ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-278 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Show

$$\sqrt{\frac{5F_{n+2}}{F_n}} = \langle 3, \overline{\underbrace{1, 1, \cdots, 1}_{n-1}, 6} \rangle$$

(Continued fraction notation, cyclic part under bar).

H-279 Proposed by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Establish the F-L identities:

(a) $$F^4_{n+6r} - (L_{4r}+1)(F^4_{n+4r} - F^4_{n+2r}) - F^4_n = F_{2r}F_{4r}F_{6r}F_{4n+12r}$$

(b) $$F^4_{n+6r+3} + (L_{4r+2}-1)(F^4_{n+4r+2} - F^4_{n+2r+1}) - F^4_n = F_{2r+1}F_{4r+2}F_{6r+3}F_{4n+12r+6}\ .$$

H-280 Proposed by S. Bruckman, Concord, California.

Prove the congruences

(1) $$F_{3\cdot 2^n} \equiv 2^{n+2} \pmod{2^{n+3}};$$

(2) $$L_{3\cdot 2^n} \equiv 2 + 2^{2n+2} \pmod{2^{2n+4}},\quad n = 1, 2, 3, \cdots$$

SOLUTIONS

SUM-ARY CONCLUSION

H-264 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{i=0}^{m-r} \binom{s+i}{i}\binom{m+n-s-i+1}{n-s} = \sum_{i=0}^{n-s} \binom{r+i}{i}\binom{m+n-r-i+1}{m-r}\ .$$

Solution by P. Bruckman, Concord, Calif.

Let

(1) $$\theta(m,n,r,s) = \sum_{i=0}^{m-r} \binom{s+i}{i}\binom{m+n-s-i+1}{n-s}\quad ,$$

and

(2) $$F(w,x,y,z) = \sum_{m,n=0}^{\infty} \sum_{r=0}^{m} \sum_{s=0}^{n} \theta(m,n,r,s) w^m x^n y^r z^s .$$

Then

$$F(w,x,y,z) = \sum_{m,n,r,s=0}^{\infty} \theta(m+r, n+s, r, s) w^{m+r} x^{n+s} y^r z^s$$

$$= \sum_{m,n,r,s=0}^{\infty} \sum_{i=0}^{m} \binom{s+i}{i} \binom{m+r+n-i+1}{n} w^m x^n (wy)^r (xz)^s$$

$$= \sum_{m,n,r,s,i=0}^{\infty} \binom{s+i}{i} \binom{m+r+n+1}{n} w^{m+i} x^n (wy)^r (xz)^s$$

$$= \sum_{m,n,r,s,i=0}^{\infty} \binom{-s-1}{i} \binom{-m-r-2}{n} w^m (-x)^n (wy)^r (xz)^s (-w)^i$$

$$= \sum_{m,r,s=0}^{\infty} (1-w)^{-s-1} (1-x)^{-m-r-2} w^m (wy)^r (xz)^s$$

$$= (1-w)^{-1}(1-x)^{-2} \sum_{m,r,s=0}^{\infty} \left(\frac{w}{1-x}\right)^m \left(\frac{wx}{1-x}\right)^r \left(\frac{xz}{1-w}\right)^s$$

$$= (1-w)^{-1}(1-x)^{-2} \left(1-\frac{w}{1-x}\right)^{-1} \left(1-\frac{wy}{1-x}\right)^{-1} \left(1-\frac{xz}{1-w}\right)^{-1}$$

or

(3) $$F(w,x,y,z) = (1-w-x)^{-1}(1-x-wy)^{-1}(1-w-xz)^{-1} .$$

From (3), the following symmetry relation is evident:

(4) $$F(w,x,y,z) = F(x,w,z,y).$$

Hence,

(5) $$F(w,x,y,z) = \sum_{m,n=0}^{\infty} \sum_{r=0}^{m} \sum_{s=0}^{n} \theta(m,n,r,s) x^m w^n z^r y^s .$$

In the last expression, we may make the following substitutions:

(6) $$m \to N, \quad n \to M, \quad r \to S, \quad s \to R .$$

Then

$$F(w,x,y,z) = \sum_{N,M=0}^{\infty} \sum_{S=0}^{N} \sum_{R=0}^{M} \theta(N,M,S,R) x^N w^M z^S y^R .$$

Now reversing the orders of summation and converting capital letters to small letters again, we obtain:

(7) $$F(w,x,y,z) = \sum_{m,n=0}^{\infty} \sum_{r=0}^{m} \sum_{s=0}^{n} \theta(n,m,s,r) w^m x^n y^r z^s .$$

Now comparing coefficients of (2) and (7), treating F as a function of each of its variables, in order, we conclude

(8) $$\theta(m,n,r,s) = \theta(n,m,s,r). \quad \text{Q.E.D.}$$

Also solved by D. Beverage and the Proposer.

