PYTHAGOREAN TRIPLES CONTAINING FIBONACCI NUMBERS: SOLUTIONS FOR $F_n^2 \pm F_k^2 = K^2$

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1. INTRODUCTION

When can Fibonacci numbers appear as members of a Pythagorean triple? It has been proved by Hoggatt [1] that three distinct Fibonacci numbers cannot be the lengths of the sides of any triangle. L. Carlitz [8] has shown that neither three Fibonacci numbers nor three Lucas numbers can occur in a Pythagorean triple. Obviously, one Fibonacci number could appear as a member of a Pythagorean triple, because any integer could so appear, but $F_{3(2m+1)}$ cannot occur in a primitive triple, since it contains a single factor of 2. However, it appears that two Fibonacci lengths can occur in a Pythagorean triple only in the two cases 3 - 4 - 5 and 5 - 12 - 13, two Pell numbers only in 5 - 12 - 13, and two Lucas numbers only in 3 - 4 - 5. Further, it is strongly suspected that two members of any other sequence formed by evaluating the Fibonacci polynomials do not appear in a Pythagorean triple.

Here, we define the Fibonacci polynomials $\{F_n(x)\}$ by

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \tag{1.1}$$

and the Lucas polynomials $\{L_n(x)\}$ by

$$L_n(x) = F_{n+1}(x) + F_{n-1}(x) \tag{1.2}$$

and form the sequences $\{F_n(a)\}$ by evaluating $\{F_n(x)\}$ at $x = a$. The Fibonacci numbers are $F_n = F_n(1)$, the Lucas numbers $L_n = L_n(1)$, and the Pell numbers $P_n = F_n(2)$.

While it would appear that $F_n(a)$ and $F_k(a)$ cannot appear in the same Pythagorean triple (except for 3 - 4 - 5 and 5 - 12 - 13), we will restrict our proofs to primitive triples, using the well-known formulas for the legs a and b and hypotenuse c,

$$a = 2mn, \quad b = m^2 - n^2, \quad c = m^2 + n^2, \tag{1.3}$$

where $(m,n) = 1$, m and n not both odd, $m > n$. We next list Pythagorean triples containing Fibonacci, Lucas, and Pell numbers. The preparation of the tables was elementary; simply set $F_k = a$, $F_k = b$, $F_k = c$ for successive values of k and evaluate all possible solutions.

Table 1

PYTHAGOREAN TRIPLES CONTAINING F_k, $1 \le k \le 18$

m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$	
2	1	4	$3 = F_4$	$5 = F_5$	
3	2	12	$5 = F_5$	$13 = F_7$	
3	1	6	$8 = F_6$	10	(not primitive)
4	1	$8 = F_6$	15	17	
7	6	84	$13 = F_7$	85	
5	2	20	$21 = F_8$	29	
11	10	220	$21 = F_8$	221	
5	3	30	16	$34 = F_9$	(not primitive)
17	1	$34 = F_9$	288	290	(not primitive)
8	3	48	$55 = F_{10}$	73	
28	27	1512	$55 = F_{10}$	1513	
8	5	80	39	$89 = F_{11}$	
45	44	3960	$F_{11} = 89$	3961	
37	35	2590	$144 = F_{12}$	2594	(not primitive)
20	16	640	$144 = F_{12}$	656	(not primitive)
15	9	270	$144 = F_{12}$	306	(not primitive)
13	5	130	$144 = F_{12}$	194	(not primitive)
9	8	$144 = F_{12}$	17	145	
72	1	$144 = F_{12}$	5183	5185	
36	2	$144 = F_{12}$	1292	1300	(not primitive)
24	3	F_{12}	567	585	(not primitive)
18	4	F_{12}	308	340	(not primitive)
12	6	F_{12}	108	180	(not primitive)
13	8	208	105	$233 = F_{13}$	
117	116	27144	$233 = F_{13}$	27145	
16	11	352	135	$377 = F_{14}$	
19	4	152	345	$377 = F_{14}$	
189	188	71064	$377 = F_{14}$	71065	
21	8	336	$377 = F_{14}$	505	
21	13	546	272	$610 = F_{15}$	(not primitive)
23	9	414	448	$610 = F_{15}$	(not primitive)
305	1	$610 = F_{15}$	93024	93026	(not primitive)
61	5	$610 = F_{15}$	3696	3746	(not primitive)
494	493	487084	$987 = F_{16}$	487085	
166	163	54116	$987 = F_{16}$	54125	
34	13	884	$987 = F_{16}$	1325	
74	67	9916	$987 = F_{16}$	9965	
34	21	1428	715	$1597 = F_{17}$	
799	798	1275204	$1597 = F_{17}$	1275205	
647	645	834630	$2584 = F_{18}$	834634	(not primitive)
325	321	208650	$2584 = F_{18}$	208666	(not primitive)
53	15	1590	$2584 = F_{18}$	3034	(not primitive)
55	21	2310	$2584 = F_{18}$	3466	(not primitive)
1292	1	$2584 = F_{18}$	1669263	1669265	
646	2	$2584 = F_{18}$	417312	417320	(not primitive)
323	4	$2584 = F_{18}$	104313	104345	

