GENERALIZED LUCAS SEQUENCES

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1. INTRODUCTION

In working with linear recurrence sequences, the generating functions are of the form

$$\frac{q(x)}{p(x)} = \sum_{n=0}^{\infty} a_n x^n , \tag{1.1}$$

where $p(x)$ is a polynomial and $q(x)$ is a polynomial of degree smaller than $p(x)$. In multisecting the sequence $\{a_n\}$ it is necessary to find polynomials $P(x)$ whose roots are the k^{th} power of the roots of $p(x)$. Thus, we are led to the elementary symmetric functions.

Let

$$p(x) = \prod_{i=1}^{n} (x - a_i) = x^n - p_1 x^{n-1} + p_2 x^{n-2} - p_3 x^{n-3} + \cdots + (-1)^k p_k x^{n-k} + \cdots + (-1)^n p_n , \tag{1.2}$$

where p_k is the sum of products of the roots taken k at a time. The usual problem is, given the polynomial $p(x)$, to find the polynomial $P(x)$ whose roots are the k^{th} powers of the roots of $p(x)$,

$$P(x) = x^n - P_1 x^{n-1} + P_2 x^{n-2} - P_3 x^{n-3} + \cdots + (-1)^n P_n . \tag{1.3}$$

There are two basic problems here. Let

$$S_k = a_1^k + a_2^k + a_3^k + \cdots + a_n^k , \tag{1.4}$$

where

$$p(x) = (x - a_1)(x - a_2) \cdots (x - a_n) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n$$

and $c_k = (-1)^k p_k$. then Newton's Identities (see Conkwright [1])

$$\begin{aligned} S_1 + c_1 &= 0 \\ S_2 + S_1 c_1 + 2c_2 &= 0 \\ &\cdots \\ S_n + S_{n-1} c_1 + \cdots + S_1 c_{n-1} + n c_n &= 0 \\ S_{n+1} + S_n c_1 + \cdots + S_1 c_n + (n+1) c_{n+1} &= 0 \end{aligned} \tag{1.5}$$

can be used to compute S_k for $S_1, S_2, \cdots, S_n$. Now, once these first n values are obtained, the recurrence relation

$$S_{n+1} + S_n c_1 + S_{n-1} c_2 + \cdots + S_1 c_n = 0 \tag{1.6}$$

will allow one to get the next value S_{n+1} and all subsequent values of S_m are determined by recursion.

Returning now to the polynomial $P(x)$,

$$P(x) = (x - a_1^k)(x - a_2^k)(x - a_3^k) \cdots (x - a_n^k) = x^n + Q_1 x^{n-1} + Q_2 x^{n-2} + \cdots + Q_n , \tag{1.7}$$

where

$$Q_1 = a_1^k + a_2^k + \cdots + a_n^k = S_k$$

and it is desired to find the $Q_1, Q_2, Q_3, \cdots, Q_n$. Clearly, one now uses the Newton identities (1.5) again, since $S_k, S_{2k}, S_{3k}, \cdots, S_{nk}$ can be found from the recurrence for S_m, where we know $S_k, S_{2k}, S_{3k}, \cdots, S_{nk}$ and wish to find the recurrence for the k-sected sequence. Before, we had the auxiliary polynomial for S_m and computed the $S_1, S_2, \cdots, S_n$. Here, we have $S_k, S_{2k}, \cdots, S_{nk}$ and wish to calculate the coefficients of the auxiliary polynomial $P(x)$. Given a sequence S_m and that it satisfies a *linear recurrence* of order n, one can use Newton's identities to obtain that recurrence. This requires only that $S_1, S_2, S_3, \cdots, S_n$ be known. If

$$S_{n+1} + (S_n c_1 + S_{n-1} c_2 + \cdots + S_1 c_n) + (n+1)c_{n+1} = 0$$

is used, then $S_{n+1} = -(S_n c_1 + \cdots + S_1 c_n)$ and $c_{n+1} = 0$.

