GENERALIZED BELL NUMBERS

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1. INTRODUCTION

In the notation of Riordan [2], the Stirling numbers of the second kind, $S(n,k)$, with arguments n and k are defined by the relation

(1.1) $$t^n = \sum_{k=0}^{n} S(n,k)(t)_k, \qquad n > 0,$$

where $(t)_n = t(t-1)\cdots(t-n+1)$ is the factorial power function. They have been utilized by Tate and Goen [4] in obtaining the distribution of the sum of zero-truncated Poisson random variables where

(1.2) $$(e^t - 1)^k/k! = \sum_{n=k}^{\infty} S(n,k)t^n/n! .$$

The Bell numbers or exponential numbers B_n can be expressed as

(1.3) $$B_n = \sum_{k=0}^{n} S(n,k), \qquad n \geq 0,$$

with $B_0 \equiv 1$. They have been investigated by many authors: see [1] and [3] for lists of references. Uppuluri and Carpenter [7] have recently studied the moment properties of the probability distribution defined by

(1.4) $$p(k) = S(n,k)/B_n, \qquad k = 1, 2, \cdots, n,$$

and give

(1.5) $$\sum_{k=1}^{n} k^r S(n,k) = \sum_{i=1}^{r} \binom{r}{i} C_i B_{n+r-i} ,$$

where the sequence $\{C_n, n = 0, 1, \cdots\}$ is defined by

(1.6) $$\sum_{k=0}^{\infty} C_k x^k/k! = \exp(1 - e^x) .$$

Tate and Goen [4] have also derived the n-fold convolution of independent random variables having the Poisson distribution truncated on the left at 'c' in terms of the generalized Stirling numbers of the second kind, $d_c(n,k)$ given by

(1.7) $$(e^t - 1 - t - \cdots - t^c/c!)^k/k! = \sum_{n=k(c+1)}^{\infty} d_c(n,k)t^n/n! ,$$

where $d_c(n,k) = 0$ for $n < k(c+1)$. They give an explicit representation for $d_c(n,k)$ too complicated to reproduce here. The $d_c(n,k)$ can be shown to satisfy the recurrence formula

(1.8) $$d_c(n+1,k) = kd_c(n,k) + \binom{n}{c} d_c(n-c, k-1),$$

where $d_c(0,0) = 1$ for all c.

Definition 1. We define the numbers $B_c(n)$ given by

(1.9) $$B_c(n) = \sum_{k=0}^{n} d_c(n,k),$$

for $c \geqslant 1$ and $n \geqslant 0$ as generalized Bell numbers. It may be noted that $B_0(n) = B_n$.

Definition 2. A random variable X is said to have the generalized Bell distribution (GBD) if its probability function is given by

(1.10) $$p_c(k) = d_c(n,k)/B_c(n), \qquad k = 0, 1, \cdots, n.$$

It may also be noted that when $c = 0$ and $n > 0$ (1.10) reduces to (1.4) as then $d_0(n,0) = 0$.

In this paper we investigate some properties of the numbers $B_c(n)$ and provide recurrence relations for the ordinary and factorial moments of the GBD. It is shown that the related results obtained by Uppuluri and Carpenter [7] follow as special cases for $c = 0$.

2. PROPERTIES OF $B_c(n)$

Property 1.

(2.1) $$\sum_{n=0}^{\infty} B_c(n)t^n/n! = \exp(e^t - 1 - t - \cdots - t^c/c!).$$

This is immediately evident upon expansion of the right-hand side making use of (1.7).

Lemma 1.

(2.2) $$d_c(n+1,k) = \sum_{m=0}^{n-c} \binom{n}{m} d_c(m, k-1).$$

Proof. Differentiating both sides of (1.7) with respect to t and expanding in powers of t we obtain

$$\sum_{r=c}^{\infty}\sum_{m=0}^{\infty} \binom{r+m}{m} d_c(m, k-1)t^{r+m}/(r+m)! = \sum_{n=0}^{\infty} d_c(n,k)t^{n-1}/(n-1)!.$$

Interchanging sums on the left-hand side and equating coefficients of t^n we are led to Lemma 1.

