# ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

*H-245 Proposed by P. Bruckman, University of Illinois, Chicago Circle, Illinois.*

Prove the identity

$$\sum_{k=0}^{n} \frac{x^{\frac{1}{2}k(k-1)}}{(x)_k(x)_{n-k}} = \frac{2\prod_{r=1}^{n-1}(1+x^r)}{(x)_n}, \quad n = 1, 2, \cdots,$$

where

$$(x)_n = (1-x)(1-x^2)(1-x^3)\cdots(1-x^n), \quad n = 1, 2, \cdots; \ (x)_0 = 1.$$

*H-246 Proposed by L. Carlitz, Duke University, Durham, North Carolina.*

Put

$$F(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} F_{i+j}F_{m-i+j}F_{i+n-j}F_{m-i+n-j}$$

$$L(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} L_{i+j}L_{m-i+j}L_{i+n-j}L_{m-i+n-j}.$$

Show that

$$L(m,n) - 25F(m,n) = 8L_{m+n}F_{m+1}F_{n+1}.$$

*H-247 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

Show that for each Fibonacci number $F_r$, there exist an infinite number of positive nonsquare integers, $D$, such that

$$F_{r+s}^2 - F_r^2 D = 1.$$

*H-248 Proposed by F.D. Parker, St. Lawrence University, New York.*

A well known identity for the Fibonacci numbers is

$$F_n^2 - F_{n-1}F_{n+1} = -(-1)^n$$

and a less well known identity for the Lucas numbers is

$$L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n.$$

More generally, if a sequence $\{y_0, y_1, \cdots\}$ satisfies the equation

$$y_n = y_{n-1} + y_{n-2},$$

and if $y_0$ and $y_1$ are integers, then there exists an integer $N$ such that

$$y_n^2 - y_{n-1}y_{n+1} = N(-1)^n.$$

Prove this statement and show that $N$ cannot be of the form $4k+2$, and show that $4N$ terminates in 0, 4, or 6.

## SOLUTIONS

### SUM SEQUENCE

*H-216 Proposed by Guy A.R. Guillotte, Cowansville, Quebec, Canada.*

Let $G_m$ be a set of rational integers such that

$$\sum_{n=1}^{\infty}\left[\log_e\left(\sum_{m=0}^{\infty}\frac{G_m}{(m)!(F_{2n+1})^m}\right)\right] = \frac{\pi}{4}.$$

Find a formula for $G_m$.

*Solution by L. Carlitz, Duke University, Durham, North Carolina.*

Put

$$e^{\arctan x} = \sum_{m=0}^{\infty} G_m \frac{x^m}{m!}, \quad G_0 = G_1 = 1.$$

Then, by differentiation

$$e^{\arctan x} = (1+x^2)\sum_{m=0}^{\infty} G_{m+1}\frac{x^m}{m!},$$

so that

$$G_m = G_{m+1} + m(m-1)G_{m-1} \qquad (m \geqslant 1).$$

It follows that the $G_m$ are rational integers.

Consider

$$S \equiv \sum_{n=1}^{\infty}\log\left[\sum_{m=0}^{\infty}\frac{G_m}{m!F_{2n+1}^m}\right] = \sum_{n=1}^{\infty}\log\left[\exp\left(\arctan\frac{1}{F_{2n+1}}\right)\right] = \sum_{n=1}^{\infty}\arctan\frac{1}{F_{2n+1}}.$$

Since

$$\arctan\frac{1}{F_{2n}} - \arctan\frac{1}{F_{2n+2}} = \arctan\left(\frac{F_{2n+2}-F_{2n}}{F_{2n}F_{2n+1}+1}\right) = \arctan\frac{1}{F_{2n+1}}$$

it follows that

$$\sum_{n=1}^{\infty}\arctan\frac{1}{F_{2n+1}} = \arctan\frac{1}{F_2} = \arctan 1 = \frac{\pi}{4}.$$

Hence $S = \pi/4$.

