SOME THEOREMS ON COMPLETENESS

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1. INTRODUCTION

The notion of completeness was introduced in [1].

Definition. A sequence of positive integers, A, is "complete" if and only if every positive integer, N, is the sum of a subsequence of A. The theorem of Brown [2] gives a necessary and sufficient condition for completeness.

Theorem 1. A sequence of monotonic increasing positive integers, A, is "complete" if and only if:

$$a_1 = 1 \qquad \text{and} \qquad a_{n+1} \le 1 + \sum_{k=1}^{n} a_k .$$

Corollary. As an easy consequence of Theorem 1, the sequence $a_n = 2^{n-1}$, $n = 1, 2, 3, \cdots$ is complete, since $2^{n+1} = 1 + (2^n + \cdots + 2 + 1)$, a well known result.

Theorem 2. The Fibonacci Sequence is complete.

Proof. The identity

$$F_{n+2} - 1 = \sum_{k=1}^{n} F_k$$

gives us

$$F_{n+1} \le 1 + \sum_{k=1}^{n} F_k = F_{n+2} ,$$

since

$$F_{n+2} = F_{n+1} + F_n .$$

2. ANOTHER SUFFICIENT CONDITION

Theorem 3. If (i) $a_1 = 1$, (ii) $a_{n+1} \ge a_n$, (iii) $a_{n+1} \le 2a_n$, then sequence A is complete.

Proof.

$$\begin{aligned} a_{n+1} &\le a_n + a_n \\ &\le a_n + a_{n-1} + a_{n-1} \\ &\le a_n + a_{n-1} + \cdots + a_1 + a_1 \end{aligned}$$

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by repeated use of conditions (ii) and (iii), thus

$$a_{n+1} \leq 1 + \sum_{k=1}^{n} a_k$$

since (i) gives $a_1 = 1$.

Corollary. The Fibonacci sequence is complete. $F_1 = 1$, $F_{n+1} \leq 2F_n$ and $F_{n+1} \geq F_n$.

Theorem 4. The sequence $\{1, p_n\}$ is complete, where p_n is the n^{th} prime.

Proof. By Bertrand's postulate there is a prime in $[n, 2n]$ for $n \geq 1$. Now $p_n < p_{n+1} \leq 2p_n$. Thus by Theorem 3, Theorem 4 is proved.

Theorem 5. The sequence $\{1, p_n\}$ is complete even when an arbitrary prime ≥ 7 is removed.

Proof. By Sierpiñski's Theorem VII in [3], we have for $n > 5$ there exists at least two primes between n and $2n$.

Thus

$$p_n < p_{n+1} < p_{n+2} < 2p_n \quad .$$

Clearly, if some p_{n+1} is deleted, then Theorem 3 is still valid. Thus Theorem 5.

Theorem 6. The sequence $\{1, p_n\}$ remains complete even if for $n > 5$ we remove an infinite subsequence of primes no two of which are consecutive.

3. COMPLETENESS OF FIBONACCI POWERS

Theorem 7. The sequence of 2^{m-1} copies of F_k^m is complete.

Proof.

$$\frac{F_{n+1}}{F_n} \leq 2 \quad \text{for} \quad n \geq 3$$

and

$$\left(\frac{F_{n+1}}{F_n}\right)^4 \leq 2^3 \quad \text{for} \quad n \geq 3 \; .$$

Thus

$$\left(\frac{F_{n+1}}{F_n}\right)^m \leq 2^{m-1} \quad \text{for} \quad m \geq 4, \; n \geq 3 \; .$$

Now:

$$F_{n+1}^m \leq 2^{m-1} F_n^m \leq 1 + 2^{m-1} \sum_{k=1}^{n} F_k^m \quad .$$

For $m = 1$, the theorem is true by Theorem 2. For $m = 2$, we have

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

shows that one copy is not enough.