Suppose that we know that $L_1, L_2, L_3, L_4, \cdots$, the Lucas sequence, satisfies a linear recurrence of order two. Then $L_1 + c_1 = 0$ yields $c_1 = -1$; $L_2 + L_1 c_1 + 2c_2 = 0$ yields $c_2 = -1$; and $L_3 + L_2 c_1 + L_1 c_2 + 3c_3 = 0$ yields $c_3 = 0$. Thus, the recurrence for the Lucas numbers is

$$L_{n+2} - L_{n+1} - L_n = 0.$$

We next seek the recurrence for $L_k, L_{2k}, L_{3k}, \cdots$. $L_{nk} = \alpha^{nk} + \beta^{nk}$ is a Lucas-type sequence and $L_k + Q_1 = 0$ yields $Q_1 = -L_k$; $L_{2k} + c_1 L_k + 2c_2 = 0$ yields $L_{2k} - L_k^2 + 2c_2 = 0$, but $L_k^2 = L_{2k} + 2(-1)^k$ so that

$$L_{2k} - L_k^2 + 2c_2 = 0$$

gives $c_2 = (-1)^k$. Thus, the recurrence for L_{nk} is

$$L_{(n+2)k} - L_k L_{(n+1)k} + (-1)^k L_{nk} = 0.$$

This one was well known. Suppose as a second example we deal with the generalized Lucas sequence associated with the Tribonacci sequence. Here, $S_1 = 1$, $S_2 = 3$, and $S_3 = 7$, so that $S_1 + c_1 = 0$ yields $c_1 = -1$;

$$S_2 + c_1 S_2 + 2c_2 = 0 \quad \text{yields} \quad c_2 = -1,$$

and

$$S_3 + c_1 S_2 + c_2 S_1 + 3c_3 = 0 \quad \text{yields} \quad c_3 = -1.$$

Here,

$$S_k = \alpha^k + \beta^k + \gamma^k,$$

where α, β, γ are roots of

$$x^3 - x^2 - x - 1 = 0.$$

Suppose we would like to find the recurrence for S_{nk}. Using Newton's identities,

$$S_k + Q_1 = 0 \qquad Q_1 = -S_k$$

$$S_{2k} + S_k(-S_k) + 2Q_2 = 0 \qquad Q_2 = \tfrac{1}{2}(S_k^2 - S_{2k})$$

$$S_{3k} + S_{2k}(-S_k) + S_k[\tfrac{1}{2}(S_k^2 - S_{2k})] + 3Q_3 = 0 \qquad Q_3 = \tfrac{1}{6}(S_k^3 - 3S_k S_{2k} + 2S_{2k})$$

This is, of course, correct, but it doesn't give the neatest value. What is Q_2 but the sum of the product of roots taken two at a time,

$$Q_2 = (\alpha\beta)^k + (\alpha\gamma)^k + (\beta\gamma)^k = \frac{1}{\gamma^k} + \frac{1}{\beta^k} + \frac{1}{\alpha^k} = S_{-k}$$

and $Q_3 = (\alpha\beta\gamma)^k = 1$. Thus, the recurrence for S_{nk} is

$$S_{(n+3)k} - S_k S_{(n+2)k} + S_{-k} S_{(n+1)k} + S_{nk} = 0. \tag{1.8}$$

This and much more about the Tribonacci sequence and its associated Lucas sequence is discussed in detail by Trudy Tong [3].

2. DISCUSSION OF E-2487

A problem in the Elementary Problem Section of the *American Mathematical Monthly* [2] is as follows:

If $S_k = a_1^k + a_2^k + \cdots + a_n^k$ and $S_k = k$ for $1 \leq k \leq n$, find S_{n+1}.

From $S_k = a_1^k + \cdots + a_n^k$, we know that the sequence S_m obeys a linear recurrence of order n. From Newton's Identities we can calculate the coefficients of the polynomial whose roots are $a_1, a_2, \cdots, a_n$. (We do not need to know the roots themselves.) Thus, we can find the recurrence relation, and hence can find S_{n+1}. This is for an arbitrary but fixed n.

Let

(2.1) $$S(x) = S_1 + S_2x + S_3x^2 + \cdots + S_{n+1}x^n + \cdots,$$

where $S_1, S_2, S_3, \cdots, S_n$ are given. In our case, $S(x) = 1/(1-x)^2$.

