system of 7 equations in the 6 variables $\{b_1, b_2, b_3, c_1, c_2, c_3\}$ yields the solution

F_n	F_{n+2}	F_{n+3}
L_{n+2}	F_{n+2}	$-F_{n-1}$
F_n	F_{n+2}	F_{n+3}

To prove that, use will be made of the well known identity $L_n = F_{n+1} + F_{n-1}$ and the defining recurrence relation for the Fibonacci numbers as needed.

For column 1:

$$2F_n + L_{n+2} = 2F_{n+2} - 2F_{n+1} + F_{n+3} + F_{n+1} = 2F_{n+2} + F_{n+3} - F_{n+1} = 3F_{n+2}.$$

For column 2: Obvious.

For column 3:

$$2F_{n+3} - F_{n-1} = 2F_{n+2} + 2F_{n+1} - F_{n+1} + F_n = 2F_{n+2} + F_{n+1} + F_n = 3F_{n+2}.$$

For row 2:

$$L_{n+2} + F_{n+2} - F_{n-1} = F_{n+3} + F_{n+1} + F_{n+2} + F_n - F_{n+1} = 2F_{n+2} + F_{n+1} + F_n = 3F_{n+2}.$$

For row 3 and for both diagonals: Same as row 1.

Thus $b_1 = L_{n+2}$.

Also solved by Paul S. Bruckman, Steve Edwards, Ovidiu Furdui, G.C. Greubel, Russell J. Hendel, Harris Kwong, Kathleen E. Lewis, H.-J. Seiffert, James A. Sellers, Pavel Trojovsky, and the proposer.

In the End, It's Just 4

B-992 Proposed by the Problem Editor (Vol. 43, no. 1, February 2005)

Prove or disprove: $L_{6n}^2 \equiv 4 \pmod{10}$ for all integers n.

Solution by James A. Sellers, Department of Mathematics, Pennsylvania State University, University Park, PA

First, we note from the Binet formulas that $L_n^2 - 5F_n^2 = 4(-1)^n$. Thus, we know $L_{6n}^2 = 5F_{6n}^2 + 4$. Since F_{6n} is divisible by $F_6 = 8$, we see that $5F_{6n}^2 \equiv 0 \pmod{5 \cdot 8^2}$. Therefore, we know $L_{6n}^2 \equiv 4 \pmod{320}$. The result follows.

Indeed, we have actually proved a much stronger result (with a larger modulus). Moreover, it is clear that one can prove the following much more general result:

For all integers n and m, $L_{2mn}^2 \equiv 4 \pmod{5F_{2m}^2}$.

Also solved by Charles Ashbacher, Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Kenneth B. Davenport, Steve Edwards, Ovidiu Furdui, G.C. Greubel, Russell J. Hendel, Emrah Kilic, Harris Kwong, Kathleen E. Lewis, David E. Manes, H. -J. Seiffert, David R. Stone, Pavel Trojovsky, and the proposer.

Much Ado About Zeros!

B-993 Proposed by Miquel Grau and José Luis Díaz-Barrero, Universitat Politécnica de Catalunya, Barcelona, Spain (Vol. 43, no. 1, February 2005)

Let n be a nonnegative integer. Prove that

$$F_n^2(F_n - 2F_{n+1} - F_{2n}) + F_{2n}(F_n + 2F_nF_{n+1} - F_{2n}) = 0,$$

$$L_n^2(L_n - 2F_{n+1} - F_{2n}) + F_{2n}(L_n + 2L_nF_{n+1} - F_{2n}) = 0.$$

Solution by David E. Manes, SUNY College at Oneonta, Oneonta, NY

Using the identity $F_{2n} = F_nL_n$ for each $n \geq 0$, note that

$$F_n^2(F_n - 2F_{n+1} - F_{2n}) + F_{2n}(F_n + 2F_nF_{n+1} - F_{2n}) = F_n^2(F_n - 2F_{n+1} - F_nL_n + L_n + 2L_nF_{n+1} - L_n^2).$$

Since $F_n + L_n = 2F_{n+1}$ for each $n \geq 0$, it follows that

$$F_n - 2F_{n+1} - F_nL_n + L_n + 2L_nF_{n+1} - L_n^2 = (F_n - 2F_{n+1} + L_n)(1 - L_n) = 0.$$

Hence

$$F_n^2(F_n - 2F_{n+1} - F_{2n}) + F_{2n}(F_n + 2F_nF_{n+1} - F_{2n}) = 0.$$

Similarly,

$$\begin{aligned} L_n^2(L_n - 2F_{n+1} - F_{2n}) &+ F_{2n}(L_n + 2L_nF_{n+1} - F_{2n}) \\ &= L_n^2(L_n - 2F_{n+1} - F_nL_n + F_n + 2F_nF_{n+1} - F_n^2) \\ &= (1 - F_n)(L_n - 2F_{n+1} + F_n) = 0. \end{aligned}$$

Also solved by Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Steve Edwards, Ovidiu Furdui, George C. Greubel, Russell J. Hendel, Emrah Kilic, Harris Kwong, H. -J. Seiffert, James A. Sellers, Pavel Trojovsky, and the proposer.

A Divisor's Condition

B-994 Proposed by Juan Pla, Paris, France (Vol. 43, no. 1, February 2005)

Under what condition(s) on k, if any, does $L_k + 2$ divide $\frac{F_{kn}}{F_k} + (-1)^n n$ for all integers $n \geq 0$?

