# ADVANCED PROBLEMS AND SOLUTIONS

Edited by
**RAYMOND E. WHITNEY**
**Lock Haven State College, Lock Haven, Pennsylvania 17745**

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

*H-255 Proposed by L. Carlitz, Duke University, Durham, North Carolina.*

Show that

$$\sum_{j=0}^{2m}\sum_{k=0}^{2n}(-1)^{j+k}\binom{2m}{j}\binom{2n}{k}\binom{2m+2n}{j+k}\binom{2m+2n}{2m-j+k} = (-1)^{m+n}\frac{(3m+3n)!(2m)!(2n)!}{m!n!(m+n)!(2m+n)!(m+2n)!},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$.

*H-256 Proposed by E. Karst, Tucson, Arizona.*

Find all solutions of

(i) $$x+y+z = 2^{2n+1}-1$$

and

(ii) $$x^3+y^3+z^3 = 2^{6n+1}-1,$$

simultaneously for $n < 5$, given that

(a) $x, y, z$ are positive rationals
(b) $2^{2n+1}-1, 2^{6n+1}-1$ are integers
(c) $n = \log_2\sqrt{t}$, where $t$ is a positive integer.

*H-257 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.*

Consider the array, $D$, indicated below in which $F_{2n+1}$ $(n = 0, 1, 2, \cdots)$ is written in staggered columns

$$D:\quad \begin{array}{cccccc} 1 \\ 2 & 1 \\ 5 & 2 & 1 \\ 13 & 5 & 2 & 1 \\ 34 & 13 & 5 & 2 & 1 \\ 89 & 34 & 13 & 5 & 2 & 1 \\ \cdots \end{array}$$

i) Show that the row sums are $F_{2n+2}$ $(n = 0, 1, 2, \cdots)$.
(ii) Show that the rising diagonal sums are $F_{n+1}F_{n+2}$ $(n = 0, 1, 2, \cdots)$.
(iii) Show that if the columns are multiplied by 1, 2, 3, $\cdots$ sequentially to the right, then the row sums are $F_{2n+3}-1$ $(n = 0, 1, 2, \cdots)$

## READER COMMENTS

Paul Bruckman noted that H-241 is identical to H-206.

Charles Wall noted that H-188 is a weaker version of B-141.

*H-239 Correction*

The given inequality should read

$$\left|\frac{c}{a}-\frac{d}{b}\right| \leqslant \frac{1}{100} \qquad \text{not} \qquad \left|\frac{c}{d}-\frac{d}{b}\right| \leqslant \frac{1}{100}\quad .$$

## SOLUTIONS

### A NEST OF SUBSETS

*H-223 Proposed by L. Carlitz and R. Scoville, Duke University, Durham, North Carolina.*

Let $S$ be a set of $k$ elements. Find the number of sequences $(A_1, A_2, \cdots, A_n)$ where each $A_i$ is a subset of $S$, and where $A_1 \subseteq A_2$, $A_2 \supseteq A_3$, $A_3 \subseteq A_4$, $A_4 \supseteq A_5$, etc.

*Solution by the Proposers.*

Let $\phi_1$ be the characteristic function of $A_1$, $\phi_2$ the characteristic function of $A'_2$, $\phi_3$ of $A'_3$, $\phi_4$ of $A'_4$, etc. The condition on the $A_i$'s is equivalent to

$$(1)' \qquad \phi_i(j) = 1 \Rightarrow \phi_{i+1}(j) = 0, \qquad \forall_{i,j}\ .$$

For instance, suppose $A_i \subseteq A_{i+1}$. Then $i+1$ is even. If $\phi_i(j) = 1$, then $j \in A_i$, $j \in A_{i+1}$, $j \notin A'_{i+1}$ and $\phi_{i+1}(j) = 0$.

The matrix $(\phi_i(j))$ has $k$ columns each of which is a sequence of 0's and 1's of length $n$ in which no 1's occur consecutively. Since there are $F_{n+2}$ such sequences, there are $F^k_{n+2}$ matrices satisfying (1)'.

### SUM LEGENDRE

*H-227 Proposed by L. Carlitz, Duke University, Durham, North Carolina.*

Show that

$$\sum_{j=0}^{m}\sum_{k=0}^{n} (-1)^{m+n-j-k}\binom{m}{j}\binom{n}{k}(aj+ck)^m(bj+dk)^n = m!n!\sum_{r=0}^{\min(m,n)}\binom{m}{r}\binom{n}{r}a^{m-r}d^{n-r}(bc)^r\ .$$

In particular, show that the Legendre polynomial $P_n(x)$ satisfies

$$(n!)^2P_n(x) = \sum_{j,k=0}^{n}(-1)^{j+k}\binom{n}{j}\binom{n}{k}(aj+ck)^n(bj+dk)^n,$$

where $ad = \frac{1}{2}(x+1)$, $bc = \frac{1}{2}(x-1)$.

