# FIBONACCI NOTES
# 2: MULTIPLE GENERATING FUNCTIONS

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1. The Hermite polynomial $H_n(x)$ may be defined by means of

$$\sum_{n=0}^{\infty} H_n(a) \cdot \frac{z^n}{n!} = e^{2az-z^2} .$$

The writer [1] has proved formulas of the following kind.

$$\sum_{m,n=0}^{\infty} H_{m+n}(a)H_m(b)H_n(c) \frac{x^m y^n}{m!\,n!} \tag{1.1}$$

$$= (1 - 4x^2 - 4y^2)^{-\frac{1}{2}} \exp\left\{\frac{-4a^2(x^2 + y^2) + 4a(bx + cy) - 4(bx + cy)^2}{1 - 4x^2 - 4y^2}\right\} ,$$

$$\sum_{m,n,p=0}^{\infty} H_{m+n+p}(a)H_m(b)H_n(c)H_p(d) \frac{x^m y^n z^p}{m!\,n!\,p!} \tag{1.2}$$

$$= (1 - 4x^2 - 4y^2 - 4z^2)^{-\frac{1}{2}} \cdot \exp\left\{\frac{-4a^2(x^2 + y^2 + z^2) + 4a(bx + cy + dz) - 4(bx + cy + dz)^2}{1 - 4x^2 - 4y^2 - 4z^2}\right\},$$

$$\sum_{m,n,p=0}^{\infty} H_{n+p}(a)H_{p+m}(b)H_{m+n}(c) \frac{x^m y^n z^p}{m!\,n!\,p!} \tag{1.3}$$

$$= d^{-\frac{1}{2}} \exp\left\{\Sigma a^2 - \frac{\Sigma a^2 - 4\Sigma a^2x^2 - 4\Sigma abz + 8\Sigma abxy}{d}\right\} ,$$

where

$$d = 1 - 4x^2 - 4y^2 - 4z^2 + 16xyz$$

and $\Sigma a^2$, $\Sigma a^2x^2$, $\Sigma abz$, $\Sigma abxy$ are symmetric functions in the indicated parameters.

The object of the present note is to prove formulas of a similar kind for the Fibonacci and Lucas numbers.

2. Consider first the sum

$$S = \sum_{m,n=0}^{\infty} F_{m+n} F_m F_n x^m y^n .$$

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Since

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \alpha + \beta = 1, \qquad \alpha\beta = -1,$$

we get

$$S = \frac{1}{\alpha - \beta} \sum_{m,n=0}^{\infty} F_m F_n (\alpha^{m+n} - \beta^{m+n}) x^m y^n$$

$$= \frac{1}{\alpha - \beta} \frac{\alpha x}{1 - \alpha x - \alpha^2 x^2} \frac{\alpha y}{1 - \alpha y - \alpha^2 y^2} - \frac{\beta x}{1 - \beta x - \beta^2 x^2} \frac{\beta y}{1 - \beta y - \beta^2 y^2}$$

$$= \frac{1}{\alpha - \beta} \left\{ \frac{\alpha^2 xy}{(1 - \alpha^2 x)(1 - \alpha\beta x)(1 - \alpha^2 y)(1 - \alpha\beta y)} - \frac{\beta^2 xy}{(1 - \alpha\beta x)(1 - \beta^2)(1 - \alpha\beta y)(1 - \beta^2 y)} \right\}$$

$$= \frac{xy}{(\alpha - \beta)(1 + x)(1 + y)} \frac{\alpha^2 [1 - \beta^2(x + y) + \beta^4 xy] - \beta^2 [1 - \alpha^2(x + y) + \alpha^4 xy]}{(1 - 3x + x^2)(1 - 3y + y^2)} .$$

Thus

$$\text{(2.1)} \qquad \sum_{m,n=0}^{\infty} F_{m+n} F_m F_n x^m y^n = \frac{xy - x^2 y^2}{(1 + x)(1 + y)(1 - 3x + x^2)(1 - 3y + y^2)} .$$

Similarly we find that

$$\text{(2.2)} \qquad \sum_{m,n=0}^{\infty} L_{m+n} F_m F_n x^m y^n = \frac{3 - 2(x + y) + 3xy}{(1 + x)(1 + y)(1 - 3x + x^2)(1 - 3y + y^2)} .$$

The sum

$$\sum_{m,n=0}^{\infty} L_{m+n} L_m L_n x^m y^n$$

is somewhat more complicated. We get

$$\frac{(2 - \alpha x)(2 - \alpha y)[1 - \beta^2(x + y) + \beta^4 xy] + (2 - \beta x)(2 - \beta y)[1 - \alpha^2(x + y) + \alpha^4 xy]}{(1 + x)(1 + y)(1 - 3x + x^2)(1 - 3y + y^2)} .$$