To get an explicit formula for $G_m$ we proceed as follows. Put

$$x = \tan u = \frac{1}{i}\,\frac{e^{iu}-e^{-iu}}{e^{iu}+e^{-iu}} = \frac{1}{i}\,\frac{e^{2iu}-1}{e^{2iu}+1}, \qquad e^{2iu} = \frac{1+ix}{1-ix},$$

that is,

$$e^{2i \arctan x} = \frac{1+ix}{1-ix}.$$

Thus

$$e^{\arctan x} = \left(\frac{1+ix}{1-ix}\right)^{-\frac{1}{2}i} = (1+ix)^{-\frac{1}{2}i}(1-ix)^{\frac{1}{2}i}$$

$$= \sum_{r=0}^{\infty} \binom{-\frac{1}{2}i}{r}(ix)^r \sum_{s=0}^{\infty} \binom{\frac{1}{2}i}{s}(-ix)^s = \sum_{m=0}^{\infty} i^m x^m \sum_{r+s=m} (-1)^s \binom{-\frac{1}{2}i}{r}\binom{\frac{1}{2}i}{s}.$$

It follows that

$$G_m = i^m m! \sum_{r+s=m} (-1)^s \binom{-\frac{1}{2}i}{r}\binom{\frac{1}{2}i}{s}$$

$$= (-1)^m \sum_{r+s=m} \binom{m}{r} (\tfrac{1}{2}i)(\tfrac{1}{2}i+1)\cdots(\tfrac{1}{2}i+r-1)(\tfrac{1}{2}i)(\tfrac{1}{2}i-1)\cdots(\tfrac{1}{2}i-s+1).$$

A simpler formula for $G_m$ would be desirable.

*Also partially solved by P. Bruckman.*

**PRIME ASSUMPTION**

*H-217 (corrected) Proposed by S. Krishnan, Orissa, India.*

(a) Show that

$$2^{4n-4x-4}\binom{2x+2}{x+1} \equiv \binom{4n-2x-2}{2n-x-1} \pmod{4n+1},$$

where $n$ is a positive integer and $-1 \leqslant x \leqslant 2n-1$, $x$ is an integer, and $4n+1$ is prime.

(b) Show that

$$2^{4n-4x-6}\binom{2x+4}{x+2} + \binom{4n-2x-2}{2n-x-1} \equiv 0 \pmod{4n+3},$$

where $n$ is a positive integer, $-2 \leqslant x \leqslant 2n-1$, $x$ is an integer, and $4n+3$ is prime.

*Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.*

Assertions (a) and (b) are *false* for general $n$; we may make them true assertions by adding the hypothesis that $4n+1$ is prime, for part (a), and $4n+3$ is prime, for part (b). We may combine the two assertions as follows: If $p$ is a positive odd prime and $x$ is an integer with $0 \leqslant x \leqslant \frac{1}{2}(p-1)$, then

$$2^{p-1-4x}\binom{2x}{x} \equiv (-1)^{\frac{1}{2}(p-1)}\binom{p-1-2x}{\frac{1}{2}(p-1)-x} \pmod{p}.$$

The following lemma is useful in the proof:

*Lemma.* If $p$ is an odd prime, then

$$(\tfrac{1}{2})^{p-1}\binom{p-1}{\frac{1}{2}(p-1)} = \frac{1\cdot3\cdot5\cdots(p-2)}{2\cdot4\cdot6\cdots(p-1)} \equiv (-1)^{\frac{1}{2}(p-1)} \pmod{p}.$$

*Proof.*

$$\frac{1\cdot3\cdots(p-2)}{2\cdot4\cdots(p-1)} = \frac{1^2 3^2\cdots(p-2)^2}{(p-1)!} \equiv \frac{1\cdot3\cdots(p-2)(-2)(-4)\cdots(1-p)}{(p-1)!} \pmod{p}$$

$$\equiv (-1)^{\frac{1}{2}(p-1)}\frac{(p-1)!}{(p-1)!} \pmod{p} \equiv (-1)^{\frac{1}{2}(p-1)} \pmod{p},$$

as asserted.

Now, let

$$U = 2^{p-1-4x}\binom{2x}{x}, \qquad V = \binom{p-1-2x}{\frac{1}{2}(p-1)-x},$$

where $p$ and $x$ are as stated above. Thus,

$$U = 2^{p-1-2x}\left\{\frac{1\cdot 3\cdots(2x-1)}{2\cdot 4\cdots(2x)}\right\}, \qquad V = 2^{p-1-2x}\left\{\frac{1\cdot 3\cdots(p-2-2x)}{2\cdot 4\cdots(p-1-2x)}\right\}.$$

Therefore,

$$V \equiv 2^{p-1-2x}\left\{\frac{(-2x-2)(-2x-4)\cdots(-p+1)}{(-2x-1)(-2x-3)\cdots(-p+2)}\right\} \pmod p \equiv 2^{p-1-2x}\left\{\frac{(2x+2)(2x+4)\cdots(p-1)}{(2x+1)(2x+3)\cdots(p-2)}\right\} \pmod p.$$