Table 1 (continued)

m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$	
76	17	$2584 = F_{18}$	5487	6065	
68	19	$2584 = F_{18}$	4263	4985	
38	34	$2584 = F_{18}$	288	2600	(not primitive)
F_{n+1}	F_n	$2F_nF_{n+1}$	$F_{n-1}F_{n+2}$	F_{2n+1}	
		$2F_k$	$F_k^2 - 1$	$F_k^2 + 1$	
		F_{6m}	$(F_{6m}^2 - 4)/4$	$(F_{6m}^2 + 4)/4$	
		$(F_{3m\pm1}^2 - 1)/2$	$F_{3m\pm1}$	$(F_{3m\pm1}^2 + 1)/2$	
F_{k+1}	F_{k-1}	$2F_{k+1}F_{k-1}$	F_{2k}	$F_k^2 + 2F_{k-1}F_{k+1}$	

Table 2

PYTHAGOREAN TRIPLES CONTAINING L_k, $1 \le k \le 18$

m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$	
2	1	$4 = L_3$	$3 = L_2$	5	
4	3	24	$7 = L_4$	25	
6	5	60	$11 = L_5$	61	
9	1	$18 = L_6$	80	82	(not primitive)
5	2	20	21	$29 = L_7$	
15	14	420	$29 = L_7$	421	
24	23	1104	$47 = L_8$	1105	
20	18	720	$76 = L_9$	724	(not primitive)
19	2	$76 = L_9$	357	365	
38	1	$76 = L_9$	1443	1445	
62	61	7564	$123 = L_{10}$	7565	
22	19	836	$123 = L_{10}$	845	
100	99	19800	$199 = L_{11}$	19801	
23	7	$322 = L_{12}$	480	578	(not primitive)
161	1	$322 = L_{12}$	25920	25922	(not primitive)
20	11	440	279	$521 = L_{13}$	
261	260	135720	$521 = L_{13}$	135721	
422	421	355324	$843 = L_{14}$	355325	
142	139	39476	$843 = L_{14}$	39485	
42	20	1680	$1364 = L_{15}$	2164	(not primitive)
342	340	232560	$1364 = L_{15}$	232564	(not primitive)
682	1	$1364 = L_{15}$	465123	465125	
341	2	$1364 = L_{15}$	116277	116285	
62	11	$1364 = L_{15}$	3723	3985	
31	22	$1364 = L_{15}$	471	1445	
1104	1103	2435424	$2207 = L_{16}$	2435425	
1786	1785	637020	$3571 = L_{17}$	6376021	
2889	1	$5778 = L_{18}$	8346320	8346322	(not primitive)

Table 2 (continued)

m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$	
963	3	$5778 = L_{18}$	927360	927378	(not primitive)
321	9	$5778 = L_{18}$	102960	103122	(not primitive)
107	27	$5778 = L_{18}$	10720	12178	(not primitive)

Table 3

PYTHAGOREAN TRIPLES CONTAINING PELL NUMBERS P_k, $1 \leq k \leq 8$

m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$	
2	1	4	3	$5 = P_3$	
3	2	$12 = P_4$	$5 = P_3$	13	
6	1	$12 = P_4$	35	37	
5	2	20	21	$29 = P_5$	
15	14	420	$29 = P_5$	421	
35	1	$70 = P_6$	1224	1226	(not primitive)
7	5	$70 = P_6$	24	74	(not primitive)
12	5	120	119	$169 = P_7$	
85	84	14280	$169 = P_7$	14281	
103	101	20806	$408 = P_8$	20810	(not primitive)
53	49	5194	$408 = P_8$	5210	(not primitive)
204	1	$408 = P_8$	41615	41617	
102	2	$408 = P_8$	10400	10408	(not primitive)
51	4	$408 = P_8$	2585	2617	
68	3	$408 = P_8$	4615	4633	
34	6	$408 = P_8$	1120	1192	(not primitive)
17	12	$408 = P_8$	145	433	
P_{n+1}	P_n	$2P_nP_{n+1}$	$P_{n-1}P_{n+2}$	P_{2n+1}	