Property 2.

(2.3) $$B_c(n+1) = \sum_{m=0}^{n-c} \binom{n}{m} B_c(m).$$

This is now immediate from Definition 1 and Lemma 1. We note that when $c = 0$ (2.3) reduces to the known relation

$$B_{n+1} = \sum_{m=0}^{n} \binom{n}{m} B_m$$

for Bell numbers.

In attempting to find a recurrence relation in c for $B_c(n)$ we first need

Lemma 2.

(2.4) $$d_c(n,k) = \sum_{i=0}^{k} [(-1)^i n!/i!(c!)^i(n-ci)!] d_{c-1}(n-ci, k-i),$$

for $c \geqslant 1$.

Proof. See Riordan [2], p. 102.

Using Lemma 2 we can now write

$$B_c(n) = \sum_{i=0}^{n} [(-1)^i \binom{n}{i} (n-i)!/(c!)^i(n-ci)!] \sum_{k=i}^{n} d_{c-1}(n-ci, k-i).$$

It follows directly from the above that we now have

Property 3.

$$B_c(n) = \sum_{i=0}^{n} [(-1)^i \binom{n}{i} (n-i)!/(c!)^i(n-ci)!] B_{c-1}(n-ci), \qquad c \geqslant 1. \tag{2.5}$$

The well-known Dobinski formula for Bell numbers has the form

$$B_{n+1} = e^{-1}(1^n + 2^n/1! + 3^n/2! + \cdots). \tag{2.6}$$

When $c = 1$ Property 1 gives us a formula similar to that of Dobinski.

Property 4.

$$B_1(n) = e^{-1}((-1)^n/1! + 1^n/2! + 2^n/3! + \cdots). \tag{2.7}$$

Property 3 suggests that we may write the generalized Bell numbers as a linear combination of the Bell numbers. Write the right-hand side of (2.1) in the form

$$\exp(e^t - 1 - t - t^2/2! - \cdots - t^c/c!) = \exp(e^t - 1)H(t), \tag{2.8}$$

where

$$H(t) = \sum_{r=0}^{\infty} b_c(r)t^r/r!, \qquad c \geqslant 1. \tag{2.9}$$

Property 5.

$$B_c(n) = \sum_{j=0}^{n} \binom{n}{j} b_c(j)B_{n-j}, \qquad c \geqslant 0. \tag{2.10}$$

Proof. Expand the right-hand side of (2.8) in powers of t. Property 5 now follows from (2.1), with $c = 0$, and (2.9).

For the purposes of enumeration the recurrence relation for $b_c(r)$,

$$b_c(r+1) = -\sum_{i=0}^{c-1} \binom{r}{i} b_c(r-i), \qquad c \geqslant 1, \tag{2.11}$$

with $b_0(j) = 0$ for all $j \geqslant 0$ and $b_c(0) = 1$, can be obtained by differentiating both sides of (2.8) with respect to t, using (2.9), and equating coefficients. With $b_1(j) = (-1)^j$ we alternately have Property 4 from Property 5.

Making use of the above properties, the first few values of $B_c(n)$ are as follows:

Table 1

Table for $B_c(n)$

c \ n	0	1	2	3	4	5	6	7
0	1	1	2	5	15	52	203	877
1	1	0	1	1	4	11	41	162
2	1	0	0	1	1	1	11	36

3. RECURRENCE RELATIONS FOR MOMENTS OF THE GBD

Let X be a random variable having the generalized Bell distribution defined by (1.10). The r^{th} ordinary moment of X is given by

$$\mu_c(x^r) = \sum_{k=0}^{n} k^r d_c(n,k)/B_c(n). \tag{3.1}$$

Let

$$B_c(n,r) = \sum_{k=0}^{n} k^r d_c(n,k). \tag{3.2}$$

Property 6.

