THE FIBONACCI PSEUDOGROUP, CHARACTERISTIC POLYNOMIALS AND EIGENVALUES OF TRIDIAGONAL MATRICES, PERIODIC LINEAR RECURRENCE SYSTEMS AND APPLICATION TO QUANTUM MECHANICS

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INTRODUCTION

There are numerous applications of linear operators and matrices that give rise to tridiagonal matrices. Such applications occur naturally in mathematics, physics, and chemistry, e.g., eigenvalue problems, quantum optics, magnetohydrodynamics and quantum mechanics. It is convenient to have theoretical as well as computational access to the characteristic polynomials of tridiagonal matrices and, if at all possible, to their roots or eigenvalues. This paper produces explicitly the characteristic polynomials of general (finite) tridiagonal matrices: these polynomials are given in terms of the Fibonacci pseudogroup F_n (of order f_n, the nth Fibonacci number), a subset of the full symmetric group $\mathfrak{S}_n$. We then turn to some interesting special cases of tridiagonal matrices, those which have periodic properties: this leads directly to periodic linear recurrence systems which generalize the two-term Fibonacci type recurrence to collections of two-term recurrences defining a sequence. After some useful lemmas concerning generating functions for these systems, we return to explicitly calculate eigenvalues of periodic tridiagonal matrices. As an example of the power of the techniques, we have a theorem which gives the eigenvalues of a six-variable periodic tridiagonal matrix of odd degree explicitly as algebraic functions of these six variables, generalizing a result of Jacobi. We end with a brief discussion of how to explicitly calculate the characteristic polynomials of certain finite dimensional representations of a Hamiltonian operator of quantum mechanics.

SECTION A. THE FIBONACCI PSEUDOGROUP

We give a few essential definitions and observations about finite sets and permutations acting upon them which will be necessary in the sequel. We may think of this section as a theory of exterior powers of sets.

Let A be a finite set and let $|A|$ denote the number of distinct elements in A. Let 2^A denote the class of all subsets of A and define $\Lambda^k A$ to be the subclass of 2^A consisting of all subsets of A with exactly k distinct elements of A. Thus for $B \in 2^A$, $B \in \Lambda^k A$ iff $|B| = k$. Clearly,

$$|\Lambda^k A| = \binom{|A|}{k} \text{ (binomial coefficient) } \quad \text{and} \quad |2^A| = 2^{|A|}.$$

We have

$$2^A = \bigcup_{0 \le k \le |A|} \Lambda^k A \text{ (disjoint class union)}$$

which implies the usual relation

$$2^n = \sum_{0 \le k \le n} \binom{n}{k}.$$

Note that $\Lambda^0 A = \{\emptyset\}$ (empty class) and that $\Lambda^{|A|} A = A$.

Let S_n denote the full symmetric group of all permutations on n elements. Assume S_n acts by permuting the set of ciphers $N = \{1, 2, \ldots, n\}$. We will write the permutation as disjoint cycles; empty products will be the identity permutation. Consider the following subset $F_n \subseteq S_n$, defined by

$$F_n = \{(i_1, i_1 + 1) \ldots (i_k, i_k + 1) | 1 < i_1 + 1 < i_2, i_2 + 1 < i_3, \ldots, i_{k-1} + 1 < i_k < n\}.$$

F_n is a certain subset of disjoint two-cycle products in S_n. Observe that $(1) \varepsilon F_n$, (1) = identity of S_n. For $\sigma \varepsilon F_n$, $\sigma^2 = (1)$, thus every element of F_n is of order two and is its own inverse. Thus, if $\sigma \varepsilon F_n$, then $\sigma^{-1} \varepsilon F_n$. Suppose $\sigma, \rho \varepsilon F_n$. Then $\sigma\rho \varepsilon F_n$ iff σ and ρ are disjoint; all the two-cycle products of F_n are not disjoint. A pseudogroup is a subset of a group which contains the group identity, closed under taking inverses, but does not always have closure. In the present case $F_n = S_n$ iff $n = 0, 1, 2$. If $n < 2$, F_n is not a group, but F_n is a pseudogroup. We call F_n the Fibonacci pseudogroup because of the following lemma.

Lemma A1: Let f_n denote the nth Fibonacci number. Then

$$|F_n| = f_n, \; n \geq 0.$$

Proof: We may write

$$F_n = \bigcup_{0 \leq k \leq [n/2]} F_{k,n} \quad \text{(disjoint union)}$$

where $F_{k,n}$ consists of k disjoint two-cycles of F_n. But observe that

$$|F_{k,n}| = \binom{n-k}{k}$$

and the lemma follows. Note that $(-1)^k$ is the sign of the permutations in $F_{k,n}$. Then there are $\sum_{k \text{ odd}} \binom{n-k}{k}$ with negative sign and $\sum_{k \text{ even}} \binom{n-k}{k}$ with even sign: this gives an alternative proof with $|F_n| = |F_{n-1}| + |F_{n-2}|$, by observing that $|F_0| = 1$, $|F_1| = 1$.