We note that in 3 - 4 - 5 and 5 - 12 - 13, the hypotenuse is a prime Fibonacci number, and one leg and the hypotenuse are Fibonacci lengths. These are the only solutions with two Fibonacci lengths where a prime Fibonacci number gives the length of the hypotenuse. If F_p is prime, then p is odd, because $F_w | F_{2w}$. If F_p is a prime of the form $4k - 1$, then there are no solutions to $m^2 + n^2 = F_p$, and if F_p is a prime of the form $4k + 1$, then $m^2 + n^2$ has exactly one solution: $m = F_{k+1}$, $n = F_k$, or, the triple

$$a = 2F_kF_{k+1}, \quad b = F_{k-1}F_{k+2}, \quad c = F_{2k+1} \quad \text{(see [2])}.$$

In either case, F_{2k+1} does not appear as the hypotenuse in a triple containing two Fibonacci numbers if F_{2k+1} is prime. These remarks also hold for the generalized Fibonacci numbers $\{F_n(a)\}$.

Also note that some triples contain numbers from more than one sequence. We have, in 3 - 4 - 5, F_4-L_3-F_5, or L_2-L_3-F_5, or F_4-L_3-P_3, while 5 - 12 - 13 has F_5-P_4-F_7, or P_3-P_4-F_7, and 20 - 21 - 29 has F_8 and L_7 or F_8 and P_5. There also are a few "near misses," which are close enough to being Pythagorean triples to fool the eye if a triangle were constructed: 55 - 70 - 89, 21 - 34 - 40, and 8 - 33 - 34. However, 3 - 4 - 5 and 5 - 12 - 13 seem to be the only Pythagorean triples which contain two members from the same sequence.

Lastly, note that numbers of the form $4m + 2$ cannot be used as members of a primitive triple, since one leg is always divisible by four, so that Fibonacci numbers of the form F_{6k+3} are excluded from primitive Pythagorean triples.

2. SQUARES AMONGST THE GENERALIZED FIBONACCI NUMBERS $\{F_n(a)\}$

Squares are very sparse amongst the sequences $\{F_n(a)\}$, beyond $F_0(a) = 0$ and $F_1(a) = 1$. In the Fibonacci sequence, the only squares are 0, 1, and 144 [3]; in the lucas sequence, 1 and 4; and in the Pell sequence, 0, 1, and 169. There are no small squares other than 0 and 1 in $\{F_n(a)\}$, $3 \le a \le 10$; it is unknown whether other squares exist in $\{F_n(a)\}$, except when $a = k^2$, of course.

Cohn [3] has proved the first two theorems below, which we shall need later.

Theorem 2.1: If $L_n = x^2$, then $n = 1$ or 3.
If $L_n = 2x^2$, then $n = 0$ or $n = \pm 6$.

Theorem 2.2: If $F_n = x^2$, then $n = 0, \pm 1, 2$, or 12.
If $F_n = 2x^2$, then $n = 0, \pm 3$, or 6.

We shall need the following lemma:

Lemma 2.1: For the Fibonacci and Lucas polynomials,

$$F_{m+2k}(x) = L_k(x)F_{m+k}(x) + (-1)^{k+1}F_m(x).$$

Proof: Lemma 2.1 appears in [4] with only a change in notation.

We will use Lemma 2.1 with $x = 2$, so that $F_n(2) = P_n$ and $L_n(2) = R_n$, the Pell numbers and their related sequence.

Conjecture 2.3: If $P_n = x^2$, $n = 0, \pm 1$, or ± 7.

Partial Proof: Let $R_k = P_{k-1} + P_{k+1}$ so that $R_k = L_k(2)$. Then

$$R_{2m} = 8P_m^2 + (-1)^m \cdot 2, \text{ or, } R_{2m} \equiv \pm 2 \pmod 8 \text{ so that } R_{2m} \ne K^2.$$

$$\begin{aligned} R_{2k+1} &= P_{2k} + P_{2k+2} = P_{2k} + 2P_{2k+1} + P_{2k} \\ &= 2(P_{2k+1} + P_{2k}) = 2(2M + 1) \end{aligned}$$

since $2|P_n$ if and only if $2|n$. Thus, $R_{2k+1} \ne K^2$ and $R_n \ne K^2$ for any n.