After some manipulation we find that

$$\text{(2.3)} \qquad \sum_{m,n=0}^{\infty} L_{m+n} L_m L_n x^m y^n$$

$$= \frac{8 - 14(x + y) - 2(x^2 + y^2) + 27xy + 7xy(x + y) + 3x^2 y^2}{(1 + x)(1 + y)(1 - 3x + x^2)(1 - 3y + y^2)} .$$

Recurrences of an unusual kind are implied by these formulas. In particular (2.1) yields

$$\begin{aligned}(2.4)\qquad F_{m+n}F_mF_n + F_{m+n-1}F_{m-1}F_n + F_{m+n-1}F_mF_{n-1} + F_{m+n-2}F_{m-1}F_{n-1} \\ = F_{2m}F_{2n} - F_{2m-2}F_{2n-2}\end{aligned},$$

while (2.2) gives

$$\begin{aligned}(2.5)\qquad L_{m+n}F_mF_n + L_{m+n-1}F_{m-1}F_n + L_{m+n-1}F_mF_{n-1} + L_{m+n-2}F_{m-1}F_{n-1} \\ = 3F_{2m+2}F_{2n+2} - 2F_{2m}F_{2n+2} - 2F_{2m+2}F_{2n} + 3F_{2m}F_{2n}\end{aligned}.$$

It may be of interest to mention that the generating functions (2.1), (2.2), (2.3) can be extended in various ways. For example we have

$$(2.6)\qquad \sum_{m,n=0}^{\infty} F_{m+n+p}F_mF_nx^my^n = \frac{F_{p+2}xy - F_pxy(x+y) + F_{p-2}x^2y^2}{(1+x)(1+y)(1-3x+x^2)(1-3y+y^2)}.$$

This in turn leads to the following extension of (2.4):

$$\begin{aligned}(2.7)\qquad F_{m+n+p}F_mF_n + F_{m+n+p-1}(F_{m-1}F_n + F_mF_{n-1}) + F_{m+n+p-2}F_{m-1}F_{n-1} \\ = F_{p+2}F_{2m}F_{2n} - F_p(F_{2m-2}F_{2n} + F_{2m}F_{2n-2}) + F_{p-2}F_{2m-2}F_{2n-2}\end{aligned}.$$

Since $L_n = F_{n+1} + F_{n-1}$, it is evident that (2.6) and (2.7) imply

$$(2.8)\qquad \sum_{m,n=0}^{\infty} L_{m+n+p}F_mF_nx^my^n = \frac{L_{p+2} - L_pxy(x+y) + L_{p-2}x^2y^2}{(1+x)(1+y)(1-3x+x^2)(1-3y+y^2)}$$

and

$$\begin{aligned}(2.9)\qquad L_{m+n+p}F_mF_n + L_{m+n+p-1}(F_{m-1}F_n + F_mF_{n-1}) + L_{m+n+p-2}F_{m-1}F_{n-1} \\ = L_{p+2}F_{2m}F_{2n} - L_p(F_{2m-2}F_{2n} + F_{2m}F_{2n-2}) + L_{p-2}F_{2m-2}F_{2n-2}\end{aligned},$$

respectively.

3. We consider next the triple sum

$$(3.1)\qquad \sum_{m,n,p=0}^{\infty} F_{m+n+p}F_mF_nF_px^my^nz^p.$$

Exactly as above we find that (3.1) is equal to

$$\frac{1}{\alpha - \beta}\left\{\frac{\alpha^3 xyz}{(1 - \alpha^2 x)(1 - \alpha\beta x)(1 - \alpha^2 y)(1 - \alpha\beta y)(1 - \alpha^2 z)(1 - \alpha\beta z)}\right.$$

$$\left. - \frac{\beta^3 xyz}{(1 - \alpha\beta x)(1 - \beta^2 x)(1 - \alpha\beta y)(1 - \beta^2 y)(1 - \alpha\beta z)(1 - \beta^2 z^2)}\right\}$$

$$= \frac{xyz}{\alpha - \beta}\left\{\frac{\alpha^3[1 - \beta^2(x + y + z) + \beta^4(yz + zx + xy) - \beta^6 xyz]}{(1 + x)(1 + y)(1 + z)(1 - 3x + x^2)(1 - 3y + y^2)(1 - 3z + z^2)}\right.$$

$$\left. - \beta^3 \frac{[1 - \alpha^2(x + y + z) + \alpha^4(yz + zx + xy) - \alpha^6 xyz]}{(1 + x)(1 + y)(1 + z)(1 - 3x + x^2)(1 - 3y + y^2)(1 - 3z + z^2)}\right\}.$$