Returning now to the finite set $N = \{1, 2, \ldots, n\}$ and the action of S_n on N, consider the convenient map

$$\text{Fix}: S_n \to 2^N$$

given for $\sigma \varepsilon S_n$ by $\text{Fix}\,\sigma = \{i \varepsilon N: \sigma(i) = i\}$, i.e., the set of elements of N fixed by σ. Thus, $\text{Fix}\,(1) = N$. We also define $\text{CoFix}\,\sigma = \{i \varepsilon N: \sigma(i) \neq i\}$ and note that $N = \text{Fix}\,\sigma \cup \text{CoFix}\,\sigma$ (disjoint union) for every $\sigma \varepsilon S_n$. If $n > 3$, then Fix can be onto.

Restricting Fix to F_n, the Fibonacci pseudogroup definition yields the handy facts that if $\sigma \varepsilon F_{k,n}$, then $|\text{Fix}\,\sigma| = n - 2k$ and $|\text{CoFix}\,\sigma| = 2k$.

It will be convenient to work with just half of the set $\text{CoFix}\,\sigma$; therefore, we define the subset of $\text{CoFix}\,\sigma$, (small c) $\text{coFix}\,\sigma = \{i \varepsilon N: \sigma(i) = i + 1\}$. Then $|\text{coFix}\,\sigma| = k$. Also, the number of elements of $\text{Fix}\,\sigma$, $\sigma \varepsilon F_{k,n}$ with $|\text{Fix}\,\sigma| = n - 2k$ is exactly $\binom{n-k}{n-2k} = \binom{n-k}{k}$. Again combining definitions, if $\sigma \varepsilon F_{k,n}$, then $|\Lambda^{\ell}\text{Fix}\,\sigma| = \binom{n-2k}{\ell}$.

SECTION B. APPLICATIONS OF THE FIBONACCI PSEUDOGROUP TO DETERMINANTS AND CHARACTERISTIC POLYNOMIALS OF TRIDIAGONAL MATRICES

We consider tridiagonal $n \times n$ matrices of the following form.

$$(1) \qquad A_n = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_2 & a_3 & b_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & c_3 & a_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-2} & b_{n-2} & \\ 0 & 0 & 0 & 0 & \cdots & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & & c_{n-1} & a_n \end{bmatrix}$$

We define vectors

$$a = (a_1, \ldots, a_n),\ b = (b_1, \ldots, b_{n-1}),\ c = (c_1, \ldots, c_{n-1})$$

and regard A_n as a function of these three vectors, $A_n = A_n(a, b, c)$ or as a function of $3n - 2$ variables. Let $\det A$ denote the determinant of A. We record some simple facts about the determinant and characteristic polynomial of A_n.

Lemma B1: Let A_n be the tridiagonal matrix defined above. Then,

a. $\det A_n = a_n \det A_{n-1} - b_{n-1}c_{n-1} \det A_{n-2}$.

b. $$\begin{aligned}\det(A_n(a, b, c) - \lambda I) &= (-1)^n \det(\lambda I - A_n(a, b, c)) \\ &= \det(A_n(a, -b, -c) - \lambda I) \\ &= (-1)^n \det(\lambda I - A\ (a, -b, -c)) \\ &= (-1)^n \det(\lambda + A\ (-a, b, c)).\end{aligned}$$

Our object is to give explicit information about $\det(A_n - \lambda I)$. We summarize this information using the notation of Section A in the result.

Theorem B1: The characteristic polynomial of a tridiagonal matrix can be written as the sum of a polynomial of codegree zero and a polynomial of codegree two as follows:

$$(2) \qquad \det(A_n(a, b, c) - \lambda I) = \prod_{1 \le k \le n} (a_k - \lambda) + P_n(\lambda; a, b, c)$$

where

$$\deg P_n(\lambda; a, b, c) = n - 2$$

and

$$P_n(\lambda; a, b, c)$$

$$(3) \qquad = (-1)^n \sum_{0 \le \mu \le n-2} \lambda^\mu \sum_{1 \le k \le [n/2]} (-1)^{n-\mu-k} \left(\sum_{\sigma \in F_{k,n}} \left(\prod_{j \in \mathrm{coFix}\,\sigma} b_j c_j \left(\sum_{A \in \Lambda^{n-4-2k}\mathrm{Fix}\,\sigma} \prod_{i \in A} a_i \right)\right)\right).$$

In particular,

$$(4) \qquad \det A_n = \sum_{\sigma \in F_n} \mathrm{sgn}(\sigma) \prod_{i \in \mathrm{Fix}\,\sigma} a_i \prod_{j \in \mathrm{coFix}\,\sigma} b_j c_j.$$

This theorem gives complete closed form information about the polynomial $P_n(\lambda)$. $P_n(\lambda)$ explicitly describes the perturbation of the characteristic polynomial of A from the characteristic polynomial of the diagonal of A. Further, consider the family of hyperbolas $x_k y_k = d_k$, $1 \leq k \leq n-1$ in $\mathbb{R}^{2n-2}$ space, $d_1, \ldots, d_{n-1}$ fixed constants. Then for fixed $a \in \mathbb{R}^n$, points on these hyperbolas parameterize a family of tridiagonal matrices $A_n(a, x, y)$ which all have exactly the same latent roots with the same multiplicities. The coefficients of the powers of λ in $P_n(\lambda)$ are elegantly expressed polynomials in the components of a, b, c and can be easily generated for computational purposes: the set F_n can be generated from $\{1, 2, \ldots, n\}$ in order $0 \leq k \leq [n/2]$, $F_{k,n}$; coFix is had immediately therefrom, and Λ^mFix can be generated from a combination subroutine.