Let $a_{2n} = a_{2n+1} = F_n^2$, then clearly

$$a_{2n+1} \le 1 + \sum_{k=1}^{2n} a_k ,$$

since

$$a_{2n+1} = a_{2n}$$

$$\sum_{k=1}^{2n} a_k = 2(F_1^2 + F_2^2 + \cdots + F_n^2) = 2F_n F_{n+1} .$$

Thus

$$a_{2n+2} = F_{n+1}^2 \le 1 + 2F_n F_{n+1} ,$$

since

$$F_{n+1} \le 2F_n .$$

Therefore by Theorem 1 it is complete. For $m = 3$, four copies of F_n^3 is complete from [5]. Theorem 7 is proved. See Brown [4].

In [5] is the following Theorem which we cite without proof:

Theorem 8. If any a_n, $n \le 6$, is deleted from the two copies of the Fibonacci Squares, the sequence remains complete, while if $n \ge 7$, the sequence becomes incomplete.

In [5] the following theorem is given:

Theorem 9. If four copies of F_n^3 forms a sequence, then the sequence remains complete if F_k^3 is removed for k odd and becomes incomplete if F_k^3 is removed for k even.

The following conjecture was given by O'Connell in [5]:

Theorem 10. If $m \ge 4$, the 2^{m-1} copies of F_n^m remains complete even if a F_k^m is removed.

Proof. Since $F_{n+1}^m \le 2^{m-1} F_n^m$ for $n \ge 3$; $m \ge 4$, then

$$F_{n+k+1}^m \le 2^{m-1} F_{n+k}^m \le 1 + 2^{m-1} \sum_{s=1}^{n+k} F_s^m - F_n^m .$$

From Theorem 8, the sequence is complete up to terms using $2^{m-1} F_n^m$ clearly if we delete one F_k^m the first possible difficulty appears at $k = 1$ above. Clearly this causes no trouble for $k \ge 0$. The result follows and the proof of Theorem 10 is finished.

Theorem 11. If $m = 4k$, then the sequence of 2^{m-1} copies of F_n^k remains complete even if 2^{k-1} of the F_n^m are deleted.

Proof.

$$\left(\frac{F_{n+1}}{F_n}\right) \le 2 \quad \text{for} \quad n \ge 3$$

then

$$\left(\frac{F_{n+1}}{F_n}\right)^{4k} \le 2^{3k}$$

$$F_{n+1}^{4k} \le 2^{3k} F_n^{4k} = F_n^{4k} + (2^{3k} - 1)F_n^{4k}$$

$$\le 2^{3k} F_{n-1}^{4k} + 2^{k-1} (2^{3k} - 1)F_n^{4k}$$

$$\le 2^{4k-1} \sum_{i=1}^{n-1} F_i^{4k} + (2^{4k-1} - 2^{k-1})F_n^{4k}$$

$$\le 1 + 2^{4k-1} \sum_{i=1}^{n} F_i^{4k} - 2^{k-1} F_n^{4k} .$$

then let $m = 4k$;

$$F_{n+1}^m \le 1 + 2^{m-1} \sum_{i=1}^{n} F_i^m - 2^{k-1} F_n^m .$$

Thus 2^{k-1} copies of F_n^m can be deleted without loss of completeness. Further:

Theorem 12.

$$\sum_{i=1}^{k} \alpha_i F_{s_i}^m$$

can be deleted without loss of completeness, and where $\alpha_i \le 2^{k-1}$

$$\sum_{i=1}^{k} \alpha_i F_{s_i}^m \le 2^{k-1} F_{s_k}^m .$$

Proof. As a consequence of Theorem 11, we have

$$\sum_{i=1}^{k} \alpha_i F_{s_i}^m \le 2^{k-1} F_{s_k}^m \qquad \alpha_i \le 2^{k-1} .$$

Thus:

$$F_{n+1}^m \le 1 + 2^{m-1} \sum_{i=1}^{n} F_i^m - \sum_{i=1}^{k} \alpha_i F_{s_i}^m$$

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