Suppose n is even. Since $P_{2k} = P_k R_k$, if $n = 4p + 2$, then

$$P_n = P_{2p+1}R_{2p+1} \text{ where } (P_{2p+1}, R_{2p+1}) = 1.$$

Then $P_n = K^2$ if and only if $R_{2p+1} = x^2$ *and* $P_{2p+1} = y^2$, but $R_{2p+1} \ne x^2$, so $P_n \ne K^2$. If $n = 4p$, then

$$P_n = P_{2p}R_{2p} \text{ where } (P_{2p}, R_{2p}) = 2,$$

so $P_n = K^2$ if $P_{2p} = 2x^2$ and $R_{2p} = 2y^2$, but since $R_{2p} = 8P_p^2 \pm 2 = 2(X^2 \pm 1)$, $R_{2p} = 2y^2$ only for $p = 0$, giving P_0 as the only solution. Thus, $P_n \ne K^2$ for n even, unless $n = 0$.

Since $P_{m+8} \equiv P_m \pmod 8$ and $P_{8m\pm1} \equiv 1 \pmod 8$ and $P_{8m\pm3} \equiv 5 \pmod 8$, since all odd squares are congruent to 1 (mod 8), if n is odd, $n = 8m \pm 1$ if $P_n = K^2$. Of course, $P_n = k^2$ for $n = \pm 1, \pm 7$. The conjecture is not resolved.

Conjecture 2.4: If $P_n = 5k^2$, then $n = 0$ or $n = \pm 3$.

Partial Proof: If $P_n = 5k^2$, then $P_n \equiv 5 \cdot 0 \equiv 0 \pmod 8$, or $P_n \equiv 5 \cdot 1 \equiv 5 \pmod 8$, or $P_n \equiv 5 \cdot 4 \equiv 4 \pmod 8$, so that $n = 8m$, $8m+4$, $8m+3$, or $8m+5$, since $P_{8m} \equiv 0 \pmod 8$, $P_{8m+4} \equiv 4 \pmod 8$, and $P_{8m\pm3} \equiv 5 \pmod 8$.

If n is even, then $n = 4k$, and $P_n = P_{4k} = P_{2k}R_{2k}$ where $(P_{2k}, R_{2k}) = 2$ and $R_{2k} \neq x^2$, $R_{2k} \neq 2x^2$, and $R_{2k} \neq 5x^2$ since $5 \nmid R_{2k}$. We have $P_{4k} \neq K^2$ unless $k = 0$, or, $P_n \neq K^2$ when n is even, unless $n = 0$.

If n is odd, then $n = 8m \pm 3$. Now, $n = \pm 3$ gives a solution. If $n \neq \pm 3$, then $n = 8m \pm 3 = 2 \cdot 4w \pm 3$, and since $P_{-3} = P_3 = 5$, both of these give $P_n = -P_3 \pmod{R_{4w}} = -5 \pmod{R_{4w}}$ by way of Lemma 2.1 and

$$(2.1) \qquad P_{m+2k} = R_k P_{m+k} + (-1)^{k+1} P_m$$

where $m = \pm 3$ and $k = 4w$. Now, if w is odd, then R_4 divides R_{4w}, and we can write, from (2.1),

$$P_{2\cdot 4w\pm 3} = R_4 \cdot K \cdot P_{4w\pm3} - P_{\pm 3}$$

so that, since $R_4 = 34$, $P_n \equiv -5 \pmod{34}$, where -5 is not a quadratic residue of 34. It is strongly suspected that -5 is not a quadratic residue of R_{4w}, but the conjecture is not established if w is even.

Theorem 2.5: If $F_n = 5x^2$, then $n = 0$ or $n = \pm 5$.

Proof: If n is even, $F_n = F_{2k} = F_k L_k = 5x^2$ if $F_k = 5x^2$ and $L_k = y^2$, or $F_k = x^2$ and $L_k = 5k^2$ (impossible), which has solutions for $k = 0$ only.