To prove the theorem, we begin with

$$\det A_n = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a^1_{\sigma(1)} \cdots a^n_{\sigma(n)},$$

where $a^i_j = a_i, b_i, c_i, 0$ for $i = j$, $i + 1 = j$, $i - 1 = j$, otherwise, respectively, $1 \leq i, j \leq n$. However, $\det A_n$ is really a sum over $F_n \subseteq S_n$, has, in general, f_n terms, and $b_i c_i$ occurs whenever b_i occurs (Lemma B1). From the partition of F_n into k two-cycles, $0 \leq k \leq [n/2]$, we have

$$\det A_n = \sum_{0 \leq k \leq [n/2]} (-1)^k \sum_{\sigma \in F_{k,n}} a^1_{\sigma(1)} \cdots a^n_{\sigma(n)} \tag{6}$$

$$= \sum_{0 \leq k \leq [n/2]} (-1)^k \prod_{i \in \mathrm{Fix}\,\sigma} a_i \prod_{j \in \mathrm{coFix}\,\sigma} b_j c_j$$

because there are three cases, $j = \sigma(j)$, $j > \sigma(j)$, and $j < \sigma(j)$. If $a^j_{\sigma(j)} \neq 0$ then $|j - \sigma(j)| \leq 1$. In case of equality, $a^j_{\sigma(j)}\, a^{\sigma(j)}_j = b_j c_j$ occurs in the product. For $\sigma \in F_{k,n}$, σ moves $2k$ elements and fixes $n - 2k$ elements and is characterized by its fixed elements. The most σ can fix for $k > 0$ is $n - 2$, so that (replacing each a_k by $a_k - \lambda$) we have $\deg P_n(\lambda, a, b, c) = n - 2$. Setting $P_n(\lambda) = P_n(\lambda, a, b, c)$, we have

$$P_n(\lambda) = \sum_{1 \leq k \leq [n/2]} (-1)^k P_{k,n}(\lambda) \tag{7}$$

where $\deg P_{k,n}(\lambda) = n - 2k$ and

$$P_{k,n}(\lambda) = \sum_{\sigma \in F_{k,n}} \prod_{i \in \mathrm{Fix}\,\sigma} (a_i - \lambda) \prod_{j \in \mathrm{coFix}\,\sigma} b_j c_j \tag{8}$$

Let $M \subseteq N$, then

$$\prod_{i \in M} (a_i - \lambda) = \sum_{0 \leq \ell \leq |M|} (-1)^{|M| - \ell} \left(\sum_{A \in \Lambda^\ell M} \prod_{i \in A} a_i \right) \lambda^{|M| - \ell} \tag{9}$$

is simply the symmetric polynomials identity rewritten in the notation of exterior powers of sets. From this fact (9) and rearranging (8) for $M = \mathrm{Fix}\,\sigma$ we have

$$P_{k,n}(\lambda) = \sum_{\sigma \in F_{k,n}} \sum_{0 \leq \ell \leq n-2k} (-\lambda)^{n-2k-\ell} \prod_{j \in \mathrm{coFix}\,\sigma} b_j c_j \sum_{A \in \Lambda^\ell \mathrm{Fix}\,\sigma} \prod_{i \in A} a_i. \tag{10}$$

For comparison, we note that combining equations (9) and (2) gives a direct evaluation of the traces of exterior powers of A_n (in this context, exterior powers of A_n are the compound matrices of A_n). This is so from the identity

$$\det(A_n - \lambda I) = \sum_{0 \le k \le n-1} (-1)^{n-k}(\operatorname{tr}\Lambda^{n-k}A_n)\lambda^k + (-1)^n\lambda^n, \tag{11}$$

where A_n can be an arbitrary $n \times n$ matrix, tr is the trace of a matrix, $\Lambda^k A_n$ is the kth exterior power of A_n $\left(\text{an } \binom{n}{k} \times \binom{n}{k} \text{ matrix}\right)$. Thus, it is possible to also give $\operatorname{tr}\Lambda^k A_n(a, b, c)$ as an explicit polynomial in the components of a, b, c for $1 \le k \le n$.

