THE RECIPROCAL PERIOD LAW

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The opinion of scientists on Bode's rule falls into several camps. The computer work of Hills [1, Fig. 2] proves that an average period ratio exists and lies in the range $9/4 < P < 3$. Some think there is a reason for this [2, 13], while others such as Lecar [3] think that the distances are random subject to the restraint of not being too near to each other. The idea that the asteroids were once a planet has been disproven [4]. The interested readership may consult any of the several summaries of physical theories of the origin of the solar system [6, 7, 8, 9, 10, 11]. Almost all theories proposing specific distance rules are discounted (e.g., Blagg's 1.73 rule [9], Dermott's rules [12]) by almost all scientists because they have too many independent parameters and lack any logical basis. More than two parameters is too many. The only models not discounted in this way are von Weizsacker's [see 5], the author's [13, 14] and perhaps Schmidt's [see 7]. Von Weizsacker proposed a system of eddies of ellipticity = ½ lubricated by smaller eddies. One can derive the equation: distance factor = $\tan^2 [\pi (N + 1)/4N]$, where N is the number of eddies in a ring. If the first law of scientific reasoning is that equations should balance dimensionally, then the zeroth law should be the principle of Occkham's razor—the paring away of unnecessary assumptions. The mathematical theory in [16] is now given a logical derivation. I begin this essay by a study of first principles.

1. PRINCIPLES

I insist that satellite and planetary systems:

i. are discrete and therefore discrete algebra should be used, namely a difference equation,

ii. have at most two boundary conditions (B.C.) and therefore

$$\delta^2 Z_m = jZ_m + cZ_{m-1}, \tag{18}$$

where j,c are constants and $\delta^2 = \Delta - \nabla = \Delta\nabla$ is the central difference operator,

iii. consist of one primary, a pair of secondaries which we ignore and the rest tertiary masses,

iv. by the Commonality Principle must all satisfy the ***same*** spacing law,

v. may equally likely have pro or retrograde outer satellites since retrograde bodies are not irregular,

vi. are stable due to weak (gravitational, tidal or gyroscopic) non-dissipative forces and hence,

vii. the relevant variable is the frequency of nearest approach, the synodic frequency, Y, where

$$Y_{m+h} = Z_{m+1} - Z_m \quad \text{with} \quad h = ½ \tag{19}$$

viii. cannot have B.C. in empty space and hence they must reside in the primary which means that the reciprocal period sequence, Z_m, must turn around near the outermost body and be asymptotic to the inner bodies upon both leaving and returning to the primary. Alone this restricts us to even order difference equations. It *forces* the roots of (18) to be reciprocal pairs and by the theory of equations $c = 0$. Thus

$$\delta^2 Z_m = jZ_m. \tag{20}$$

To elucidate, values of $a = j + 2$ equal to ±2 give arithmetic progressions, and ±3 gives finite exponential (E_n) or alternate FL numbers, and ±6 gives alternate Pell numbers. The sequence ··· 11, 12, 16, 24, ··· is given by $j = ¼$. The data on near-commensurabilities are not significant [15] if the peculiar ratios of 2 and 4 are omitted.

ix. Intuitive considerations of stability require the minimization of the number of mutual perturbations between adjacent satellite orbits. This will obtain if adjacent periods are coprime. This, as is proven later, determines j to be an integer. Now we can determine the value of j.

x. The forces are attractive so the largest root of (20) should be as small as allowed; thus $a = ±3$. Assuming that the Sun-Jupiter distance is fixed then a better way to state point (x) is that it is the minimization of the potential energy of the tertiaries with respect to their secondary. Thus satellites try to get as close to their secondaries as other conditions (ix) will allow. Now $a = 0$ gives two constant sequences and so is trivial and $a = \pm 1$ gives cyclical sequences of periodicities 6 and 3 and so is also trivial. Arithmetic progressions, $a = \pm 2$, are also trivial. Hence $a = \pm 3$, *i.e.*, $j = +1$ or -5. I first used bisected FL (Fibonacci Lucas) sequences in a letter [17].