$$B_c(n, r+1) = B_c(n+1, r) - \binom{n}{c} \sum_{j=0}^{r} \binom{r}{j} B_c(n-c, j). \tag{3.3}$$

Proof. Multiply both sides of (1.8) by k^r and sum over k. We have for every choice of c

$$B_c(n+1, r) = B_c(n, r+1) + \binom{n}{c} \sum_{k=0}^{n} k^r d_c(n-c, k-1)$$

$$= B_c(n, r+1) + \binom{n}{c} \sum_{j=0}^{r} \binom{r}{j} B_c(n-c, j).$$

Property 6 follows immediately. When $c = 0$, $B_0(n,r)$ becomes $B_n^{(r)}$ in [7] with Property 6 replaced by

Property 7.

$$B_n^{(r+1)} = B_{n+1}^{(r)} - \sum_{j=0}^{r} \binom{r}{j} B_n^{(j)} . \tag{3.4}$$

Property 7 is not given however by Uppuluri and Carpenter.

In attempting to express $B_c(n,r)$ as a linear combination of the generalized Bell numbers we are led after expanding (3.3) for the first few values of r to the following:

Property 8.

$$B_c(n,r) = \sum_{i=0}^{r} \sum_{j=0}^{i} a_{i,j}(n,r,c) B_c(n+r-i-jc), \tag{3.5}$$

where $a_{i,j}(n,r,c)$ satisfies the recurrence relation

$$a_{i,j}(n, r+1, c) = a_{i,j}(n+1, r, c)$$

$$- \binom{n}{c} \sum_{s=r-i+j}^{r} \binom{r}{s} a_{i+s-r-1,j-1}(n-c, s, c), \tag{3.6}$$

with $a_{0,0}(n,r,c) = 1$ and $a_{i,j}(n,r,c) = 0$ if $i > r$, $j > i$, or $j = 0$ and $i > 0$.

The proof consists of substituting (3.5) into (3.3) and equating appropriate coefficients.

Comparing (3.5) with (1.5) when $c = 0$ we must have

$$\sum_{j=0}^{i} a_{i,j}(n,r,0) = \binom{r}{i} C_i, \tag{3.7}$$

independent of n for $i = 1, 2, \cdots, r$. By starting with (3.6) and summing out j one can show that

$$C_{k+1} = -\sum_{i=0}^{k} \binom{k}{i} C_i \tag{3.8}$$

which agrees with Proposition 3 in [7]. We note also when $c = 0$

$$a_{i,j}(n,r,0) = (-1)^j \binom{r}{i} S(i,j), \tag{3.9}$$

independent of n, as (3.6) is then equivalent to

$$S(i,j) = \sum_{k=0}^{i-1} \binom{i-1}{k} S(k, j-1), \tag{3.10}$$

a property of Stirling numbers of the second kind.

Now let

$$W_c(n,r) = \sum_{j=0}^{n} (j)_r d_c(n,j). \tag{3.11}$$

Then the factorial moments of the generalized Bell distribution are given by

$$\nu_c((x)_r) = W_c(n,r)/B_c(n). \tag{3.12}$$

We now seek a recurrence formula for $W_c(n,r)$ and investigate the special case $c = 0$.

Property 9.

$$W_c(n, r+1) = W_c(n+1, r) - rW_c(n,r) - \binom{n}{c} [W_c(n-c, r) + rW_c(n-c, r-1)]. \tag{3.13}$$

Proof. From (3.11)

$$W_c(n, r+1) = \sum_{j=0}^{n} (j)_{r+1} d_c(n,j) = \sum_{j=0}^{n} j(j)_r d_c(n,j) - rW_c(n,r).$$

Hence

$$\sum_{j=0}^{n} j(j)_r d_c(n,j) = W_c(n, r+1) + rW_c(n,r). \tag{3.14}$$

Using (1.8) we can write, with $c \geq 1$,

$$W_c(n, r+1) = \sum_{j=0}^{n} (j)_r [d_c(n+1, j) - \binom{n}{c} d_c(n-c, j-1)] - rW_c(n,r)$$

$$= W_c(n+1, r) - rW_c(n,r) - \binom{n}{c} \sum_{j=0}^{n-1} (j+1)_r d_c(n-c, j).$$