We conclude this section with two examples. The first arose in a problem of positive definiteness of certain quadratic forms of interest in a plasma physics energy principle analysis.

a. Let $1 \le m \le n$ and choose $a_m = a/m$, $b_m c_m = b$. Then

$$n! \det A_n = \sum_{0 \le k \le [n/2]} (-1)^k B_{k,n}\, a^{n-2k} b^k \tag{12}$$

where the $B_{k,n}$ are certain integers

$$B_{k,n} = \sum_{\sigma \in F_{k,n}} \prod_{m \in \text{CoFix}\,\sigma} m, \tag{13}$$

(note the upper case C on CoFix here, $|\text{CoFix}\,\sigma| = 2k$). See Table 1 for a few of these integers.

b. Let $1 \le m \le n$ and choose $a_m = a$, $b_m c_m = b$. Then

$$\det A_n = \sum_{0 \le k \le [n/2]} (-1)^k C_{k,n}\, a^{n-2k} b^k \tag{14}$$

where the $C_{k,n}$ are certain integers

$$C_{k,n} = \sum_{\sigma \in F_{k,n}} \prod_{m \in \text{Fix}\,\sigma} m. \tag{15}$$

Table 1 also contains a few of these integers.

Table 1. The First Few CoFix; Fix Integers $B_{k,n}$; $C_{k,n}$ Defined by Equations (13); (15), Respectively; $0 \le k \le [n/2]$

$n \backslash k$	0	1	2	3	4
1	1; 1				
2	1; 2	2; 1			
3	1; 6	8; 4			
4	1; 24	20; 18	24; 1		
5	1; 120	40; 96	184; 9		
6	1; 720	70; 600	784; 72	720; 1	
7	1; 5040	112; 4320	2464; 600	8448; 16	
8	1; 40320	168; 36480	6384; 5400	42272; 196	40320; 1

SECTION C. PERIODIC LINEAR RECURRENCE SYSTEMS

It is now possible to use the results and notation of Sections A and B to draw conclusions about periodic linear recurrence systems. Of course, these generalize the usual linear recurrences; however, it is surprising that the Fibonacci pseudogroup is the key idea in their description. We first state a natural corollary to Theorem B1 without restriction of periodicity.

Theorem C1: Given a pair of arbitrary sequences a_1, a_2, a_3, ... and b_1, b_2, b_3, ..., then the one-parameter class of linear recurrences

$$f_n(t) = a_n f_{n-1} + t b_{n-1} f_{n-2} \tag{16}$$

with $f_0 = 1$, $f_i = a_i$, has the general solution $n > 1$

$$f_n(t) = \sum_{0 \le k \le [n/2]} t^k \sum_{\sigma \in F_{k,n}} \prod_{i \in \mathrm{Fix}\,\sigma} a_i \prod_{j \in \mathrm{coFix}\,\sigma} b_j. \tag{17}$$

For example, taking $t = 1$, $a_k = a$, $b_k = b$, $k \ge 1$, and recalling that for $\sigma \in F_{k,n}$, $|\mathrm{Fix}\,\sigma| = n - 2k$, $|\mathrm{coFix}\,\sigma| = k$, and $|F_{k,n}| = \binom{n-k}{k}$ yields

$$f_n = \sum_{0 \le k \le [n/2]} \binom{n-k}{k} a^{n-2k} b^k \tag{18}$$

the general solution of $f_n = a_{n-1} + bf_{n-2}$, $f_0 = 1$, $f_1 = a$. Taking $a = b = 1$ yields the well-known sum over binomial coefficients expression for the Fibonacci sequence. On the other hand, writing the generating function

$$G(t) = \sum_{n \ge 0} f_n t^n \tag{19}$$

and recognizing that $G(t)$ is a rational function of at most two poles, indeed $G(t) = 1/(1 - at - bt^2)$, yields the alternative solution

$$f_n = \frac{1}{\sqrt{a^2 + 4b}} \left\{ \left(\frac{a + \sqrt{a^2 + 4b}}{2} \right)^{n+1} - \left(\frac{a - \sqrt{a^2 + 4b}}{2} \right)^{n+1} \right\}. \tag{20}$$

Of course, from (18) we may regard $f_n = f_n(a, b)$ as a polynomial in a and b. In particular $f_n(a - \lambda, b)$ as a polynomial in λ can be written

$$f_n(a - \lambda, b) = \sum_{0 \le m \le n} (-1)^m \left(\sum_{0 \le k \le [n/2]} \binom{n-k}{k} \binom{n-2k}{m} a^{n-m-2k} b^k \right) \lambda^m. \tag{21}$$

We see now that the zeros, λ , $1 \le k \le n$, of polynomial (21) are precisely

$$\lambda_k = a + 2\sqrt{-b} \cos(\pi k/(n + 1)), \quad 1 \le k \le n. \tag{22}$$

This follows from equation (20), for $f_n = 0$ implies that

$$a + \sqrt{a^2 + 4b} = \left(a - \sqrt{a^2 + 4b}\right) e^{2\pi 2k/n+1}$$

so that

$$\sqrt{a^2 + 4b} = -\sqrt{-1}\, a^2 \tan \pi k/n + 1.$$

Squaring gives $a^2 \sec^2 (\pi k/n + 1) = -4b$. Replacing a by $a - \lambda$ gives equation (22). We have basically done the case of a period of length one.