xi. I assert that only one physical B.C. exists which must equal both matematical B.C. Therefore

(21) $$Z_O = \Delta Z_M = \text{B.C.} \quad \text{or} \quad \nabla Z_O = Z_M = \text{B.C.}$$

which differ only in notation. This is equivalent to $G_O = G_N$ in [16]. And from point (x) we have

(22a,b) $$\delta^2 Z_m = Z_m \quad \text{or} \quad \delta^2 Z_m = -5Z_m ,$$

where the "-5" case corresponds to outer satellites that are alternately prograde and retrograde. When M is infinite, Eqs. (21) and (22a) give sequence S of [16]. Writing $v = \sqrt{5}$ for brevity we have

(23a)	$5v+7$	$2v+3$	$v+2$	$v+3$	$2v+7$	$5v+18$	$13v+47$	$34v+123$	$89v+322$
(23b)	$-3v-4$	$-v-1$	$0+1$	$v+4$	$3v+11$	$8v+29$	$21v+76$	$55v+199$	
(24)		Nept	X	Uran	Satur	(Jup)	Astrea	Mars	
(25) $i =$	$-7h$	$-3h$	h	$5h$	$9h$	$13h$	$17h$		

where *either* sequence may be regarded as the first-order differences (synodic frequencies) of the other. Sequence (23) gives an earth value of 521. For convenience, not rigor, sequence (24) has been placed parallel to (23). The index $h = \frac{1}{2}$.

2. CONCEPTS

A FL sequence, H_n, cannot be expressed as a function of δ^2 and I alone since

(26) $$(\Delta + \nabla - I)H_n = 0 .$$

But a finite exponential (bisected FL) sequence, E_n, satisfies

(27) $$(\delta^2 - I)E_n = 0 .$$

Further, define a sequence, E'_n, such that

(27a) $$(\delta^2 + 5I)E'_n = 0 .$$

Now let Z_n be a bisection of G_n (Eq. 1 and Table 1 of [16]). Then Z_n satisfies (27). If Z_n represents the real reciprocal periods of satellites or planets this can be written as a minimum principle,

(28) $$\sum (\delta^2 - I)Z_m \to 0 .$$

We may state this in words. *A system of satellites (planets) much lighter than their primary tries to act as if their synodic frequencies correspond to real bodies with* their *synodic frequencies in turn being the reciprocal periods of the original bodies.* This is true even if all the bodies do not revolve in the same sense. If they are alternately pro- and retro-grade we can use (31). Thus (28) gives a closed system having a finite number of sidereal (true) and synodic frequencies.

Now in point (xi) we could not have written $Z_O = \Delta Z_O$ since that leads to monotonically increasing sequences. Now this point, namely (21) which is the same as Eq. (1) $G_O = G_N$ led via the theorem in [16] to the beautiful closure relation (14) $\Sigma^\dagger S_i = (-1)^{i-h} S_{-i}$. This immediately gives by taking ratios

(29) $$(S_{i+2} + S_i)/(S_i + S_{i-2}) = S_{-i-1}/S_{1-i} = (S_{-i-2} - S_{-i})/(S_{-i} - S_{2-i}) .$$

Now if satellites are alternately pro- and retro-grade then we may interpret the first pair of (29) to mean that the ratio of adjacent synodic frequencies (since S_i is now negative) equals the ratio of the sidereal frequencies of two other members of the bisection of S aside from a (-1). Real satellite systems have a finite number of bodies but the difference in the ratios given by $\{S\}$ and $\{G_{33}\}$ for example is less than 10^{-6}. Hence (29) is an excellent approximation to the finite cases.