Let

(2.2) $$C(x) = c_1x + c_2x^2 + \cdots + c_nx^n + \cdots.$$

These coefficients c_n are to be calculated from the $S_1, S_2, \cdots, S_n$.

From Newton's Identities (1.5),

$$S_{n+1} + S_nc_1 + S_{n-1}c_2 + \cdots + S_1c_n + (n+1)c_{n+1} = 0.$$

These are precisely the coefficients of x^n in

$$S(x) + S(x)C(x) + C'(x) = 0.$$

The solution to this differential equation is easily obtained by using the integrating factor. Thus

$$C(x)e^{\int S(x)dx} = \int e^{\int S(x)dx}(-S(x))dx + C$$

so that

$$C(x) = -1 + ce^{-\int S(x)dx} = -1 + e^{-(S_1x + S_2x^2/2 + \cdots + S_nx^n/n + \cdots)}$$

since $C(0) = 0$.

In this problem, $S(x) = 1/(1-x)^2$ so that

$$C(x) = -1 + e^{-x/(1-x)}.$$

If one writes this out,

$$-1 + e^{-x/(1-x)} = -1 + 1 - \frac{x}{1!(1-x)} + \frac{x^2}{2!(1-x)^2} - \frac{x^3}{3!(1-x)^3} + \cdots.$$

From Waring's Formula (See Patton and Burnside, *Theory of Equations*, etc.)

$$c_n = \sum \frac{(-1)^{r_1+r_2+\cdots+r_n}S_1^{r_1}S_2^{r_2}\cdots S_n^{r_n}}{r_1!\,r_2!\,r_3!\cdots r_n!\,1^{r_1}2^{r_2}\cdots n^{r_n}},$$

where the summation is over all non-negative solutions to

$$r_1 + 2r_2 + 3r_3 + \cdots + nr_n = n.$$

In our case where $S_k = k$ for $1 \leqslant k \leqslant n$, this becomes

$$c_n = \sum \frac{(-1)^{r_1+r_2+\cdots+r_n}}{r_1!\,r_2!\cdots r_n!}$$

over all nonnegative solutions to

$$r_1 + 2r_2 + 3r_3 + \cdots + nr_n = n,$$

so that

$$\sum_{r_1+2r_2+\cdots+nr_n=n} \frac{(-1)^{r_1+r_2+\cdots+r_n}}{r_1!\,r_2!\,r_3!\cdots r_n!} = \sum_{k=1}^{n} \frac{(-1)^k\binom{n-1}{k-1}}{k!}.$$

Then

$$c_1 = \frac{-1}{1!} = -1$$

$$c_2 = \frac{-1}{1!} + \frac{1}{2!} = -1/2$$

$$c_3 = \frac{-1}{1!} + \frac{2}{2!} - \frac{1}{3!} = -1/6$$

$$c_4 = \frac{-1}{1!} + \frac{3}{2!} - \frac{3}{3!} + \frac{1}{4!} = 1/24$$

... ...

$$c_n = -\frac{\binom{n-1}{0}}{1!} + \frac{\binom{n-1}{1}}{2!} - \frac{\binom{n-1}{2}}{3!} + \cdots + \frac{(-1)^n\binom{n-1}{n-1}}{n!}$$

so that

(2.3) $$c_n = \sum_{k=1}^{n} \frac{(-1)^k\binom{n-1}{k-1}}{k!}$$

Here we have an explicit expression for the c_n for $S_k = k$ for $1 \leqslant k \leqslant n$.

We now return to the problem E-2487. From the Newton-Identity equation

$$S_{n+1} + c_1S_n + \cdots + c_nS_1 + (n+1)c_{n+1} = 0.$$

We must make a careful distinction between the solution to E-2487 for n and values of the S_m sequence for larger n. Let S_n^* be the solution to the problem; then

$$S_n^* + c_1S_n + c_2S_{n-1} + \cdots + c_nS_1 = 0,$$

where $S_k = k$ for $1 \leqslant k \leqslant n$ and the c_k for $1 \leqslant k \leqslant n$ are given by the Newton Identities using these S_k. We note two diverse things here. Suppose we write the next Newton-Identity for a higher value of n,

$$S_{n+1} + c_1S_n + \cdots + c_nS_1 + (n+1)c_{n+1} = 0;$$

then

$$(n+1) - S_n^* + (n+1)c_{n+1} = 0$$

so that

(2.4) $$S_n^* = (n+1)(1+c_{n+1}) = (n+1)\left[1 + \sum_{k=1}^{n+1} \frac{(-1)^k\binom{n}{k-1}}{k!}\right].$$