We now take up the case of period two.

Lemma C1: Let $\{f_n\}$, $n \geq 0$, be a sequence defined by

$$f_n = a_n f_{n-1} + b_{n-1} f_{n-2},\ f_0 = 1,\ f_1 = a_1$$

and the sequences $\{a_n\}$, $\{b_n\}$, have period two, i.e.,

$$a_{2n} = a_2,\ a_{2n-1} = a_1,\ b_{2n-1} = b_1,\ b_{2n} = b_2,\ n \geq 1.$$

Then the generating function is rational with at most four poles:

$$G(t) = \sum_{n \geq 0} f_n t^n \tag{23}$$

$$= \frac{1 + a_1 t - b_2 t^2}{1 - (b_1 + b_2 + a_1 a_2) t^2 + b_1 b_2 t^4} \tag{24}$$

$$= \frac{A(\alpha,\ \beta)}{1 - \alpha t} + \frac{A(-\alpha,\ \beta)}{1 + \alpha t} + \frac{A(\beta,\ \alpha)}{1 - \beta t} + \frac{A(-\beta,\ \alpha)}{1 + \beta t} \tag{25}$$

where for $D = b_1 + b_2 + a_1 a_2$,

$$2\alpha^2 = D + \sqrt{D^2 - 4b_1 b_2},\quad 2\beta^2 = D - \sqrt{D^2 - 4b_1 b_2} \tag{26}$$

and

$$A(\alpha,\ \beta) = (\alpha^2 + a_1\alpha - b_2)/2(\alpha^2 - \beta^2). \tag{27}$$

Proof: Write $G(t)$ in terms of its even and odd parts (two functions). Then substitute the period two relations in to get the rationality of $G(t)$ from the pair of relations

$$\left(1 - \frac{a_2 - a_1}{2} t - \frac{b_1 + b_2}{2} t^2\right) G(t) + \left(\frac{a_2 - a_1}{2} t + \frac{b_2 - b_1}{2} t^2\right) G(-t) = 1 \tag{28}$$

$$\left(-\frac{a_2 - a_1}{2} t + \frac{b_2 - b_1}{2} t^2\right) G(t) + \left(1 + \frac{a_2 + a_1}{2} t - \frac{b_2 + b_1}{2} t^2\right) G(-t) = 1 \tag{29}$$

where the determinant of this system is the denominator of the right-hand side of equation (24).

Of course, comparing coefficients will give an expression for f_n as a linear combination of powers of poles of $G(t)$ analogous to equation (20). On the other hand, there are polynomial expressions in the four variables a_1, a_2, b_1, b_2 of the type (18) which follow directly from Theorem B.

We give only one example of the former.

Let $f_{2n} = f_{2n-1} + f_{2n-2}$, $f_{2n+1} = f_{2n} + 2f_{2n-1}$, $f_0 = 1$, $f_1 = 1$, so that f_n is the sequence 1, 1, 2, 4, 6, 14, 20, 48, 68, 166, 234, ... Then, we have

$$f_{2n} = \frac{1}{2}\left((2 + \sqrt{2})^n + (2 - \sqrt{2})^n\right), \tag{30}$$

$$f_{2n+1} = \frac{1}{2\sqrt{2}}\left((2 + \sqrt{2})^{n+1} - (2 - \sqrt{2})^{n+1}\right). \tag{31}$$

Alternatively (30) and (31) can be shown by induction to satisfy the linear recurrence of period two.

We now consider the general case of rationality of generating functions of arbitrary periodic systems of linear recurrences.

Lemma C2: Let $f_n = a_n f_{n-1} + b_{n-1} f_{n-2}$ be given with $f_0 = 1$, $f_1 = a_1$. Suppose that $a_n = a_\ell$ and $b_n = b_\ell$ if $n \equiv \ell \pmod{k}$ and that

$$a_\ell,\ 1 \le \ell \le k, \quad b_\ell,\ 0 \le \ell \le k-1$$

are given as the first elements of the sequences $\{a_n\}$ and $\{b_n\}$ which are not in two k-periods. Call the system a period k system. Set

$$G(t) = \sum_{n \ge 0} f_n t^n \tag{32}$$

then $G(t)$ is a rational function of t where

$$G(t) = P(t)/Q(t) \tag{33}$$

and $P(t)$, $Q(t)$ are polynomials in t, deg $P(t) \le 2k - 1$, deg $Q(t) \le 2k$.

