ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania 17745

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-261 Proposed by A. J. W. Hilton, University of Reading, Reading, England.

It is known that, given k a positive integer, each positive integer n has a unique representation in the form

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t},$$

where $t = t(n,k)$, $a_i = a_i(n,k)$, $(i = t, \cdots, k)$, $t \geq 1$ and, if $k > t$, $a_k > a_{k-1} > \cdots > a_t$. Call such a representation the *k-binomial representation* of n.

Show that, if $k \geq 2$, $n = r + s$, where $r \geq 1$, $s \geq 1$ and if the k-binomial representations of r and s are

$$r = \binom{b_k}{k} + \binom{b_{k-1}}{k-1} + \cdots + \binom{b_u}{u}, \qquad s = \binom{c_k}{k} + \binom{c_{k-1}}{k-1} + \cdots + \binom{c_v}{v}$$

then

$$\binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_t}{t-1} \leq \binom{b_k}{k-1} + \binom{b_{k-1}}{k-2} + \cdots + \binom{b_u}{b-1} + \binom{c_k}{k-1} + \binom{c_{k-1}}{k-2} + \cdots + \binom{c_v}{v-1}$$

H-262 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$L_{p^2} \equiv 1 \pmod{p^2} \Leftrightarrow L_p \equiv 1 \pmod{p^2}.$$

H-263 Proposed by G. Berzsenyi, Lamar University, Beaumont, Texas.

Prove that $L^2_{2mn} \equiv 4 \pmod{L^2_m}$ for every $n,m = 1, 2, 3, \cdots$.

SOLUTIONS

WFFLE!

H-234 Proposed by R. E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Suppose an alphabet, $A = \{x_1, x_2, x_3, \cdots\}$, is given along with a binary connective, P (in prefix form). Define a well formed formula (wff) as follows: a wff is

(1) x_i for $i = 1, 2, 3, \cdots$, or

(2) If A_1, A_2 are wff's, then PA_1A_2 is a wff and

(3) The only wff's are of the above two types.

A wff of order n is a wff in which the only alphabet symbols are $x_1, x_2, \cdots, x_n$ in that order with each letter occurring exactly once. There is one wff of order 1, namely x_1. There is one wff of order 2, namely Px_1x_2. There are two wff's of order 3, namely $Px_1Px_2x_3$ and $PPx_1x_2x_3$, and there are five wff's of order 4, etc.

Define a sequence $\left\{G_i\right\}_{i=1}^{\infty}$ as follows:

G_i is the number of distinct wff's of order i.

(a). Find a recurrence relation for $\left\{G_i\right\}_{i=1}^{\infty}$ and

(b). Find a generating function for $\left\{G_i\right\}_{i=1}^{\infty}$.

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

Let F_k denote any arbitrary wff of order k. In order to form F_n $(n = 2, 3, \cdots)$, we need to apply P to all possible distinct "products" of the form $F_k F_{n-k}$, $(k = 1, 2, \cdots, n-1)$. Hence, we obtain the recursion:

$$\text{(a)} \qquad G_n = \sum_{k=1}^{n-1} G_k G_{n-k}, \qquad n = 2, 3, 4, \cdots, \text{ with } G_1 = 1 .$$

The above recursion is the well known relation which yields the Catalan numbers, defined by:

$$\text{(b)} \qquad G_{n+1} = \frac{\binom{2n}{n}}{n+1}; \qquad \text{thus,} \qquad (G_n) = (1, 1, 2, 5, 14, 42, 132, 429, \cdots).$$

We shall give a brief derivation of (b) from (a), using generating functions. Let us define the generating function for the G_n's:

$$\text{(1)} \qquad y = \sum_{n=0}^{\infty} G_{n+1} x^n = \sum_{n=1}^{\infty} G_n x^{n-1};$$

then

$$y^2 = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} G_{k+1}G_{n-k+1} = \sum_{n=0}^{\infty} x^n \sum_{k=1}^{n+1} G_k G_{n+2-k} = \sum_{n=2}^{\infty} x^{n-2} \sum_{k=1}^{n-1} G_n G_{n-k}$$

$$= \sum_{n=2}^{\infty} G_n x^{n-2},$$

using (a). Hence,

$$xy^2 = \sum_{n=2}^{\infty} G_n x^{n-1} = y - G_1 = y - 1$$

(using (1)). Thus, y is a solution of the quadratic equation

$$\text{(2)} \qquad xy^2 - y + 1 = 0; \quad \text{we note that} \quad y(0) = G_1 = 1.$$

Solving the quadratic, we obtain two solutions:

$$y = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

The positive sign must be rejected, since this solution is not defined at $x = 0$. Thus, $y = [1 - (1-4x)^{1/2}]/2x$; it is easy to verify, by L'Hospitâl's rule, that $\lim_{x \to 0} y = 1$. Expanding this expression by the binomial theorem, or otherwise, we find

$$y = \frac{1}{2x} - \frac{1}{2x}\sum_{n=0}^{\infty}\binom{1/2}{n}(-4x)^n = -\frac{1}{2x}\sum_{n=1}^{\infty}\binom{1/2}{n}(-4x)^n = 2\sum_{n=0}^{\infty}\binom{1/2}{n+1}(-4x)^n .$$

Comparing coefficients with (1), we have:

$$G_{n+1} = 2\binom{\frac{1}{2}}{n+1}(-4)^n = 2\cdot\tfrac{1}{2}\cdot\frac{\binom{-\frac{1}{2}}{n}}{n+1}\cdot(-4)^n = \frac{\binom{2n}{n}}{n+1},$$

which establishes (b).