We can also get a solution in another way.

$$S_n^* = -[c_1S_n + \cdots + c_nS_1]$$

is the n^{th} coefficient in the convolution of $S(x)$ and $C(x)$ which was used earlier (2.1), (2.2). Thus

$$S^*(x) = -C(x)S(x) = [1 - e^{-x/(1-x)}]/(1-x)^2 = \frac{x}{1!\,(1-x)^3} - \frac{x^2}{2!\,(1-x)^4} + \frac{x^3}{3!\,(1-x)^5} - \cdots$$

$$S_1^* = 1/1! = 1$$

$$S_2^* = 3/1! - 1/2! = 5/2$$

$$S_3^* = 6/1! - 4/2! + 1/3! = 25/6$$

and

(2.5) $$S_n^* = \sum_{k=1}^{n} \frac{(-1)^{k+1}\binom{n+1}{k+1}}{k!}.$$

It is not difficult to show that the two formulas (2.4) and (2.5) for S_n^* are the same.

3. A GENERALIZATION OF E-2487

If one lets $S(x) = 1/(1-x)^{m+1}$, then

(3.1) $$C(x) = -1 + e^{\frac{1}{m}[1-1/(1-x)^m]}$$

and

(3.2) $$S^*(x) = \frac{1 - e^{\frac{1}{m}[1-1/(1-x)^m]}}{(1-x)^{m+1}}.$$

We now get explicit expressions for S_n, c_n, and S_n^*.

First,

$$S(x) = \frac{1}{(1-x)^{m+1}} = \sum_{n=0}^{\infty} \binom{n+m}{n} x^n,$$

so that

$$S_{n+1} = \binom{n+m}{n}. \tag{3.3}$$

We shall show that

Theorem 3.1.

$$c_n = \sum_{k=1}^{n} \frac{1}{k!\,m^k} \sum_{\alpha=1}^{k} (-1)^k \binom{k}{\alpha} \binom{\alpha m + n - 1}{n}$$

and

$$S_n^* = \binom{n+m}{n} + (n+1)c_{n+1} = \binom{n+m}{n} + (n+1) \sum_{k=1}^{n+1} \frac{1}{k!\,m^k} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} \binom{m\alpha + n}{n+1}$$

Proof. From Schwatt [4], one has the following. If $y = g(u)$ and $u = f(x)$, then

$$\frac{d^n y}{dx^n} = \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} u^{k-\alpha} \frac{d^n u^\alpha}{dx^n} \frac{d^k y}{du^k}.$$

We can find the Maclaurin expansion of

$$y = e^{1/m} e^{-1/m(1-x)^m} = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n y}{dx^n}\right|_{x=0} x^n.$$

Let $y = e^{1/m} e^u$, where $u = -1/m(1-x)^m$; then $u^\alpha = (-1)^\alpha/m^\alpha(1-x)^{m\alpha}$ and

$$\frac{d^n u^\alpha}{dx^n} = \frac{(-1)^\alpha}{m^\alpha} \frac{(m\alpha)(m\alpha+1)\cdots(m\alpha+n-1)}{(1-x)^{m\alpha+n}},$$

$$\frac{d^k y}{du^k} = e^{1/m} e^u, \qquad \text{and} \qquad \left.\frac{d^k y}{dx^k}\right|_{x=0} = 1.$$

Thus,

$$\frac{1}{n!} \left.\frac{d^n y}{dx^n}\right|_{x=0} = \sum_{k=1}^{k} \frac{(-1)^k}{k!} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} \frac{(-1)^{k-\alpha}}{m^{k-\alpha}} \frac{(-1)^\alpha}{m^\alpha} \binom{m\alpha + n - 1}{n}$$

so that

$$c_n = \sum_{k=1}^{n} \frac{1}{k!\,m^k} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} \binom{m\alpha + n - 1}{n}.$$

Thus, since $S_n^* = S_{n+1} + (n+1)c_{n+1}$, then

$$S_n^* = \binom{n+m}{n} + (n+1) \sum_{k=1}^{n+1} \frac{1}{k!\,m^k} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} \binom{m\alpha + n}{n+1}$$

which concludes the proof of Theorem 3.1.