Simplifying we get

$$\text{(3.2)} \qquad \sum_{m,n,p=0} F_{m+n+p} F_m F_n F_p x^m y^n z^p$$

$$= \frac{2 - (x + y + z) + (yz + zx + xy) - 2xyz}{(1 + x)(1 + y)(1 + z)(1 - 3x + x^2)(1 - 3y + y^2)(1 - 3z + z^2)}$$

$$= \frac{(1 - x)(1 - y)(1 - z) + 1 - xyz}{(1 + x)(1 + y)(1 + z)(1 - 3x + x^2)(1 - 3y + y^2)(1 - 3z + z^2)}$$

The general formula of this kind can now be stated, namely

$$\text{(3.3)} \qquad \sum_{n_1, \cdots, n_k = 0}^{\infty} F_{n_1 + \cdots + n_k} F_{n_1} \cdots F_{n_k} x_1^{n_1} \cdots x_k^{n_k}$$

$$= \frac{\displaystyle\sum_{0 \le 2j \le k} (-1)^j F_{k-2j}(c_j - c_{k-j})}{\displaystyle\prod_{j=1}^{k} (1 + x_j) \cdot \prod_{j=1}^{k} (1 - 3x_j + x_j^2)},$$

where $c_j$ is the $j^{th}$ elementary symmetric function of $x_1, x_2, \cdots, x_k$.

To prove (3.3) it is enough to observe that the numerator is equal to

$$\frac{1}{\alpha - \beta}\left\{\alpha^k \prod_{j=1}^{k} (1 - \beta^2 x_j) - \beta^k \prod_{j=1}^{k} (1 - \alpha^2 x_j)\right\}$$

$$= \frac{1}{\alpha - \beta}\left\{\alpha^k \sum_{j=0}^{k} (-1)^j c_j \beta^{2j} - \beta^k \sum_{j=0}^{k} (-1)^j c_j \alpha^{2j}\right\}$$

$$= \frac{1}{\alpha - \beta} \sum_{j=0}^{k} (-1)^j c_j (\alpha^k \beta^{2j} - \alpha^{2j} \beta^k)$$

$$= \sum_{2j<k} (-1)^j c_j F_{k-2j} - (-1)^k \sum_{2j>k} (-1)^j c_j F_{2j-k}$$

$$= \sum_{2j<k} (-1)^j c_j F_{k-2j} - \sum_{2j<k} (-1)^j c_{k-j} F_{k-2j}$$

$$= \sum_{2j<k} (-1)^j (c_j - c_{k-j}) F_{k-2j} \quad .$$

In exactly the same way we can prove the more general result

$$\sum_{n_1,\cdots,n_k=0}^{\infty} F_{n_1+\cdots+n_k+p} F_{n_1} \cdots F_{n_k} x_1^{n_1} \cdots x_k^{n_k} = \frac{\sum_{j=0}^{k} (-1)^j c_j F_{k+p-2j}}{\prod_{j=1}^{k} (1+x_j) \cdot \prod_{j=1}^{k} (1-3x_j+x_j^2)} \quad . \tag{3.4}$$

Hence we also have

$$\sum_{n_1,\cdots,n_k=0}^{\infty} L_{n_1+\cdots+n_k+p} F_{n_1} \cdots F_{n_k} x_1^{n_1} \cdots x_k^{n_k} = \frac{\sum_{j=0}^{k} (-1)^j c_j L_{k+p-2j}}{\prod_{j=1}^{k} (1+x_j) \cdot \prod_{j=1}^{k} (1-3x_j+x_j^2)} \quad . \tag{3.5}$$

4. We consider next the series

$$\sum_{m,n,p=0}^{\infty} F_{n+p} F_{p+m} F_{m+n} x^m y^n z^p = \frac{1}{(\alpha-\beta)^3} \sum_{m,n,p=0}^{\infty} (\alpha^{n+p}-\beta^{n+p})(\alpha^{p+m}-\beta^{p+m})(\alpha^{m+n}-\beta^{m+n}) x^m y^n z^p$$

$$= \frac{1}{(\alpha-\beta)^3} \left\{ \frac{1}{(1-\alpha^2 x)(1-\alpha^2 y)(1-\alpha^2 z)} - \sum \frac{1}{(1-\alpha^2 x)(1+y)(1+z)} + \sum \frac{1}{(1-\beta^2 x)(1+y)(1+z)} - \frac{1}{(1-\beta^2 x)(1-\beta^2 y)(1-\beta^2 z)} \right\}$$