If n is odd, then $n \equiv 3 \pmod 4$ or $n \equiv 1 \pmod 4$. If $n \equiv 3 \pmod 4$, then write $n = 3 + 4M = 3 + 2 \cdot 3^n \cdot k$, where $2 \mid k$, $3 \nmid k$, and

$$5F_n \equiv -5F_3 \equiv -10 \pmod{L_k},$$

but $L_k \equiv 3 \pmod 4$ if $2 \mid k$, $3 \nmid k$, so -10 is not a quadratic residue, and

$$5F_n \neq k^2 \text{ so } F_n \neq 5k^2.$$

If $n \equiv 1 \pmod 4$, $n = 5$ is a solution. If $n \neq 5$

$$n = 1 + 4M = 1 + 2 \cdot 3^r \cdot k,$$

where $2 \mid k$, $3 \nmid k$, and

$$5F_n \equiv -5F_1 \equiv -5 \pmod{L_k},$$

but -5 is not a quadratic residue, and

$$5F_n \neq k^2 \text{ so } F_n \neq 5K^2 \text{ when } n \text{ is odd, unless } n = 5.$$

Since $F_{-n} = (-1)^{n+1}F_n$, $n = -5$ is also a solution. Thus, $F_n \neq 5x^2$ unless $n = 0, \pm 5$.

We will find another relationship between squares of the generalized Fibonacci numbers useful.

Theorem 2.6:

$$F_n^2(x) = (-1)^{n+k} F_k^2(x) + F_{n-k}(x) F_{n+k}(x).$$

Proof: For simplicity, we will prove Theorem 2.6 for Fibonacci numbers, or for $x = 1$, noting that every identity used is also an identity for the Fibonacci polynomials [4]. In particular, we use

$$(2.2)\qquad (-1)^{n+1}F_n(x) = F_{-n}(x)$$

$$(2.3)\qquad F_{p+r}(x) = F_{p-1}(x)F_r(x) + F_p(x)F_{r+1}(x)$$

$$(2.4)\qquad F_n^2(x) = (-1)^{n+1} + F_{n-1}(x)F_{n+1}(x)$$

$$(2.5)\qquad F_{n+1}^2(x) + F_n^2(x) = F_{2n+1}(x)$$

Proof is by mathematical induction. Theorem 2.6 is true for $k = 1$ by (2.4) Set down the theorem statement as $P(k)$ and $P(k + 1)$:

$$P(k):\quad F_n^2 = (-1)^{n+k}F_k^2 + F_{n-k}F_{n+k}$$

$$P(k+1):\quad F_n^2 = (-1)^{n+k+1}F_{k+1}^2 + F_{n-k-1}F_{n+k+1}$$

Equating $P(k)$ and $P(k + 1)$,

$$\begin{aligned}(-1)^{n+k+1}\left(F_{k+1}^2 + F_k^2\right) &= F_{n-k}F_{n+k} + F_{n-k-1}F_{n+k+1}\\ &= (-1)^{k-n+1}F_{k-n}F_{n+k} + (-1)^{k-n+1}F_{k+1-n}F_{n+k+1}\end{aligned}$$

by (2.2). By (2.5) and (2.3), the left-hand and right-hand members become

$$(-1)^{n+k+1}F_{2k+1} = (-1)^{k-n+1}F_{2k+1}.$$

Since all the steps reverse,

$$(-1)^{n+k+1}F_{k+1}^2 + F_{n-k-1}F_{n+k+1} = (-1)^{n+k}F_k^2 + F_{n-k}F_{n+k} = F_n^2$$

so that $P(k + 1)$ is true whenever $P(k)$ is true. Thus, Theorem 2.6 holds for all positive integers n.

3. SOLUTIONS FOR $F_n^2(a) + F_n^2(a) = K^2$

By Theorem 2.6, when n and k have opposite parity,

$$(3.1)\qquad F_n^2(a) + F_k^2(a) = F_{n-k}(a)F_{n+k}(a).$$

Since $(F_n(a), F_k(a)) = 1 = F_{(n,k)}(a)$ by the results of [5], $(n,k) = 1$ and opposite parity for n and k means that $(n - k, n + k) = 1$ so that

$$(F_{n-k}(a), F_{n+k}(a)) = 1.$$

Thus, $F_{n-k}(a)F_{n+k}(a) = K^2$ if and only if both $F_{n-k}(a) = x^2$ and $F_{n+k}(a) = y^2$. We would expect a very limited number of solutions, then, since squares are scarce amongst $\{F_n(a)\}$.

Since one leg is divisible by 4 in a Pythagorean triple, one of n or k is a multiple of 6 if a is odd, and a multiple of 2 if a is even; thus, n and k cannot both be odd. Also, n and k cannot both be even, since $F_2(a)$ is a factor of $F_{2m}(a)$ and $F_2(a) > 1$ for all sequences except $F_n(1) = F_n$.

Restated,

Theorem 3.1: Any solution to $F_n^2(a) + F_k^2(a) = K^2$ in positive integers, $a \geq 2$, occurs only for such values of n and k that $F_{n-k}(a) = x^2$ *and* $F_{n+k}(a) = y^2$.