Proof: First write

$$G(t) = \sum_{1 \le \ell \le k} G_\ell(t) \tag{34}$$

where

$$G_\ell(t) = \sum_{n \equiv \ell \ (\mathrm{mod}\ k)} f_n t^n \tag{35}$$

and where the sum is over integers $n \ge 0$, n congruent to ℓ modulo k. From the relations

$$f_n = a_\ell f_{n-1} + b_{\ell-1} f_{n-2} \quad \text{if} \quad n \equiv \ell \pmod{k}, \tag{36}$$

we have that

$$G_\ell(t) = a_\ell t G_{\ell-1}(t) + b_{\ell-1} t^2 G_{\ell-2}(t). \tag{37}$$

Using the modulo k relations we can write the following equations

$$G_1(t) = a_1 t G_0(t) + b_0 t^2 G_{-1}(t) = a_1 t + a_1 t G_k(t) + b_0 t\ G_{k-1}(t), \tag{38}$$

$$G_2(t) = a_2 t G_1(t) + b_1 t^2 G^0(t) = a_2 t G_1(t) + b_1 t^2 + b_1 t^2 G_k(t), \tag{39}$$

$$G_3(t) = a_3 t G_2(t) + b_2 t^2 G_1(t) \tag{40}$$

$$\vdots$$

$$G_k(t) = a_k t G_{k-1}(t) + b_{k-1} t^2 G_{k-2}(t) \tag{41}$$

This gives the system of equations in matrix form as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & -b_0 t^2 & -a_1 t \\ -a_2 t & 1 & 0 & 0 & 0 & \cdots & 0 & -b_1 t^2 \\ -b_2 t^2 & -a_3 t & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -b_3 t^2 & -a_4 t & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -b_4 t^2 & -a_5 t & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -b_{k-2} t^2 & -a_{k-1} t & & 1 & 0 \\ 0 & 0 & \cdots & 0 & -b_{k-1} t^2 & & -a_k t & 1 \end{bmatrix} \begin{bmatrix} G_1(t) \\ G_2(t) \\ G_3(t) \\ G_4(t) \\ G_5(t) \\ \vdots \\ G_{k-1}(t) \\ G_k(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{42}$$

We rewrite equation (42) as

$$HG = J, \tag{42}$$

with the obvious interpretation. Now H is invertible (in the indeterminant t) and we can solve for $G_1(t)$, ..., $G_k(t)$ separately as rational functions, their sum is $G(t)$. But, clearly, deg det $H(t) = 2k$, so that the denominator of $G(t)$ must divide this, i.e., $\deg Q(t) \leq 2k$. Also, the adjoint of H is given by polynomials of degree $\leq 2k - 1$, thus, $\deg P(t) \leq 2k - 1$.

This rationality result is the starting point to produce further facts of which Lemma B1 and equation (20) are examples. The central difficulty lies in analyzing the denominator of the rational function to display sums of powers of its roots. We will apply the technique to tridiagonal matrices of periodic type in the next section.

SECTION D. APPLICATIONS OF PERIODIC RECURRENCES TO TRIDIAGONAL MATRICES

We return to tridiagonal matrices to apply the results of Section C first to recover a result of Jacobi and second to give a generalization of Jacobi's theorem.

Theorem D1 (Jacobi): The latent roots of the tridiagonal $n \times n$ matrix

$$\begin{bmatrix} a & b & 0 & 0 & 0 & \cdots & 0 & 0 \\ c & a & b & 0 & 0 & \cdots & 0 & 0 \\ 0 & c & a & b & 0 & \cdots & 0 & 0 \\ 0 & 0 & c & a & b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & 0 & 0 & \cdots & c & a \end{bmatrix} \tag{43}$$

are given for $1 \leq k \leq n$ by

$$\lambda_k = a - 2\sqrt{bc}\cos\frac{\pi k}{n+1}.$$

Proof: This follows directly from Lemma B1 and equation (22), by recognizing that the matrix (43) defines a (period one) linear recurrence system.

Theorem D2: The latent roots of the $(2n+1) \times (2n+1)$ tridiagonal matrix

$$\begin{bmatrix} a & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ d & c & e & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & f & a & b & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & d & c & e & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & f & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & d & c & e \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & f & a \end{bmatrix} \tag{44}$$

lie among the values ($1 \leq k \leq n + 1$ with the plus sign, $1 \leq k \leq n$ with the minus sign):

$$(45)\qquad \lambda = \frac{a+c}{2} \pm \sqrt{\left(\frac{a-c}{2}\right)^2 + bd + ef + 2\sqrt{bdef}\cos\frac{\pi k}{n+1}}\,.$$

Proof: Note that when $a = c$, $b = e$, and $d = f$ this reduces to the case of the period one theorem. By Lemma B1, we recognize (44) as defining a period *two* linear recurrence system. Take therefore the odd case in Lemma C1, thus $(-1)^{2n-1} = -1$ and

$$(46)\qquad \frac{A(\alpha,\ \beta) - A(-\alpha,\ \beta)}{A(\beta,\ \alpha) - A(-\beta,\ \alpha)} = \frac{\alpha}{\beta}\,.$$

Then f_n is zero iff $(\alpha/\beta)^{2n+2} = e^{2\pi ik}$, $0 \le k \le n+1$. Reasoning as with equation (22) yields

$$(47)\qquad bd + ef + ac = 2\sqrt{bdef}\cos\frac{\pi k}{n+1}\,.$$

Replacing ac by $(a-\lambda)(c-\lambda)$ and solving for λ gives (45). Thus we have all latent roots of a five-parameter family of matrices.