But

$$S^*(x) = -C(x)/(1-x)^{m+1}$$

so that we can get yet another expression for S_n^*, (3.4) $$S_n^* = -\sum_{j=1}^{n}(S_jc_{n-j+1}) = -\sum_{j=1}^{n}S_{n-j+1}c_j\,,$$

where c_n is as above and

$$S_n = \binom{n+m-1}{m} = \binom{n+m-1}{n-1}.$$

4. RELATIONSHIPS TO PASCAL'S TRIANGLE

An important special case deserves mention. If we let $S_k = m$ for $1 \leqslant k \leqslant n$, then $S(x) = m/(1-x)$ and

$$C(x) = -1 + e^{-\int[m/(1-x)]\,dx} = -1 + (1-x)^m\,.$$

Therefore,

$$c_k = (-1)^k\binom{m}{k}$$

for $1 \leqslant k \leqslant m \leqslant n$ or for $1 \leqslant k \leqslant n < m$, and $c_k = 0$ for $n < k \leqslant m$, and $c_k = 0$ for $k > n$ in any case. Now, let $S_k = -m$ for $1 \leqslant k \leqslant n$; then

$$S(x) = -m/(1-x) \qquad \text{and} \qquad C(x) = -1 + 1/(1-x)^m,$$

and we are back to columns of Pascal's triangle.

If we return to

$$c_k = \frac{(-1)^k}{k!}\begin{vmatrix} m & 1 & 0 & 0 & 0 & \cdots \\ m & m & 2 & 0 & 0 & \cdots \\ m & m & m & 3 & 0 & \cdots \\ m & m & m & m & 4 & \cdots \\ m & m & m & m & m & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{k\times k}$$

then we have rows of Pascal's triange, while with

$$c_k = \frac{(-1)^k}{k!}\begin{vmatrix} -m & 1 & 0 & 0 & 0 & \cdots \\ -m & -m & 2 & 0 & 0 & \cdots \\ -m & -m & -m & 3 & 0 & \cdots \\ -m & -m & -m & -m & 4 & \cdots \\ -m & -m & -m & -m & -m & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{k\times k}$$

we have columns of Pascal's triangle.

Suppose that we have this form for c_k in terms of general S_k but that the recurrence is of finite order. Then, clearly, $c_k = 0$ for $k > n$. To see this easily, consider, for example, $S_1 = 1, S_2 = 3, S_3 = 7,$

$$S_{n+3} = S_{n+2} + S_{n+1} + S_n\,.$$

$$c_k = \frac{(-1)^k}{k!}\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 2 & 0 & 0 & 0 & \cdots \\ 7 & 3 & 1 & 3 & 0 & 0 & \cdots \\ 11 & 7 & 3 & 1 & 4 & 0 & \cdots \\ 21 & 11 & 7 & 3 & 1 & 5 & \cdots \\ 39 & 21 & 11 & 7 & 3 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{k\times k}$$

$$\begin{aligned} 1-1 &= 0 \\ 3-1-2 &= 0 \\ 7-3-1-3 &= 0 \\ 11-7-3-1 &= 0 \\ 21-11-7-3 &= 0 \\ 39-21-11-7 &= 0, \text{ etc.} \end{aligned}$$

Thus, in this case, we can get the first column all zero with multipliers c_1, c_2, c_3, each of which is -1.

5. THE GENERAL CASE AND SOME CONSEQUENCES

Returning now to

(5.1) $$C(x) = -1 + e^{-(S_1x+S_2x^2/2+S_3x^3/3+\cdots+S_nx^n/n+\cdots)}$$

which was found in Riordan [6], we can see some nice consequences of this neat formula.