Conjecture 3.2: $F_n^2(2) + F_k^2(2) = K^2$, $n > k > 0$, where $F_n(2) = P_n$, the nth Pell number, has the unique solution $n = 4$, $k = 3$, giving 5 - 12 - 13.

Proof: Apply Theorems 3.1 and Conjecture 2.3.

Theorem 3.3: If $F_n^2 + F_k^2 = K^2$, $n > k > 0$, then both n and k are even.

Proof: Apply Theorems 3.1 and 2.2.

Theorem 3.4: If $F_n^2 + F_k^2 = K^2$, $n > k > 0$, then $F_{10} = 55$, $F_8 = 21$, $F_{18} = 2584$, $F_6 = 8$, and $F_4 = 3$ each divide either F_n or F_k, and 13 is the smallest prime factor possible for K.

Proof: Since 3 divides one leg of a Pythagorean triple, F_4 divides F_k or F_n. Since 4 divides one leg of a Pythagorean triple, and the smallest F_n divisible by 4 is F_6, F_6 divides F_k or F_n. That F_{10} divides either F_n or F_k follows by examining the quadratic residues of 11. The quadratic residues of 11 are 1, 3, 4, 5, and 9. It is not difficult to calculate

$$F_{10w}^2 \equiv 0 \pmod{11}$$
$$F_{10w\pm 2}^2 \equiv 1 \pmod{11}$$
$$F_{10w\pm 4}^2 \equiv 9 \pmod{11}$$

where we need only consider even subscripts by Theorem 3.3. Notice that $F_{10w}^2 + F_{10w\pm 2}^2 \equiv 1 \pmod{11}$ and $F_{10w}^2 + F_{10w\pm 4}^2 \equiv 9 \pmod{11}$, where 1 and 9 are quadratic residues of 11, so that these are possible squares, but $F_{10w\pm 2}^2 + F_{10w\pm 4}^2 \equiv 10 \pmod{11}$, where 10 is not a residue. $F_{10w\pm 2}^2 + F_{10w\pm 2}^2$ produces the nonresidue 2, and similarly $F_{10w\pm 4}^2 + F_{10w\pm 4}^2 \equiv 7 \pmod{11}$, so that either $F_n = F_{10w}$ or $F_k = F_{10w}$. In either case, F_{10} divides one of F_n or F_k.

Similarly, we examine the quadratic residues of 7, which are 0, 1, 2, and 4. We find

$$F_{8m}^2 \equiv 0 \pmod{7}$$
$$F_{8m\pm 2}^2 \equiv 1 \pmod{7}$$
$$F_{8m\pm 4}^2 \equiv 2 \pmod{7}$$

where $F_{8m}^2 + F_{8m\pm 2}^2 \equiv 1 \pmod{7}$ and $F_{8m}^2 + F_{8m\pm 4}^2 \equiv 2 \pmod{7}$ are possible squares but $F_{8m\pm 2}^2 + F_{8m\pm 4}^2 \equiv 3 \pmod{7}$ is not a possible square. But, F_{8m}^2 and $F_{8m\pm 4}^2$, or F_{8m}^2 and $F_{8m^*}^2$, or $F_{8m\pm 4}^2$ and $F_{8m^*\pm 4}^2$, cannot occur in the same primitive triple, since they have common factor F_4. $F_{8m\pm 2}^2$ and $F_{8m^*\pm 2}^2$ cannot be in the same triple, because F_4 divides one leg, and neither subscript is divisible by 4. Thus, F_{8m} is one leg in the only possible cases, forcing F_8 to be a factor of F_n or of F_k.

Using 17 for the modulus, with quadratic residues 0, 1, 2, 4, 8, 9, 13, 15, 16, we find

$$F_{18m}^2 \equiv 0 \pmod{17}$$
$$F_{18m\pm 2}^2 \equiv 1 \pmod{17}$$
$$F_{18m\pm 4}^2 \equiv 9 \pmod{17}$$
$$F_{18m\pm 6}^2 \equiv 13 \pmod{17}$$
$$F_{18m\pm 8}^2 \equiv 16 \pmod{17}$$

