

$$\sum_{i=1}^n [\xi_i(a_i^k - 1)] a_i^{s+j} = 0 \quad (0 \leq j).$$

Setting  $x_i = \xi_i(a_i^k - 1)$

$$\sum_{i=1}^n a_i^{s+j} x_i = 0 \quad (0 \leq j \leq n-1).$$

The coefficient determinant

$$\det \begin{bmatrix} a_1^s & & a_n^s \\ a_1^{s+1} & & a_n^{s+1} \\ \vdots & \dots & \vdots \\ a_1^{s+n-1} & & a_n^{s+n-1} \end{bmatrix} = (a_1 \dots a_n)^s \det \begin{bmatrix} a_1^0 & & a_n^0 \\ a_1^1 & & a_n^1 \\ \vdots & \dots & \vdots \\ a_1^{n-1} & & a_n^{n-1} \end{bmatrix} \neq 0,$$

since this is the Vandermonde matrix and the  $a_i$  are distinct since  $p$  is irreducible. Hence the  $n$  linear forms are independent, so

$$x_i = 0 \quad (1 \leq i \leq n).$$

Some  $\xi_i \neq 0$  since  $U_{n-1} \neq 0$ . For that  $i$ ,  $a_i^k = 1$ . Since the  $a$ 's are roots of an irreducible polynomial, by Galois theory  $a_j^k = 1$  for  $1 \leq j \leq n$ .

*Q.E.D.*

*Corollary.* Kronecker's Theorem holds even if  $p$  is not irreducible.

*Proof.* We factor  $p(x) = \prod p_i(x)$ , where the  $p_i$  are irreducible. All roots  $\alpha$  of  $p_i$  are roots of  $p$  so  $|\alpha| = 1$  so all roots are roots of unity. But all roots of  $p$  are roots of some  $p_i$  and hence roots of unity.

David Cantor has noted that the proof after (\*\*\*) can be shortened using generating functions. For

$$\sum_{i=0}^n U_i x^i = \frac{x^{n-1}}{1 - \sum_{i=1}^n a_i x^i} = \frac{A(x)}{x^k - 1}$$

Hence

$$x^n p(x^{-1}) = 1 - \sum_{i=1}^n a_i x^i \mid x^k - 1$$

$p(\alpha) = 0$  implies  $p(\alpha^{-1}) = 0$  implies  $\alpha^{-k} - 1 = 0$ ,  $\alpha^{-k} = 1$ , so  $\alpha^k = 1$ .

\*\*\*\*\*

[Continued from page 8.]

I must tell you that I am short of proofs and most of the propositions would have to be presented as observations or conjectures. Co-authors with proofs are welcome.

Thank you for your attention to this letter. Please write and let me know whether the subject is of interest. You are free, of course, to publish this letter or any part of it.

Sincerely,  
John W. Jameson,  
P. O. Box 205  
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