Again, to apply similar techniques to families of matrices with more parameters involves analyzing the denominator in Lemma C2. We point out that for large periodic matrices of special type (particular sparse matrices) the root analysis is relatively easy to do numerically, say, for periods small relative to the size of the matrix.

SECTION E. THE APPLICATION TO A HAMILTONIAN OPERATOR OF QUANTUM MECHANICS

The differential equation of the quantum mechanical asymmetric rotor may be written as $(D - E\ )\Psi\ = 0$. (Schroedinger equation) where the matrix corresponding to the inertia tensor is

$$(48)\qquad \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

Define the variables α, β, δ by the equation

$$(49)\qquad \begin{bmatrix} A \\ B \\ C \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \delta \end{bmatrix}$$

so that $\beta = C - \ \ (A + B)$, and the differential equation becomes (single variable representation)

$$(50)\qquad P(x)\frac{d^2}{dz^2} + A(z)\frac{d}{dz} + R(z) = 0$$

where

$$(51)\qquad \begin{aligned} P(z) &= \alpha z^6 + \beta z^4 + \alpha z^2, \\ Q(z) &= 2\alpha(j+2)z^5 + \beta z^3 = 2(j+1)z, \\ R(z) &= \ \ (j+1)(j+2)z^4 - E\ z^2 + \alpha(j+1)(j+2). \end{aligned}$$

After choosing a convenient z-basis of eigenfunctions, getting the corresponding difference equation with respect to that basis we have a tridiagonal matrix appear. This tridiagonal matrix, however, is tridiagonal with the main diagonal and second upper and lower diagonals, but it is possible to reduce it to direct sums of the usual tridiagonals that we have already treated in Section B. We are not concerned here with giving the representation theory, and so we will sketch briefly the facts we need.

The difference equation alluded to above becomes

$$P_{j,m}A_{m+2} + (Q_{j,m} - E)A_m - R_{j,m}A_{m-2} = 0, \tag{52}$$

where

$$\begin{aligned} P_{j,m} &= (j-m)(j-m-1),\\ Q_{j,m} &= \beta m^2,\\ R_{j,m} &= (j+m)(j+m-1). \end{aligned}$$

We have here for convenience replaced $\frac{\beta}{\alpha}$ by β, $\frac{E}{\alpha}$ by E; note that $P_{j,m} = R_{j,m}$, where m varies through $-j \le m \le j$, j may be a half integer. We choose the variable $n = 2j + 1$, so that $j = \frac{n-1}{2}$ and the matrix of interest is the $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} \beta\,\dfrac{n-2i+1}{2} & i = j,\\ (n-i)(n-i-1) & j = i+2,\\ (i-1)(i-2) & i = j+2,\\ 0 & \text{otherwise} \end{cases} \tag{53}$$

This is a nonstandard tridiagonal matrix with off diagonal integer entries. Generalizing this situation slightly, we define

$$A = \begin{vmatrix} a_1 & 0 & b_{n-2} & 0 & 0 & 0 & 0 & 0\\ 0 & a_2 & 0 & b_{n-3} & \ddots & & & \\ b_1 & 0 & a_3 & 0 & \ddots & & & \\ 0 & b_2 & 0 & a_4 & \ddots & & & \ddots\\ 0 & \ddots & & \ddots & a_4 & 0 & b_2 & 0\\ 0 & & & & 0 & a_3 & 0 & b_1\\ 0 & & & & b_{n-3} & 0 & a_2 & 0\\ 0 & & & & 0 & b_{n-2} & 0 & a_1 \end{vmatrix} \tag{54}$$

We see immediately that the directed graph of this matrix has two components each of which is the directed graph of a standard tridiagonal matrix. This observation will give the first direct sum splitting: we shall see that each of these splits for sufficiently large n.

Lemma E1: The $n \times n$ matrix A is similar to a direct sum of four tridiagonal matrices if n is not trivially small. Alternatively, the characteristic polynomial of the $n \times n$ matrix A factors into four polynomials whose degrees differ by no more than one.

Proof: It is sufficient to exhibit the similarity transformations that convert the generalized supertridiagonal matrix A into similar standard tridiagonal matrices. For the first stage define the permutation σ,

$$\sigma(k) = \begin{cases} 2k - 1 & \text{if} \quad k \le \dfrac{n+1}{2}\\[2ex] 2k - \left[\dfrac{n}{2}\right] & \text{if} \quad k > \dfrac{n+1}{2} \end{cases} \tag{55}$$

where $1 \leq k \leq n$ and $[x]$ denotes the greatest integer in x function. Associated with σ is an $n \times n$ permutation matrix S_σ. Then, $S_\sigma A S_\sigma^{-1}$ will be a standard tridiagonal matrix, i.e., zero entries everywhere except the main diagonal, first above and first below diagonals. Further, setting $B = S_\sigma A S_\sigma^{-1}$, B will be, in general, $(n \geq 3)$, a direct sum of two tridiagonals:

$$k \times k \quad \text{and} \quad (n - k) \times (n - k) \quad \text{where} \quad m = \left[\frac{n+1}{2}\right].$$