ANOTHER CONGRUENCE

H-265 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Show that

$$F_{2^3 \cdot 3^{k-1}} \equiv 0 \pmod{3^k}, \quad \text{where} \quad k \geqslant 1 .$$

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Let p be an odd prime, $p \neq 5$ and let m be a positive integer such that $p \mid F_m$. We shall prove that

(*) $$F_{mp^{k-1}} \equiv 0 \pmod{p^k} \quad (k = 1, 2, 3, \cdots).$$

Proof of (*). We have (Binet representation)

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad F_{pn} = \frac{\alpha^{pn} - \beta^{pn}}{\alpha - \beta},$$

so that

$$\frac{F_{pn}}{F_n} = \frac{\alpha^{pn} - \beta^{-n}}{\alpha^n - \beta^n} \quad (\alpha^n - \beta^n)^{p-1} \pmod{p}.$$

Thus

(**) $$\frac{F_{pn}}{F_n} \equiv (\alpha - \beta)^{p-1} F_n^{p-1} \pmod{p}.$$

Now assume that (*) holds up to and including the value k. By (**),

$$\frac{F_{mp^k}}{F_{mp^{k-1}}} \quad (\alpha - \beta)^{p-1} F^{p-1}_{mp^{k-1}} \equiv 0 \pmod{p},$$

$$F_{mp^k} \equiv 0 \pmod{p\, F_{mp^{k-1}}}.$$

Hence, by the inductive hypothesis,

$$F_{mp^k} \equiv 0 \pmod{p^{k+1}}.$$

This evidently completes the proof.

It is known that the smallest positive m such that $p \mid F_m$ is a divisor of $\tfrac{1}{2}(p^2 - 1)$. It follows that

$$F_M \equiv 0 \pmod{p^k}, \quad (M = \tfrac{1}{2}(p^2 - 1)p^{k-1}, \ k > 1).$$

Indeed, if $p \equiv \pm 1 \pmod 5$, then

$$F_M \equiv 0 \pmod{p^k} \quad (M = (p-1)p^{k-1}, \ k > 1).$$

In particular we have

$$F_{4 \cdot 3^{k-1}} \equiv 0 \pmod{3^k}.$$

Also solved by P. Bruckman and D. Beverage.

IDENTIFY!

H-266 Proposed by G. Berzsenyi, Lamar University, Beaumont, Texas.

Find all identities of the form

$$\sum_{k=0}^{n} \binom{n}{k} F_{rk} = s^n F_{tn}$$

with positive integral r, s and t.

Solution by P. Bruckman, Concord, California

(1) $$\sum_{k=0}^{n} \binom{n}{k} F_{rk} = \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} (\alpha^{rk} - \beta^{rk}) = \frac{(1+\alpha^r)^n - (1+\beta^r)^n}{\sqrt{5}} .$$

If this expression is to equal $s^n F_{nt}$, for some natural triplet (r,s,t), it must hold for *all* non-negative n. The case $n = 0$ yields no information, merely confirming the trivial identity $0 = 0$. The cases $n = 1$, 2 and 3 yield, respectively:

(2) $$F_r = sF_t ;$$

(3) $$2F_r + F_{2r} = s^2 F_{2t} ;$$

(4) $$3F_r + 3F_{2r} + F_{3r} = s^3 F_{3t} .$$

Using (2) and (3), we obtain:

$$2F_r + F_r L_r = s^2 F_t L_t = sF_r L_t ,$$

or, since $r > 0$,

(5) $$L_r + 2 = sL_t .$$

Finally, using (2), (4) and the identity:

$$F_{3m} = F_m(L_m^2 - (-1)^m),$$

we have:

$$3F_r + 3F_r L_r + F_r(L_r^2 - (-1)^r) = s^3 F_t (L_t^2 - (-1)^t) = s^2 F_r (L_t^2 - (-1)^t);$$

dividing throughout by F_r and using the result of (5), we obtain:

$$3 + 3L_r + L_r^2 - (-1)^r = (L_r + 2)^2 - s^2(-1)^t ,$$

or upon simplification:

(6) $$1 + (-1)^r + L_r = s^2(-1)^t .$$

We consider two mutually exclusive and exhaustive cases:

CASE I : r is even

From (5) and (6),

$$L_r + 2 = s^2(-1)^t = sL_t ;$$

hence, since $s > 0$,

$$s = (-1)^t L_t .$$

Since also t and $L_t > 0$, thus t is even, and $s = L_t$. Then by (2),

$$F_r = L_t F_t = F_{2t},$$

which implies $r = 2t$. We have shown that the triplet $(4m, L_{2m}, 2m)$ is a solution of the desired identity for $n = 0, 1, 2, 3$. It remains to verify this as a solution for all n. Substituting $r = 4m$ in the right member of (1), that expression becomes:

$$\frac{1}{\sqrt{5}} \{(1+\alpha^{4m})^n - (1+\beta^{4m})^n\} = \frac{1}{\sqrt{5}} \{\alpha^{2mn}(\alpha^{2m} + \beta^{2m})^n - (\alpha^{2m} + \beta^{2m})^n \beta^{2mn}\} = L_{2m}^n F_{2mn} ,$$

which is of the desired form, with $s = L_{2m}$, $t = 2m$. Hence,

(7) $$(r,s,t) = (4m, L_{2m}, 2m), \qquad m = 1, 2, 3, \cdots$$

is a sequence of solutions, the only ones yielded by this case.

CASE II : r is odd

From (6),

$$L_r = s^2(-1)^t .$$

Hence, t must be even and $L_r = s^2$. Substituting this result in (5), we obtain: $sL_t - s^2 = 2$, which implies $s|2$, and so $s = 1$ or 2.

SUBCASE A : $s = 1$

Thus, $L_r = 1^2 = 1$, and $r = 1$. Thus, by (2), $F_1 = 1 = F_t$. Since t must be even, thus $t = 2$. Hence, (1,1,2) is another possible solution. Since

$$\frac{1}{\sqrt{5}}\left\{(1+\alpha)^n - (1+\beta)^n\right\} = \frac{1}{\sqrt{5}}\left\{\alpha^{2n} - \beta^{2n}\right\} = F_{2n} = 1^n F_{2n},$$

thus (1,1,2) is a valid solution, the only one yielded by this subcase.

SUBCASE B : $s = 2$

Then $L_r = 2^2 = 4$, so $r = 3$. Thus, by (2), $F_3 = 2 = 2F_t$. As in Subcase A above, $t = 2$. This yields the possible solution (3,2,2). Now

$$(1+\alpha^3) = 2\alpha + 2 = 2\alpha^2;$$

similarly, $(1+\beta^3) = 2\beta^2$. Hence,

$$\frac{1}{\sqrt{5}}\left\{(1+\alpha^3)^n - (1+\beta^3)^n\right\} = \frac{2n}{\sqrt{5}}(\alpha^{2n} - \beta^{2n}) = 2^n F_{2n},$$

which shows that (3,2,2) is indeed a valid solution, the only one yielded by this subcase.

Therefore, all solutions (r,s,t) of the desired identity are given by (7), and also by (1,1,2) and (3,2,2).

Also solved by the Proposer.

Late Acknowledgements:

P. Bruckman solved H-258, H-259, H-262, H-263.

S. Singh solved H-263.

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[Continued from page 87.]

Proof. From Corollary 2 and [4, p. 205] we have $s(p^2) = s(p)$ if and only if $f(p^2) = f(p)$ if and only if

$$\phi_{(p-1)/2}(5/9) \equiv 2k(3/2) \pmod{p}.$$

From Wall's remark we note that $\phi_{(p-1)/2}(5/9) \not\equiv 2k(3/2) \pmod{p}$ for all primes p such that $5 < p < 10{,}000$.

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