Now with (3.14) and the fact that

$$(j+1)_r = j(j)_{r-1} + (j)_{r-1}$$

we have the desired recurrence relation stated in Property 9. One can verify directly that when $c = 0$ we have

Property 10.

$$W_0(n, r+1) = W_0(n+1, r) - (r+1)W_0(n,r) - rW_0(n, r-1), \tag{3.15}$$

so that (3.13) is true for all c.

The $W_0(n,r)$ may also be expressed as a linear combination of the Bell numbers. In fact using the same substitution procedure as before for Property 8 one can prove

Property 11.

$$W_0(n,r) = \sum_{i=0}^{r} a(r,i)B_{n+r-i}, \tag{3.16}$$

where $a(r,i)$ satisfies the recurrence relation

$$a(r+1, i) = a(r,i) - (r+1)a(r, i-1) - ra(r-1, i-2), \tag{3.17}$$

with $a(r,0) = 1$, $a(r,i) = 0$ if $i > r$, and $a(r,r) = (-1)^r$. A table of the $a(n,k)$ is as follows:

Table 2

Table for $a(n,k)$

(3.18)

n \ k	0	1	2	3	4	5	6
0	1						
1	1	−1					
2	1	−3	1				
3	1	−6	8	−1			
4	1	−10	29	−24	1		
5	1	−15	75	−145	89	−1	
6	1	−21	160	−545	814	−415	1

We note that the $a(n,k)$ are the coefficients of a special case of the Poisson-Charlier polynomials (cf. Szegö [6], p. 34). Touchard [5] gives formulas for the first seven polynomials corresponding to the coefficients in the table above. The polynomials take the form

$$h_n(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x)_{n-i}. \tag{3.19}$$

If we write

$$(x)_{n-i} = \sum_{k=0}^{n-i} s(n-i, k)x^k, \qquad n-i > 0, \tag{3.20}$$

where the $s(n,k)$ are the Stirling numbers of the first kind (see Riordan [2] p. 33), then

$$h_n(x) = \sum_{k=0}^{n} \left[\sum_{i=0}^{n-k} (-1)^i \binom{n}{i} s(n-i, k) \right] x^k. \tag{3.21}$$

Hence $a(n,k)$ has the representation

$$a(n,k) = \sum_{i=0}^{k} (-1)^i \binom{n}{i} s(n-i, n-k). \tag{3.22}$$

Investigating the general case using similar procedures as before one can easily prove

Property 12.

$$W_c(n,r) = \sum_{i=0}^{r} \sum_{j=0}^{i} b_{i,j}(n,r,c)B_c(n+r-i-jc), \tag{3.23}$$

where $b_{i,j}(n,r,c)$ satisfies the recurrence relation

$$b_{i,j}(n, r+1, c) = b_{i,j}(n+1, r, c) - rb_{i-1,j}(n,r,c) - \binom{n}{c}\left[b_{i-1,j-1}(n-c, r, c) - rb_{i-2,j-1}(n-c, r-1, c)\right], \tag{3.24}$$

with $b_{r,j}(n,r,c) = 0$, for $j = 0, 1, \cdots, r-1$, $b_{0,0}(n,r,c) = 1$, and $b_{r,r}(n,r,c) = (-1)^r n!/(c!)^n (n-rc)!$.

Comparing (3.16) and (3.23) when $c = 0$, we have

$$a(r,i) = \sum_{j=0}^{i} b_{i,j}(n,r,0). \tag{3.25}$$

Hence in view of (3.22)

$$b_{i,j}(n,r,0) = (-1)^j \binom{r}{j} s(r-j, r-i) \tag{3.26}$$

independent of n.

Recurrence relations for the ordinary and factorial moments are readily obtained from (3.3), (3.4), (3.13), and (3.15).

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