Now, F_{18m}^2 can be added to any of the other forms to make a quadratic residue (mod 17). $F_{18m\pm2}^2 + F_{18m\pm2}^2 \equiv 2 \pmod{17}$, but one subscript must be divisible by 6. $F_{18m\pm2}^2 + F_{18m\pm4}^2 \equiv 10 \pmod{17}$ is not a residue. $F_{18m\pm2}^2 + F_{18m\pm6}^2 \equiv 14 \pmod{17}$ is not a residue. $F_{18m\pm2}^2 + F_{18m\pm8}^2 \equiv 0 \pmod{17}$, but one subscript must be divisible by 6. $F_{18m\pm4}^2 + F_{18m\pm6}^2 \equiv 5 \pmod{17}$ is not a residue, while $F_{18m\pm4}^2 + F_{18m\pm8}^2 \equiv 8 \pmod{17}$, but one subscript must be divisible by 6. $F_{18m\pm4}^2 + F_{18m\pm4}^2$ and $F_{18m*\pm8}^2 + F_{18m\pm8}^2$ are also discarded because one subscript is not divisible by 6. $F_{18m\pm6}^2 + F_{18m\pm6}^2$ have a common factor of F_6 so cannot be in the same primitive triple, and $F_{18m\pm6}^2 + F_{18m\pm8}^2$ produce the nonresidue 12 (mod 17). The only possibility, then, is that F_{18m} appears as one leg, or that F_{18} divides either F_n or F_k.

Since K cannot have any factors in common with F_n or with F_k, we note that the prime factors 2, 3, 5, 7, and 11 occur in F_{10}, F_8, F_{18}, F_6, and F_4, but 13 does not, making 13 the smallest possible prime factor for K.

Theorem 3.5: If $F_n^2 + F_k^2 = K^2$, $n > k > 0$, has a solution in positive integers, then the smallest leg $F_k \geq F_{50}$, which has 11 digits.

Proof: Consider the required form of the subscripts n and k in the light of Theorem 3.4. Because $4|F_n$ or $4|F_k$, and both subscripts are even, we can write $F_{6m}^2 + F_{2p}^2$, where $p = 3j \pm 1$, making the required form $F_{6m}^2 + F_{6j\pm2}^2$. Since 3 divides one subscript or the other, 4 divides one subscript or the other, leading to

(i) $F_{6m}^2 + F_{12w\pm4}^2$, for j odd,

and to

(ii) $F_{12m}^2 + F_{12w\pm2}^2$, for j even.

First, consider (i). Since $F_8 = 21$ divides one leg or the other, F_8 must divide $F_{12w\pm4}$ to avoid a common factor of $F_4 = 3$, so w is odd, making $F_{6m}^2 + F_{24q\pm8}^2$ the required form. Next, F_{18} divides a leg. If F_{18} divides $F_{12w\pm4}$, then $F_6|F_{12w\pm4}$, but $6\nmid(12w \pm 4)$. So, $F_{18}|F_{6m}$, making the required form become $F_{18m}^2 + F_{24q\pm8}^2$. Next, since F_{10} divides a leg, we obtain the two final forms,

(1) $F_{90m}^2 + F_{24q\pm8}^2$ or (2) $F_{18m}^2 + F_{120s\pm40}^2$.

Next, consider (ii). Since $F_8 = 21$ divides a leg, we must have $F_8|F_{12m}$ to avoid a common factor of $F_4 = 3$, making the form become $F_{24m}^2 + F_{12w\pm2}^2$. Also, F_{18} divides a leg, but must divide F_{24m} to avoid a common factor of F_6, making the form be $F_{72m}^2 + F_{12m\pm2}^2$. Since we also have F_{10} as the divisor of a leg, we have the two possible final forms

(3) $F_{360r}^2 + F_{12w\pm2}^2$ or (4) $F_{72m}^2 + F_{60p\pm10}^2$.

Now, if F_k is the odd leg, then $F_k = m^2 - n^2$, and the even leg is $F_n = 2mn$. The largest value for $2mn$ occurs for $(m + n) = F_k$ and $(m - n) = 1$, so we do not need to know the factors of F_k. Solving to find the largest values of m and n, we find $m = (F_k + 1)/2$ and $n = (F_k - 1)/2$, making the largest possible even leg $F_n = 2mn = (F_k^2 - 1)/2$. We have available a table of Fibonacci numbers F_n, $0 \leq n \leq 571$ [6].

We look at the four possible forms again. In form (1), F_{90} has 19 digits, the smallest possible even leg. Possible odd legs are F_{16}, F_{32}, F_{40}, F_{56}, ... where F_{40} has 9 digits, so that $(F_{40}^2 - 1)/2$ has less then 19 digits, making the smallest possible leg in form (1) be F_{56}. In form (2), $F_{18m}^2 + F_{120q \pm 40}^2$, the smallest leg occurs for $m = 1$, known not to occur in such a triple from Table 1; $m = 2$ gives a common factor of 4 with the other subscript, making $m = 3$ the smallest usable value, or the smallest possible leg F_{54}. Now, form (3) has F_{360}, a number of 75 digits, as the smallest value for the even leg, making the smallest possible odd leg greater than F_{170}, which has 36 digits. Lastly, form (4) has its smallest leg F_{50}, which has 11 digits. Comparing smallest legs in the four forms, we see that the smallest leg possible is F_{50}.