Solution by H.-J. Seiffert, Berlin, Germany

We prove that k must be an even integer.

For the fixed positive integer k, let $A_n(k) = F_{kn}/F_k + (-1)^n n$, $n \geq 0$.

1. Case: k is odd.

Then (compare [2], eqn. (2.16)), $F_{3k}/F_k = L_k^2 + 1$, and therefore $A_3(k) = L_k^2 - 2 \equiv 2 \pmod{L_k + 2}$. From $L_k + 2 > 2$, it follows that $A_3(k)$ is not divisible by $L_k + 2$.

2. Case: k is even.

The Fibonacci polynomials are defined by $F_0(x) = 0, F_1(x) = 1$, and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$ for $n \geq 0$. Since k is even, we have (see [1], eqn. (3.22)) $F_{k(n+2)} = L_k F_{k(n+1)} - F_{kn}$ and a simple induction argument shows that $F_n(-iL_k) = (-i)^{n-1} F_{kn}/F_k$, $n \geq 0$, where $i = \sqrt{-1}$.

Now, the idenitty ([3], Theorem 1)

$$F_n(x) = \sum_{j=0}^{n-1} \binom{n+j}{2j+1} i^{n-1-j}(x - 2i)^j, \ n \geq 0,$$

with $x = -iL_k$ almost immediately gives

$$A_n(k) = \sum_{j=1}^{n-1} (-1)^{n-1-j} \binom{n+j}{2j+1} (L_k + 2)^j, \ n \geq 0,$$

which obviously implies that $A_n(k)$ is divisible by $L_k + 2$ for all $n \geq 0$.

References:

1. A.F. Horadam and Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985): 7-20.
2. P. Ribenboim. "FFF:(Favorite Fibonacci Flowers)." *The Fibonacci Quarterly* **43.1** (2005): 3-14.
3. H.-J. Seiffert. "Solution to Problem H-510." *The Fibonacci Quarterly* **35.2** (1997): 191-92.

Also solved by Paul S. Bruckman, G.C. Greubel, Ovidiu Furdui, Pavel Trojovsky, and the proposer.

Two Lucas Polynomials Identities

B-995 (Corrected) **Proposed by Mario Catalani, University of Torino, Torino, Italy (Vol. 43, no. 1, February 2005)**

Let $L_n \equiv L_n(x,y)$ be the Lucas polynomials defined by $L_0 = 2, L_1 = x$ and for $n \geq 2$, $L_n = xL_{n-1} + yL_{n-2}$. Assume $x \neq 0$, $y \neq 0$, and $x^2+4y \neq 0$. Prove the following identities $(n \geq 1)$:

1. $$\sum_{k=1}^{n} \binom{n}{k} kL_{k-1}(x,y) = nL_{n-1}(x+2, y-x-1),$$

2. $$\sum_{k=1}^{n} \binom{n}{k} kx^{k-1}y^{n-k}L_k(x,y) = nL_{2n-1}(x,y).$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

Our proof uses the observation that

$$\sum_{k=1}^{n} \binom{n}{k} ks^{k-1}t^{n-k} = \frac{d}{ds}\left[\sum_{k=0}^{n} \binom{n}{k} s^k t^{n-k}\right] = \frac{d}{ds}(s+t)^n = n(s+t)^{n-1},$$

and that the Binet's formula for L_n takes the form of $L_n = \alpha^n + \beta^n$, where

$$\alpha = \alpha(x,y) = \frac{x+\sqrt{x^2+4y}}{2} \quad \text{and} \quad \beta = \beta(x,y) = \frac{x-\sqrt{x^2+4y}}{2}.$$

Notice that

$$\alpha + 1 = \frac{x+2+\sqrt{x^2+4y}}{2}, \quad \beta + 1 = \frac{x+2-\sqrt{x^2+4y}}{2},$$

in which we can write

$$x^2+4y = (x+2)^2 + 4(y-x-1).$$

Furthermore, it follows from

$$x^2+4y = (2\alpha - x)^2 = 4\alpha^2 - 4\alpha x + x^2$$

that $\alpha x + y = \alpha^2$. Likewise, we have $\beta x + y = \beta^2$. These results lead to

$$\begin{aligned}\sum_{k=1}^{n} \binom{n}{k} kL_{k-1}(x,y) &= \sum_{k=1}^{n} \binom{n}{k} k(\alpha^{k-1} + \beta^{k-1}) \\ &= n[(\alpha+1)^{n-1} + (\beta+1)^{n-1}] \\ &= nL_{n-1}(x+2, y-x-1),\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n} \binom{n}{k} kx^{k-1}y^{n-k}L_k(x,y) &= \sum_{k=1}^{n} \binom{n}{k} kx^{k-1}y^{n-k}(\alpha^k + \beta^k) \\ &= n[\alpha(\alpha x + y)^{n-1} + \beta(\beta x + y)^{n-1}] \\ &= n(\alpha^{2n-1} + \beta^{2n-1}) \\ &= nL_{2n-1}(x,y).\end{aligned}$$

Also solved by Paul S. Bruckman, Kenneth Davenport, Ovidiu Furdui, George C. Greubel, H. -J. Seiffert, Pavel Trojovsky, and the proposer.

We would like to acknowledge the receipt of a late solution to problem B-988 by Kenneth Davenport.