Since $v_k(x)$ satisfies

$$\begin{cases} v_n(x) = v_{n-1}(x) + v_{n-2}(x), \ n \text{ even}, \\ v_n(x) = xv_{n-1}(x) + v_{n-2}(x), \ n \text{ odd}, \end{cases}$$

this is $\{S_k(a, b, c, d)\}$ with $a = b = d = 1$ and $c = x$. Then, by Theorem 4.2,

$$(v_n(x), v_m(x)) = v_{(m, n)}(x).$$

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THE GOLDEN SECTION IN THE EARLIEST NOTATED WESTERN MUSIC
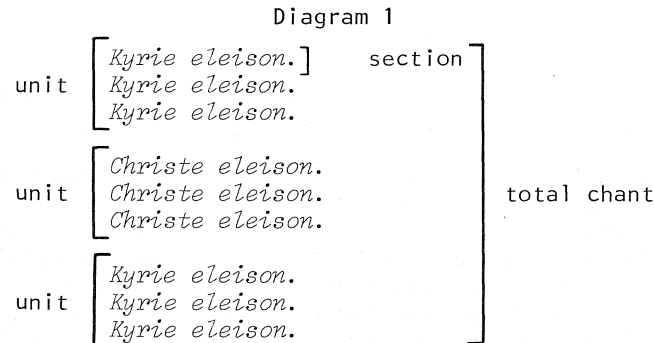
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The persistent use of the golden section as a proportion in Western Art is well recognized. Architecture, the visual arts, sculpture, drama, and poetry provide examples of its use from ancient Greece to the present day. No similar persistence has been established in music. One possible reason is that what ancient Greek music has survived is of such a fragmentary nature that it is not possible to make reliable musical deductions from it. However, beginning with the early Middle Ages a large body of music has survived in manuscripts that from ca. 10th century can be read and the music can be performed. This body of music is known as Roman liturgical chant or, more commonly, as Gregorian chant. These chants have not previously been analyzed from the standpoint of the golden section. Acknowledging the probability of the pres-

ence of a number of structural designs and proportions in these chants, it is the author's intention to establish the musical use of the golden section as an organizing principle in them.

The official collection of Roman liturgical chant is the *Liber Usualis*. The chants selected for the present study are "Kyrie" chants of which there are 30 in the collection. The chants span at least 600 years, having been written beginning with the 10th century.

The basic structure of a "Kyrie" is determined by the text, as shown in Diagram 1:

Diagram 1

$$
\text{unit} \begin{bmatrix} \textit{Kyrie eleison.}] & \text{section} \\ \textit{Kyrie eleison.} \\ \textit{Kyrie eleison.} \end{bmatrix}
$$

$$
\text{unit} \begin{bmatrix} \textit{Christe eleison.} \\ \textit{Christe eleison.} \\ \textit{Christe eleison.} \end{bmatrix} \quad \text{total chant}
$$

$$
\text{unit} \begin{bmatrix} \textit{Kyrie eleison.} \\ \textit{Kyrie eleison.} \\ \textit{Kyrie eleison.} \end{bmatrix}
$$

Each chant falls into nine separate sections. The three repetitions of the sections form three larger units which, in turn, make up the complete chant. While there is considerable variety in the melodic treatment of the text, the text itself had remained constant in the above form since ca. 900.

The actual nature of the rhythm of these chants is still open to question. Because music is a time art, any analysis that does not account for the proportional movement of the pitches in time cannot pretend to be a statement about the total nature of the music. In this sense, the following findings, though factual, remain theoretical to the degree that while pitches in succession imply time, exact temporal proportions are not deducible from that succession alone. In addition, the reader should be advised that there are more than 200 "Kyrie" melodies known to exist. In this light, the chants analyzed for this study represent a sampling of the repertory.

METHOD OF ANALYSIS

Because different treatments of the same text are usually set to different pitches, 146 distinct musical sections are present in the 30 chants, the remaining being exact repetitions of other sections. The pitches in each section were totaled, and ϕ was determined for each section. A section was examined to determine if any significant musical event occurred at either the major or minor mean. A significant event was defined as the beginning or ending of a musical phrase. The three statements of the "Kyrie," the "Christe," and the "Kyrie" tend to form larger units; these were analyzed according to the same procedure. Finally, the pitches in the complete chant were totaled, i.e., nine separate sections of text.

THE FINDINGS

Applying the analytical method described above revealed the presence of the golden section in 105 of the 146 individual sections of the "Kyries" in the *Liber Usualis*. These 105 sections make up .72 of the cases. The major

mean precedes the minor mean twice as often as the minor mean precedes the major mean.   Example 1 is a section of chant conforming to the M:m proportion.

Example 1*

Example 2 shows the proportion in reverse.

Example 2*

Twenty-one sections have phrase divisions occurring at the arithmetic mean.

The same method was applied to the next larger formal unit, i.e., the three repetitions of each exclamation. In 30 chants there are 90 such units. ϕ is found in 53 (.59) of these units. Where the musical phrase either falls short of the exact mean or extends beyond it, a tolerance of .02 of the total number of pitches was maintained in defining the unit as a golden section.

A performance of an entire chant includes nine sections as shown in Diagram 1. An analysis of the 30 chants revealed that 20 (.66) exhibit the golden section proportion. In more than half of the cases, the mean occurs at the end of the first or at the beginning of the second "Christe eleison."

CONCLUSION

At this stage, these findings tend to establish the presence of the golden section in one of the earliest notated forms of Western music, i.e., the "Kyrie" chants. To establish the presence of the golden section in chants other than the "Kyrie," requires further analysis of the general body of Gregorian chant.

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ON FIBONACCI NUMBERS WHICH ARE POWERS

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INTRODUCTION

Let $F(n)$, $L(n)$ denote the nth Fibonacci and Lucas numbers, respectively. (This slightly unconventional notation is used to avoid the need for second-order subscripts.) Consider the equation

(0) $F(m) = c^p,$

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*Source: *Liber Usualis* (Desclee & Co., Tournai [Belb.], 1953), p. 25.