It is easy to show that the ratio of adjacent terms in (23b),

$$S_{nh+1}/S_{nh-1} = (L_n + v)/L_{n-2} = L_{n+2}/(L_n - v),$$

where mod $(n, 4) = 3$. Similarly the ratio of adjacent terms in (23a) is

$$S_{nh+1}/S_{nh-1} = (L_n - v)/L_{n-2} = L_{n+2}/(L_n + v),$$

where mod $(n, 4) = 1$ and where (25) is the index. For completeness we may define the double bisection of an FL sequence, D_n, by

(30) $$(\delta^2 - 5I)D_n = 0.$$

Now a system of alternately pro- and retro-grade satellites satisfies an E primed sequence, E'_n. But the synodic frequencies are no longer differences (since every other term is negative) but sums. Application of the summing of adjacent terms twice is equivalent to the operator $(\delta^2 + 4I)$. Hence in place of (28) we may write

(31) $$\sum (\sigma^2 - I)Z'_m \to 0,$$

where σ is the central sum operator defined by

$$\sigma f_n = f_{n+h} + f_{n-h},$$

where f_n is any sequence whatsoever. It is then easy to show that

(32) $$\sigma^2 = 4I + \delta^2.$$

Z' is a bisection of S or G but with alternate terms multiplied by (-1). A Z' sequence satisfies (27a).

The theory herein has been predicated upon: The Commonality Principle, The Simplicity Principle, and the assumption that the physical reason for the stability of tertiary orbits is the avoidance of low-order commensurabilities (ALOC). J. C. Maxwell approached the motion of molecules in air in a similar vein of which James Jeans wrote [19, pp. 97–98] "...by a train of argument which seems to bear no relation at all to molecules or to their dynamics,... or to common sense, reached a formula which according... to all the rules of scientific philosophy ought to have been hopelessly wrong. ...was shown to be exactly right."

3. PREDICTIONS

Dermott [12] ignored the outer Jovian and Saturnian satellites. I have chosen to give them an important place in this paper. The reciprocal period law is the only theory to make very narrow predictions. There is a blank midway between Saturn's Phoebe and Iapetus in Table 2 of [16]. Hence a Saturnian satellite(s) of (mean) period $207.84 \leqslant P \leqslant 208.03$ day is predicted. I propose to call it Aurelia. If it is ever found, it would constitute proof positive of the theory herein. The allowed range is 0.1 percent of the numbers but I regard 1 percent as acceptible. Similarly a stable orbit in the Jovian system is likely at 97 day with much less likelihood of another at 37 day because of its proximity to the Galilean quadruplet (secondaries).

For the Sun, Jupiter, Saturn their secondaries are Jupiter, the Galilean quadruplet, Titan+Hyperion, respectively. The theory says little about the secondaries. Hence the distance between the primaries and secondaries and their mass ratio must be determined by the properties of the proto-solar system cloud, namely its mass, spin, moment of inertia and magnetic field. We infer that the proto-solar system soon formed two clouds of cold dust and gas. The larger became the Sun and the smaller became Jupiter and Saturn. These then captured enough material to form the other planets and comets by coalesence. During the late phases when dissipative forces were no longer important, the reciprocal period rule would begin to operate. The Kirkwood gaps have prevented the coalesence of asteroids into a planet. Gaps exist at 3/8, 4/9, 5/11 of Jupiter's period as well as at 1/3, 2/5, 3/7 and 1/2. In fact the gap at 3/8 is only 2 percent from the predicted asteroidal planet. See [18, p. 97].

The physical B.C. (point XI) may: (a) lie in the mean angular velocity of the primary, (b) be a mean of the spins of the primary and secondary, (c) lie in the tertiaries as a whole in which case they constitute a self—enclosed system, (d) be the period of a hypothetical satellite that skims the primary's surface, or (e) otherwise. At the moment, I prefer (c).

4. VENUS

The synodic period, y, of a superior (exterior) body of period $z > 1$ is given by

(33) $$1/z + 1/y = 1.$$

The following relations [18, p. 51] are interesting. I use ratios for clarity. Choose the Venusian sidereal (true) year, 224.701 day, to be the unit of time. Then to better than 5 significant figures the earth's period is 395/243 (13/8 is less accurate) and the rotation period of Venus is −79/73 (clockwise). Thinking of ourselves as Venusians, then Venus is fixed and the Sun and Earth appear to revolve around us. We have three frequencies: 1, 73/79, 243/395 to be added in pairs. The first pair gives 79/152 for one Venusian solar day. The first and third using (33) gives 395/152 for the earth's synodic period (584 da). The latter two give 395/608 for the time between successive Earth transits. These frequences 152/79, 608/395, 152/395 are in the exact ratios 5, 4, 1. Hence during every 584 day the same spot on Venus faces the Sun 5 times and the Earth 4 times. Venus must be aspherical so that torque forces can cause this. Tidal forces tend to pull a body apart and are inverse cube. But to align two prolate bodies one of whose axes is θ away from the line joining them requires a $\sin \theta/d^4$ force which is very weak, yet over long time periods must be sufficient.