It is easy to establish that the regular Lucas numbers have generating function

(5.2) $$\frac{1+2x}{1-x-x^2} = S(x) = \sum_{n=0}^{\infty} L_{n+1}x^n$$

$$e^{-[(1+2x)/(1-x-x^2)]\,dx} = e^{\ln(1-x-x^2)} = 1-x-x^2 = 1+C(x).$$

Here we know that $c_1 = -1$, $c_2 = -1$, and $c_m = 0$ for all $m > 2$. This implies that the Lucas numbers put into the formulas for c_m $(m > 2)$ yield zero, and furthermore, since $L_k, L_{2k}, L_{3k}, \cdots$, obey $1 - L_kx + (-1)^kx^2$, then it is true that $S_n = L_{nk}$ put into those same formulas yield non-linear identities for the k-sected Lucas number sequence. However, consider

(5.3) $$e^{(L_1x+L_2x^2/2+\cdots+L_nx^n/n+\cdots)} = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+1}x^n$$

and

$$e^{(L_kx+L_{2k}x^2/2+\cdots+L_{nk}x^n/n+\cdots)} = \frac{1}{1-L_kx+(-1)^kx^2} = \sum_{n=0}^{\infty} \frac{F_{(n+1)k}}{F_k}x^n .$$

Let us illustrate. Let $S_1, S_2, S_3, \cdots$ be generalized Lucas numbers,

$$c_1 = -S_1$$
$$c_2 = \frac{1}{2}(S_1^2 - S_2)$$
$$c_3 = \frac{1}{6}(S_1^3 - 3S_1S_2 + 2S_3)$$
$$c_4 = \frac{1}{24}(S_1^4 - 6S_1^2S_2 + 8S_1S_3 + 3S_2^2 - 6S_4)$$
$$\cdots \quad \cdots$$

Let $S_n = L_{nk}$ so that $c_m = 0$ for $m > 2$.

$$\frac{1}{6}[L_k^3 - 3L_kL_{2k} + 2L_{3k}] = 0$$

while

$$\frac{1}{6}[L_k^3 + 3L_kL_{2k} + 2L_{3k}] = F_{4k}/F_k .$$

In Conkwright [1] was given

$$c_m = \frac{(-1)^m}{m!}\begin{vmatrix} S_1 & 1 & 0 & 0 & 0 & \cdots \\ S_2 & S_1 & 2 & 0 & 0 & \cdots \\ S_3 & S_2 & S_1 & 3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{m-1} & \cdots & \cdots & \cdots & \cdots & m-1 \\ S_m & S_{m-1} & S_{m-2} & \cdots & S_2 & S_1 \end{vmatrix}$$

which was derived in Hoggatt and Bicknell [5].

Thus for $m > 2$

(5.5) $$c_m = \frac{(-1)^m}{m!}\begin{vmatrix} L_k & 1 & 0 & 0 & 0 & \cdots \\ L_{2k} & L_k & 2 & 0 & 0 & \cdots \\ L_{3k} & L_{2k} & L_k & 3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{(m-1)k} & L_{(m-2)k} & \cdots & \cdots & \cdots & k-1 \\ L_{mk} & L_{(m-1)k} & \cdots & \cdots & L_{2k} & L_k \end{vmatrix} = 0$$

for all $k > 0$, where L_k is the k^{th} Lucas number. This same formula applies, since $c_m = 0$ for $m > 3$, if $S_m = \mathcal{L}_{mk}$ where

$$\mathcal{L}_1 = 1, \quad \mathcal{L}_2 = 3, \quad \mathcal{L}_3 = 7, \quad \text{and} \quad \mathcal{L}_{m+3} = \mathcal{L}_{m+2} + \mathcal{L}_{m+1} + \mathcal{L}_m$$

are the generalized Lucas numbers associated with the Tribonacci numbers T_n

$$(T_1 = T_2 = 1, \quad T_3 = 2, \quad \text{and} \quad T_{n+3} = T_{n+2} + T_{n+1} + T_n.)$$

If $\mathcal{L}_m$ are the Lucas numbers associated with the generalized Fibonacci numbers F_n whose generating function is

$$\text{(5.6)} \qquad \frac{1}{1 - x - x^2 - x^3 - \cdots - x^r} = \sum_{n=0}^{\infty} F_{n+1}x^n ,$$

then if $S_m = \mathcal{L}_{mk}$, then the corresponding $c_m = 0$ for $m > r$, yielding (5.5) for $m > r$ with L_{mk} everywhere replaced by $\mathcal{L}_{mk}$.