*Solution by the Proposer.* We have

$$\sum_{j=0}^{m}\sum_{k=0}^{n}(-1)^{m+n-j-k}\binom{m}{j}\binom{n}{k}(aj+ck)^m(bj+dk)^n$$

$$=\sum_{j=0}^{m}\sum_{k=0}^{n}(-1)^{m+n-j-k}\binom{m}{j}\binom{n}{k}\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}a^{m-r}c^rb^{n-s}d^sj^{m+n-r-s}k^{r+s}$$

$$=\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}a^{m-r}c^rb^{n-s}d^sS_{m,n}\ ,$$

where

$$S_{m,n} = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^{m+n-r-s} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^{r+s} .$$

Since

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^t = \begin{cases} m! & (t = m) \\ 0 & (t < m), \end{cases}$$

it follows that $S_{m,n} = 0$ unless

$$\begin{cases} m+n-r-s \geq m \\ r+s \geq n \end{cases},$$

that is, $r+s = s$. Hence

$$\sum_{j=0}^{m} \sum_{k=0}^{n} (-1)^{m+n-j-k} \binom{m}{j}\binom{n}{k} (aj+ck)^m (bj+dk)^n = m!n! \sum_{r+s=n} \binom{m}{r}\binom{n}{s} a^{m-r}c^r b^{n-s} d^s$$

$$= m!n! \sum_{r=0}^{\min(m,n)} \binom{m}{r}\binom{n}{r} a^{m-r} d^{n-r} (bc)^r .$$

Since (see for example G. Szegö's *Orthogonal Polynomials,* p. 67)

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k},$$

the second assertion follows at once.

## A TRIANGULAR ARRAY

*H-229 Proposed by L. Carlitz, Duke University, Durham, North Carolina.*

A triangular array $A(n,k)$ $(0 \leq k \leq n)$ is defined by means of

$$(*) \qquad \begin{cases} A(n+1, 2k) = A(n, 2k-1) + aA(n, 2k) \\ A(n+1, 2k+1) = A(n, 2k) + bA(n, 2k+1) \end{cases}$$

together with

$$A(0,0) = 1, \qquad A(0,k) = 0 \qquad (k \neq 0)$$

Find $A(n,k)$ and show that

$$\sum_k A(n, 2k)(ab)^k = a(a+b)^{n-1},$$

$$\sum_k A(n, 2k+1)(ab)^k = (a+b)^{n-1} .$$

*Solution by the Propsoer.*

It follows from the definition that

$$A(n,0) = a^n \qquad (n = 0, 1, 2, \cdots).$$

Then

$$A(n,1) = a^{n-1} + bA(n-1, 1)$$

so that

$$A(n,1) = \frac{a^n - b^n}{a - b} .$$

Put

$$A_k(x) = \sum_{n=k}^{\infty} A(n,k)x^n .$$

Then by (*)

$$A_{2k}(x) = \sum_{n=k}^{\infty} (A(n-1, 2k-1) + aA(n-1, 2k))x^n = xA_{2k-1}(x) + axA_{2k}(x),$$

so that

$$(1 - ax)A_{2k} = xA_{2k-1}(x).$$

Similarly

$$(1 - bx)A_{2k+1}(x) = xA_{2k}(x).$$

It follows that

$$(**) \qquad \begin{cases} A_{2k+1}(x) = x^{2k+1}(1-ax)^{-k-1}(1-bx)^{-k-1} \\ A_{2k}(x) = x^{2k}(1-ax)^{-k-1}(1-bx)^{-k} \end{cases} .$$

Since

$$(1 - ax)^{-k-1} = \sum_{r=0}^{\infty} \binom{k+r}{k} a^r x^r ,$$

we get

$$\begin{cases} A(n, 2k+1) = \displaystyle\sum_{r=0}^{n-2k-1} \binom{k+r}{k}\binom{n-k-r-1}{k} a^r b^{n-2k-r-1} \\ A(n, 2k) = \displaystyle\sum_{r=0}^{n-2k} \binom{k+r}{k}\binom{n-k-r-1}{k-1} a^r b^{n-2k-r} . \end{cases}$$

It follows from (**) that

$$(***) \qquad \begin{cases} \displaystyle\sum_{k=0}^{\infty} A_{2k}(x)y^{2k} = \frac{1-bx}{(1-ax)(1-bx)-x^2y^2} \\ \displaystyle\sum_{k=0}^{\infty} A_{2k+1}(x)y^{2k+1} = \frac{xy}{(1-ax)(1-bx)-x^2y^2} \end{cases} .$$

Hence

$$\sum_{k=0}^{\infty} A_k(x)y^k = \frac{1-bx+xy}{(1-ax)(1-bx)-x^2y^2} .$$

For $a = b$ this reduces to

$$\frac{1}{1-ax-xy}$$

which is correct.

Finally, taking $y^2 = ab$ in (***), we get

$$\sum_k A(n, 2k)(ab)^k = a(a+b)^{n-1}, \qquad \sum_k A(n, 2k+1)(ab)^k = (a+b)^{n-1} .$$

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