But these tridiagonals are of a special kind, in fact, of the form

$$B' = \begin{bmatrix} \cdots & & & & \\ & a_{m-1} & b_{m+1} & 0 & 0 \\ & b_{m-1} & a_m & b_m & 0 \\ & 0 & b_m & a_m & b_{m-1} \\ & 0 & 0 & b_{m+1} & a_{m-1} \\ & & & & & \cdots \end{bmatrix} \tag{56}$$

for the even case and

$$B'' = \begin{bmatrix} \cdots & & & \\ & a_{m-1} & b_{m+2} & 0 \\ & b_{m-1} & a_m & b_{m+1} \\ & 0 & b_m & a_{m+1} \\ & & & & \cdots \end{bmatrix} \tag{57}$$

for the odd case. Because of the special up and down features, we can split these matrices by means of the similarity matrices:

$$P' = \left[\begin{array}{c|c} I & J \\ \hline -J & I \end{array}\right] \text{ for } n \text{ even; } P'' = \left[\begin{array}{c|c|c} I & 0 & J \\ \hline 0 & 1 & 0 \\ \hline -J & 0 & I \end{array}\right] \text{ for } n \text{ odd;} \tag{58}$$

where I is the identity matrix of appropriate size and J is zero everywhere except for ones on the main cross diagonal. Thus, PBP^{-1} (with appropriate primes on the P and B) is a direct sum of two matrices and of the form

$$\begin{bmatrix} \cdots & & & & \\ & a_{m-1} & b_{m+1} & & \\ & b_{m-1} & a_m - b_m & & \\ & & & a_m + b_m & b_{m-1} \\ & & & b_{m+1} & a_{m-1} \\ & & & & & \cdots \end{bmatrix} \text{ for } n \text{ even, and} \tag{59}$$

$$\begin{bmatrix} \cdots & & & & \\ & a_{m-1} & b_{m+2} & & \\ & b_{m-1} & a_m & & \\ & & & a_{m+1} & 2b_m \\ & & & b_{m+1} & a_m \\ & & & & & \cdots \end{bmatrix} \text{ when } n \text{ is odd.} \tag{60}$$

We can now apply the lemmas of Section B to write down explicitly the characteristic polynomials of these quantum mechanical Hamiltonian operators; from such explicit forms one expects to elicit information about energy levels and spectra, viz., the eigenvalues are roots of these polynomials.

REFERENCES

1. D. H. Lehmer, "Fibonacci and Related Sequences in Periodic Tridiagonal Matrices," *The Fibonacci Quarterly*, Vol. 13, No. 2 (1975), pp. 150-157.
2. R. T. Pack, "Single Variable Realization of the Asymmetric Rotor Problem" (unpublished manuscript, February 1975).
3. M. O. Scully, H. Shih, P. V. Arizonis, & W. H. Loisell, "Multimode Approach to the Physics of Unstable Laser Resonators," *Appl. Optics*, 1974.
4. R. S. Varga, *Matrix Iterative Analysis* (New York: Prentice-Hall, 1962).
5. F. M. Arscott, "Latent Roots of Tri-Diagonal Matrices," *Proc. Edinburgh Math. Soc.*, Vol. 12 (1961), pp. 5-7.

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VECTORS WHOSE ELEMENTS BELONG TO A GENERALIZED FIBONACCI SEQUENCE

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1. INTRODUCTION

In a recent paper, D. V. Jaiswal [1] considered some geometrical properties associated with Generalized Fibonacci Sequences. In this paper, we shall extend some of his concepts to n dimensions and generalize his Theorems 2 and 3. We do this by considering column vectors with components that are elements of a G(eneralized) F(ibonacci) S(equence) whose indices differ by fixed integers. We prove two theorems: first, the "area" of the "parallelogram" determined by any two such column vectors is a function of the differences of the indices of successive components; second, any column vectors of the same type form a matrix of rank 2.

2. PRELIMINARY RESULTS

We shall be considering submatrices of an $N \times N$ matrix $T = [T_{i+j-1}]$ where T_s is an element of a GFS with $T_1 = a$ and $T_2 = b$. For the special case $a = b = 1$, we denote the sequence as F_s. We shall indicate the kth column vector of the matrix T as $T_{0k} = [T_{i+k-1}]$. In particular, the first two column vectors of T are $T_{01} = [T_i]$ and $T_{02} = [T_{i+1}]$. We shall now prove a basic property of the matrix T.

Lemma 2.1: The matrix $T = [T_{i+j-1}]$ is of rank 2.

From the fundamental identity for GFS,

$$T_{r+s} = F_{r+1}T_s + F_r T_{s-1},$$

it follows that

$$T_{0k} = F_{k-1}T_{02} + F_{k-2}T_{01}.$$