In passing we give the continued fraction expansion of the distance factor derived from Kepler's III law.

$$1.8995476269\cdots = 1 + \frac{1}{1+} + \frac{1}{8+} + \frac{1}{1+} + \frac{1}{21+} + \frac{1}{4+} + \frac{1}{1+} + \frac{1}{7+} + \frac{1}{1+} + \frac{1}{1+} + \frac{1}{1+} + \cdots$$

whose convergent is 1 + 25253/28073. The first useful convergent is 416/219.

5. COPRIME SEQUENCES

If the recurrence $P_{n+1} = (\text{integer})P_n \pm P_{n-1}$ holds we have a Coprime sequence because it satisfies the following theorem which is a generalization of one in [20, p. 30]. As an example *viz.* 0, 1, 4, 15, 56, 11·19, 780, 41·71, ⋯ . Consider $P_{n+1} = bP_n + cP_{n-1}$.

Theorem. Of all two-point recurrences only those with the middle coefficient b an integer and $c = \pm 1$ have coprime adjacent terms given that an initial pair, P_0 and P_1 say, are coprime.

Proof. The proof obtains by postulating the contrariwise proposition. Let $c = 1$. Let P_{n+1} and P_n be divisible by some integer d. Then bP_n is divisible by d and so also is $P_{n-1} = P_{n+1} - bP_n$. But then

$$P_{n-2} = P_n - bP_{n-1}$$

is divisible by d and likewise all earlier terms by induction. Hence both P_0 and P_1 are divisible by d which contradicts the assumption which says that at most one of P_0 and P_1 are divisible by any number. Hence the theorem must be true.

Choosing $c = -1$ changes no essential part of the argument.

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BINET'S FORMULA GENERALIZED

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Any generalization of the Fibonacci sequence $\{F_n\} = 1, 1, 2, 3, 5, 8, 13, 21, \cdots$ necessarily involves a change in one or both of the defining equations

(1) $$F_1 = F_2 = 1, \qquad F_{n+2} = F_{n+1} + F_n \qquad (n \geqslant 1).$$

Here, however, we seek such a generalization indirectly, by starting with *Binet's formula*

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \qquad (n \geqslant 1)$$

instead of (1). Suppose we define, for any positive integer p, the sequence G_n by

(2) $$G_n = \frac{\left(\frac{1+\sqrt{p}}{2}\right)^n - \left(\frac{1-\sqrt{p}}{2}\right)^n}{\sqrt{p}} \qquad (n \geqslant 1).$$

Thus $\{G_n\} = \{F_n\}$ in the case $p = 5$. We can also write

(3) $$G_n = \frac{\alpha^n - \beta^n}{\sqrt{p}} \qquad (n \geqslant 1),$$

where

$$\alpha = \frac{1+\sqrt{p}}{2}, \qquad \beta = \frac{1-\sqrt{p}}{2}$$

are roots of the equation

(4) $$x^2 - x - \left(\frac{p-1}{4}\right) = 0 .$$

Corresponding to (1), we now have the equations

(5) $$G_1 = G_2 = 1, \qquad G_{n+2} = G_{n+1} + \left(\frac{p-1}{4}\right) G_n \qquad (n \geqslant 1).$$

Proof. Clearly $\alpha - \beta = \sqrt{p}$ and $\alpha + \beta = 1$, so that (3) implies

$$G_1 = \frac{\alpha - \beta}{\sqrt{p}} = 1, \qquad G_2 = \frac{(\alpha-\beta)(\alpha+\beta)}{\sqrt{p}} = 1.$$

[Continued on page 14.]