Also solved by A. Shannon, R. L. Goodstein, and the Proposer.

SUM DIFFERENTIAL EQUATION!

H-235 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

a. Find the second-order ordinary differential equation whose power series solution is

$$\sum_{n=0}^{\infty} F_{n+1}x^n .$$

b. Find the second-order ordinary differential equation whose power series solution is

$$\sum_{n=0}^{\infty} L_{n+1}x^n .$$

Solution by A. G. Shannon, University of New England, Armidale, N.S.W.

Consider $\{H_n\} : H_n = H_{n-1} + H_{n-2}$

and $\{H_n\} = F_n$ when $H_1 = H_2 = 1$

and $\{H_n\} = L_n$ when $H_1 = 1,\ H_2 = 3.$

Let $Y = y(x).$

Then

$$y = \sum_{n=0}^{\infty} H_{n+1}x^n \quad\text{and}\quad y' = \sum_{n=0}^{\infty}(n+1)H_{n+2}x^n \quad\text{and}\quad y'' = \sum_{n=0}^{\infty}(n+1)(n+2)H_{n+3}x^n .$$

Thus

$$(x^2+x-1)y'' + 2(2x+1)y' + 2y = \sum_{n=2}^{\infty} n(n-1)H_{n+1}x^n + \sum_{n=1}^{\infty} n(n+1)H_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)(n+2)H_{n+3}x^n$$

$$+4\sum_{n=1}^{\infty} nH_{n+1}x^n + 2\sum_{n=0}^{\infty}(n+1)H_{n+2}x^n + 2\sum_{n=0}^{\infty} H_{n+1}x^n$$

$$= \sum_{n=2}^{\infty}(n^2+3n+2)(H_{n+1}+H_{n+2}-H_{n+3})x^n .$$

Thus

$$(x^2+x-1)y'' + 2(2x+1)y' + 2y = 0 \quad \text{for } all \quad \{H_n\}.$$

Also solved by P. Bruckman, F. D. Parker, and the Proposer.

SUM PRODUCT!

H-236 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

(1) $$\sum_{n=0}^{\infty} (-1)^n x^{n^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(x)_{2n}} \prod_{k=1}^{\infty} (1-x^k),$$

(2) $$\sum_{n=0}^{\infty} (-1)^n x^{(n+1)^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(x)_{2n+1}} \prod_{k=1}^{\infty} (1-x^k),$$

where $(x)_k = (1-x)(1-x^2)\cdots(1-x^k)$, $(x)_0 = 1$.

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

We begin with the well known Jacobi "triple-product" formula:

(1) $$\prod_{r=1}^{\infty} (1+x^{2r-1}w)(1+x^{2r-1}w^{-1})(1-x^{2r}) = \sum_{n=-\infty}^{\infty} x^{n^2}w^n = 1+\sum_{n=1}^{\infty} x^{n^2}(w^n+w^{-n});$$

the following treatment is formal, and avoids questions of convergence, but it may be shown that the manipulations are valid in the unit disk $|x| < 1$. Setting $w = -1$ in (1), the left-hand side becomes:

$$\prod_{r=1}^{\infty} (1-x^{2r-1})^2(1-x^{2r}) = \prod_{r=1}^{\infty} (1-x^{2r-1})(1-x^r) = \prod_{r=1}^{\infty} \frac{(1-x^{2r-1})(1-x^{2r})}{(1+x^r)}$$

$$= \prod_{r=1}^{\infty} \frac{(1-x^r)}{(1+x^r)}.$$

Hence, we obtain the identity

(2) $$\prod_{r=1}^{\infty} \left(\frac{1-x^r}{1+x^r}\right) = -1+2\sum_{n=0}^{\infty} (-1)^n x^{n^2} = 1-2\sum_{n=0}^{\infty} (-1)^n x^{(n+1)^2}$$

Next, we will establish the following identity:

(3) $$\prod_{r=1}^{\infty} (1-x^r w)^{-1} = \sum_{n=0}^{\infty} \frac{x^n w^n}{(x)_n},$$

where

$$(x)_0 = 1, \qquad (x)_n = (1-x)(1-x^2)\cdots(1-x^n), \qquad n = 1, 2, 3, \cdots.$$

Letting

$$f(w,x) = \prod_{r=1}^{\infty} (1-x^r w)^{-1} = \sum_{n=0}^{\infty} A_n(x) w^n,$$

we first note that $f(0,x) = 1 = A_0(x)$; also, we observe that

$$f(wx,x) = \prod_{r=1}^{\infty} (1-x^{r+1}w)^{-1} = \prod_{r=2}^{\infty} (1-x^r w)^{-1} = (1-xw)f(w,x).$$

Hence, by substituting into the series form, we obtain the recursion:

$$x^n A_n(x) = A_n(x) - xA_{n-1}(x), \qquad n = 1, 2, 3, \cdots, \qquad A_0(x) = 1,$$

i.e.,

$$A_n(x) = x/(1-x^n)A_{n-1}(x).$$

By an easy induction, we establish that $A_n(x) = x^n/(x)_n$ for all n, where $(x)_n$ is defined in (3). This establishes (3).