$$= \frac{1}{(\alpha-\beta)^3} \frac{(1-\beta^2 x)(1-\beta^2 y)(1-\beta^2 z) - (1-\alpha^2 x)(1-\alpha^2 y)(1-\alpha^2 z)}{(1-3x+x^2)(1-3y+y^2)(1-3z+z^2)} - \frac{1}{(\alpha-\beta)^3} \sum \frac{(\alpha^2 - \beta^2)x}{(1-3x+x^2)(1+y)(1+z)} \quad .$$

It follows that

(4.1)
$$\sum_{m,n,p=0}^{\infty} F_{n+p}F_{p+m}F_{m+n}x^m y^n z^p$$

$$= \frac{1}{5}\,\frac{\sum x - 3\sum xy + 8xyz}{(1 - 3x + x^2)(1 - 3y + y^2)(1 - 3z + z^2)} - \frac{1}{5}\sum \frac{x}{(1 - 3x + x^2)(1 + y)(1 + z)}\,.$$

It can be shown that the right member of (4.1) is equal to

(4.2)
$$\frac{q - 5r + 2pr + 2qr + r^2 - q^2}{(1 + x)(1 + y)(1 + z)(1 - 3x + x^2)(1 - 3y + y^2)(1 - 3z + z^2)}\,,$$

where

$$p = \sum x, \qquad q = \sum xy, \qquad r = xyz.$$

A somewhat more general result than (4.1) is

(4.3)
$$\sum_{m,n,p=0}^{\infty} F_{n+p+r}F_{p+m+r}F_{m+n+r}x^m y^n z^p$$

$$= \frac{1}{5}\,\frac{F_{3r} - F_{3r-2}\sum x + F_{3r-4}\sum xy - F_{3r-6}\,xyz}{(1 - 3x + x^2)(1 - 3y + y^2)(1 - 3z + z^2)}$$

$$- \frac{(-1)^r}{5}\sum \frac{F_r - F_{r-2}\,x}{(1 - 3x + x^2)(1 + y)(1 + z)}\,.$$

Similarly we can show that

(4.4)
$$\sum_{m,n,p=0}^{\infty} L_{n+p+r}L_{p+m+r}L_{m+n+r}x^m y^n z^p$$

$$= \frac{L_{3r} - L_{3r-2}\sum x + L_{3r-4}\sum xy - L_{3r-6}\,xyz}{(1 - 3x + x^2)(1 - 3y + y^2)(1 - 3z + z^2)}$$

$$+ (-1)^r\sum \frac{L_r - L_{r-2}\,x}{(1 - 3x + x^2)(1 + y)(1 + z)}\,,$$

We remark that the left member of (4.3) can be transformed in a rather interesting way. Put

$$m' = n + p, \qquad n' = p + m, \qquad p' = m + n .$$

Then

$$\left\{\begin{array}{l} -m' + n' + p' = 2m \\ m' - n' + p' = 2n \\ m' + n' - p' = 2p \end{array}\right. , \tag{4.5}$$

so that

$$m' + n' + p' \equiv 0 \pmod 2 \tag{4.6}$$

and

$$m' \le n' + p', \qquad n' \le p' + m', \qquad p' \le m' + n' . \tag{4.7}$$

Conversely if $m'$, $n'$, $p'$ are nonnegative integers satisfying (4.6) and (4.7) then $m$, $n$, $p$ as defined by (4.5) are also nonnegative integers. Hence replacing $x$, $y$, $z$ by $vw$, $wu$. $uv$, (4.3) becomes

$$\sum_{m',n',p'} F_{m'+r}F_{n'+r}F_{p'+r}u^{m'}v^{n'}w^{p'} \tag{4.8}$$

$$= \frac{1}{5}\,\frac{F_{3r} - F_{3r-2}\sum uv + F_{3r-4}uvw\sum u - F_{3r-6}u^2v^2w^2}{(1 - 3vw + v^2w^2)(1 - 3wu + w^2u^2)(1 - 3uv + u^2v^2)}$$

$$- \frac{(-1)^r}{5}\sum \frac{F_r - F_{r-2}vw}{(1 - 3vw + v^2w^2)(1 + wu)(1 + uv)} .$$

A similar result can be stated for (4.4).

## REFERENCE

1. L. Carlitz, "Some Extensions of the Mehler Formula," Collectanea Mathematica, Vol. 21 (1970), pp. 117-130.