Since all the factors in the last expression are relatively prime to $p$, $V \not\equiv 0 \pmod p$; therefore, $V^{-1}$ exists, and

$$UV^{-1} \equiv \frac{2^{p-1-2x}}{2^{p-1-2x}}\left\{\frac{1\cdot 3\cdots(2x-1)(2x+1)(2x+3)\cdots(p-2)}{2\cdot 4\cdots(2x)(2x+2)(2x+4)\cdots(p-1)}\right\} \pmod p.$$

Thus,

$$UV^{-1} \equiv \frac{1\cdot 3\cdots(p-2)}{2\cdot 4\cdots(p-1)} \pmod p \equiv (-1)^{\frac{1}{2}(p-1)} \pmod p,$$

by the lemma. Therefore,

$$U \equiv (-1)^{\frac{1}{2}(p-1)}V \pmod p,$$

which is equivalent to our assertion.

*Also solved by P. Tracy.*

## STAGGERING PASCAL

*H-218 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, California.*

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & \\ 0 & 1 & 0 & \cdots & \\ 0 & 1 & 1 & 0 & \cdots \\ \cdots & \cdots & 2 & 1 & \cdots \\ & \cdots & & & \end{pmatrix}_{n\times n}$$

represent the matrix which corresponds to the staggered Pascal Triangle and

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ & & \cdots & & \end{pmatrix}_{n\times n}$$

represent the matrix which corresponds to the Pascal Binomial Array.

Finally let

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 2 & 5 & 9 & 14 & \cdots \\ & & \cdots & & \end{pmatrix}_{n\times n}$$

represent the matrix corresponding to the Fibonacci Convolution Array. Prove $A\cdot B = C$.

*Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.*

Presumably, the matrix $A$ should look as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} .$$

By inspection, or otherwise, we obtain the formulas

(1) $$a_{ij} = \binom{j-1}{i-j}, \quad \text{for } j \leqslant i \leqslant 2j-1; \quad a_{ij} = 0 \text{ otherwise}$$

(2) $$b_{ij} = \binom{i+j-2}{j-1} .$$

Let $D = AB.$ Then,

$$d_{ij} = \sum_{k=1+[\frac{1}{2}i]}^{i} \binom{k-1}{i-k} \binom{k+j-2}{j-1} .$$

For convenience, let $i-1=r$ and $j-1=a$; also, let $m=i-k.$ Then,

$$d_{ij} = \theta_{rs} = \sum_{m=0}^{[\frac{1}{2}r]} \binom{r-m}{m} \binom{r+s-m}{s} .$$

Now, let

$$f_j(x) = \sum_{i=1}^{\infty} d_{ij}x^{i-1} = \sum_{r=0}^{\infty} \theta_{rs}x^r ;$$

then $f_j(x)$ is the generating function for the $j^{th}$ column of $D.$

Thus,

$$f_j(x) = \sum_{r=0}^{\infty} x^r \sum_{m=0}^{[\frac{1}{2}r]} \binom{r-m}{m} \binom{r+s-m}{r-m} = \sum_{m=0}^{\infty} x^{2m} \sum_{r=0}^{\infty} \binom{r+m}{m} \binom{r+s+m}{r+m} x^r$$

$$= \sum_{m=0}^{\infty} x^{2m} \sum_{r=0}^{\infty} \binom{s+m}{m} \binom{r+s+m}{r} x^r = \sum_{m=0}^{\infty} \binom{-s-1}{m} (-x^2)^m \sum_{r=0}^{\infty} \binom{-s-m-1}{r} (-x)^r$$

$$= \sum_{m=0}^{\infty} \binom{-s-1}{x} (-x^2)^m (1-x)^{-s-m-1} = (1-x)^{-s-1} \left(1 - \frac{x^2}{1-x}\right)^{-s-1} = (1-x-x^2)^{-s-1} ,$$

i.e.,

$$f_j(x) = (1-x-x^2)^{-j} .$$

Since

$$f_1(x) = (1-x-x^2)^{-1} ,$$

the familiar generating function for the Fibonacci numbers, $f_j(x)$ is the column generator for the Fibonacci convolution matrix, i.e., $C$. Thus, $D = AB = C$.

*Also solved by the Proposer.*

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