Further, let

$$F(x) = 1 - x - x^2 - x^3 - \cdots - x^r ;$$

then

$$F'(x) = -1 - 2x - 3x^2 - \cdots - rx^{r-1}$$

and

$$\text{(5.7)} \qquad -\frac{F'(x)}{F(x)} = \frac{1 + 2x + 3x^2 + \cdots + rx^{r-1}}{1 - x - x^2 - x^2 - \cdots - x^r} = \sum_{n=0}^{\infty} \mathcal{L}_{n+1}x^n ,$$

where $\mathcal{L}_n$ is the generalized Lucas sequence associated with the generalized Fibonacci sequence whose generating function is $1/F(x)$. Thus, any of these generalized Fibonacci sequences is obtainable as follows:

$$e^{-\int [F'(x)/F(x)]\,dx} = \frac{1}{1 - x - x^2 - x^3 - \cdots - x^r} = \sum_{n=0}^{\infty} F_{n+1}x^n$$

and we have

Theorem 5.1.

$$e^{\mathcal{L}_1 x + \mathcal{L}_2 x^2/2 + \cdots + \mathcal{L}_n x^n/n + \cdots} = 1/F(x) = \sum_{n=0}^{\infty} F_{n+1}x^n .$$

The generalized Fibonacci numbers F_n generated by (5.6) appear in Hoggatt and Bicknell [7] and [8] as certain rising diagonal sums in generalized Pascal triangles.

Write the left-justified polynomial coefficient array generated by expansions of

$$(1 + x + x^2 + \cdots + x^{r-1})^n, \quad n = 0, 1, 2, 3, \cdots, r \geq 2.$$

Then the generalized Fibonacci numbers $u(n; p,q)$ are given sequentially by the sum of the element in the left-most column and the n^{th} row and the terms obtained by taking steps p units up and q units right through the array. The simple rising diagonal sums which occur for $p = q = 1$ give

$$u(n; 1,1) = F_{n+1}, \quad n = 0, 1, 2, \cdots.$$

The special case $r = 2, p = q = 1$ is the well known relationship between rising diagonal sums in Pascal's triangle and the ordinary Fibonacci numbers,

$$\sum_{i=0}^{[(n+1)/2]} \binom{n-i}{i} = F_{n+1}$$

while

$$\sum_{i=0}^{[(n+1)/2]} \binom{n-i}{i}_r = F_{n+1}$$

where

$$\binom{n-i}{i}_r$$

is the polynomial coefficient in the i^{th} column and $(n-i)^{st}$ row of the left-adjusted array.

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[Continued from p. 122.]

From this we have that

$$L(F(n)) = \frac{f(n+1) - (-1)^{F(n+2)} f(n-2)}{f(n-1)} \tag{3}$$

Now, letting $a = F(n)$, $b = F(n+1)$ in (2), we have

$$5f(n)f(n+1) = L(F(n+2)) - (-1)^{F(n)} L(F(n-1)). \tag{4}$$

Finally, substituting (3) for each term on the right of (4) and rearranging gives the required recursion.

It is interesting to note that a 5^{th} order recursion for $f(n)$ exists, but it is much more complicated.

Proposition.

$$f(n) = \frac{(5f(n-2)^2 + 2(-1)^{F(n+1)})f(n-3)^2 f(n-4) + f(n-2)(f(n-2) - (-1)^{F(n-1)} f(n-5))(f(n-1) - (-1)^{F(n)} f(n-4))}{2f(n-4)f(n-3)}$$

Proof. Use Equation (2) and the identity

$$L(a)L(b) = L(a+b) + (-1)^a L(b-a), \tag{5}$$

to obtain

$$5f(n)f(n+1) = 2L(F(n+2)) - L(F(n))L(F(n+1)).$$

Using (3) on the right-hand side and rearranging gives the required recursion.

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