If, in (3), we replace w by $-w$, we obtain:

(4) $$\prod_{r=1}^{\infty}(1+x^rw)^{-1} = \sum_{n=0}^{\infty}\frac{x^n(-1)^nw^n}{(x)_n}.$$

Adding, and then subtracting, both sides of (3) and (4), we obtain:

(5) $$\prod_{r=1}^{\infty}(1-x^rw)^{-1} + \prod_{r=1}^{\infty}(1+x^rw)^{-1} = 2\sum_{n=0}^{\infty}\frac{x^{2n}w^{2n}}{(x)_{2n}},$$

and

(6) $$\prod_{r=1}^{\infty}(1-x^rw)^{-1} - \prod_{r=1}^{\infty}(1+x^rw)^{-1} = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}w^{2n+1}}{(x)_{2n+1}}.$$

If, in (5) and (6), we set $w = 1$, and multiply throughout by

$$\prod_{r=1}^{\infty}(1-x^r),$$

we obtain:

(7) $$1+\prod_{r=1}^{\infty}\left(\frac{1-x^r}{1+x^r}\right) = 2\sum_{n=0}^{\infty}\frac{x^{2n}}{(x)_{2n}}\prod_{r=1}^{\infty}(1-x^r),$$

and

(8) $$1-\prod_{r=1}^{\infty}\left(\frac{1-x^r}{1+x^r}\right) = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(x)_{2n+1}}\prod_{r=1}^{\infty}(1-x^r).$$

Now if, in (7) and (8), we substitute the expression obtained in (2) for the infinite product on the left-hand side, simplify and divide by 2, we obtain the desired results:

(9) $$\sum_{n=0}^{\infty}(-1)^nx^{n^2} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(x)_{2n}}\prod_{r=1}^{\infty}(1-x^r),$$

and

(10) $$\sum_{n=0}^{\infty}(-1)^nx^{(n+1)^2} = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(x)_{2n+1}}\prod_{r=1}^{\infty}(1-x^r).$$

Also solved by P. Tracy and the Proposer.

SUM RECIPROCAL!

H-237 Proposed by D. A. Miller, High School Student, Annville, Pennsylvania.

Prove

$$\sum_{k=0}^{\infty}\frac{1}{F_{2^k}} = \frac{7-\sqrt{5}}{2}.$$

Editorial Note: A solution to this problem appears in a note (accepted Feb. 27, 1973) appearing in the Dec. '74 issue o the *Quarterly*, p. 346.

Solution by A. G. Shannon, University of New England, Armidale, N.S.W.

Let

$$F(x) = \sum_{k=1}^{\infty}\frac{x^{2^{k-1}}}{F_{2^k}}.$$

Then

$$F(\alpha x) = \sum_{k=1}^{\infty} \frac{\alpha^{2^{k-1}} x^{2^{k-1}}}{F_{2^k}}$$

and so

$$F(\alpha x) + F(\beta x) = \sum_{k=0}^{\infty} \frac{\alpha^{2^{k-1}} + \beta^{2^{k-1}}}{F_{2^k}} x^{2^{k-1}} = \sum_{k=1}^{\infty} \frac{x^{2^{k-1}}}{F_{2^{k-1}}} = \sum_{k=0}^{\infty} \frac{x^{2^k}}{F_{2^k}} = x + F(x^2).$$

So

$$x + F(x^2) = F(\alpha x) + F(\beta x).$$

Put $x = -\beta$:

$$-\beta + F(\beta^2) = F(-\beta^2) + F(-\alpha\beta)$$

or

$$F(1) = -\beta + 2\beta^2$$

since

$$\alpha\beta = -1$$

and

$$F(\beta^2) = F(-\beta^2) + 2\beta^2 \ .$$

Thus

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{1}{F_{2^k}} &= 1 + F(1) \\ &= 1 - \beta + 2\beta^2 \\ &= 2 - (1 + \beta - \beta^2) + \beta^2 \\ &= 2 + \frac{(3 - \sqrt{5})}{2} = \frac{7 - \sqrt{5}}{2} \ .\end{aligned}$$

Also solved by I. J. Good (see note), J. Shallit, W. Brady, P. Bruckman, F. Higgins, L. Carlitz, and the Proposer.

Editorial Note. Kurt Mahler reports

$$\sum_{n=0}^{\infty} \frac{1}{L_{2^n}}$$

is transcendental.

****