Theorem 3.6: $L_n^2 + L_k^2 = K^2$, $n > k > 0$, has the unique solution $n = 3$, $k = 2$, or the triple 3 - 4 - 5.

Proof: Since $4|L_n$ or $4|L_k$, either $n = 3(2k + 1)$ or $k = 3(2k + 1)$, so that one subscript is odd. Since 3 divides one leg in a Pythagorean triple, one leg has to have a subscript of $2(2k + 1)$, which is even, since $L_p|L_q$ if and only if $q = (2k + 1)p$ (see [1]). Thus, n and k must have opposite parity. If n and k have opposite parity, then $(n - k)$ is odd. Since $L_{-n} = (-1)^n L_n$, from [1] we have both

$$L_{n-k}L_{n+k} - L_n^2 = 5(-1)^{n+k}F_k^2, \tag{3.2}$$

$$(-1)^{n-k}L_{n-k}L_{n+k} - L_k^2 = 5(-1)^{n+k}F_n^2,$$

where $n - k$ is odd. Adding the two forms of (3.1),

$$L_n^2 + L_k^2 = 5(F_k^2 + F_n^2) = 5F_{n-k}F_{n+k}$$

by (3.1). Now, $5F_{n-k}F_{n+k} = K^2$ if and only if either $F_{n-k} = 5x^2$ and $F_{n+k} = y^2$ or $F_{n-k} = y^2$ and $F_{n+k} = 5x^2$. By Theorems 2.5 and 2.2, either $n + k = 1$ and $n - k = 5$ or $n - k = 1$ and $n + k = 5$, making the only solution $n = 3$, $k = 2$.

4. SOLUTIONS FOR $F_n^2(a) - F_k^2(a) = K^2$

By Theorem 2.6, when n and k have the same parity,

$$F_n^2(a) - F_k^2(a) = F_{n-k}(a)F_{n+k}(a). \tag{4.1}$$

As in Section 3, $F_{n-k}(a)F_{n+k}(a) = K^2$ if and only if both $F_{n-k}(a) = x^2$ and $F_{n+k}(a) = y^2$, indicating a limited number of solutions in positive integers. Note that n and k cannot both be even if $a \geq 2$, because $F_{2p}(a)$ and $F_{2r}(a)$ have the common factor $F_2(a)$, precluding a primitive triple.

Lemma 4.1: If a is odd, $2|F_{3k}(a)$, $3|F_{4k}(a)$, and $4|F_{6k}(a)$.

Proof: We list $F_0(a) = 0$, $F_1(a) = 1$, $F_2(a) = a$, $F_3(a) = a^2 + 1$, $F_4(a) = a^3 + 2a$, $F_5(a) = a^4 + 3a^2 + 1$, and $F_6(a) = a^5 + 4a^3 + 3a$. If a is odd, then $F_3(a)$ is even. If $a = 2m + 1$, then

$$\begin{aligned} F_4(a) &= (8m^3 + 12m^2 + 6m + 1) + (4m + 2) \\ &= (8m^3 + 4m) + (12m^2 + 6m + 3) \\ &= 4m(2m^2 + 1) + 3(4m^2 + 2m + 1) \\ &= 3M + 3K = 3W, \end{aligned}$$

since either $3|m$ or $3|(2m^2 + 1)$. Also, $a = 2m + 1$ makes

$$\begin{aligned} F_6(a) &= (2m+1)^5 + 4(2m+1)^3 + 3(2m+1) \\ &= (4K + 10m + 1) + 4M + (6m + 3) \\ &= 4K + 4M + 16m + 4 = 4P. \end{aligned}$$

Since $F_m(a)|F_{mk}(a)$, $m > 0$, the lemma follows.

Lemma 4.2: If a is even, $2|F_{2k}(a)$, $3|F_{4k}(a)$, and $4|F_{4k}(a)$.

Proof: Refer to the proof of Lemma 4.1 and let $a = 2m$. Then $F_2(a) = 2m$, and $F_4(a) = 8m^3 + 4m = 4[m(2m^2 + 1)] = 4 \cdot 3M$, and the Lemma follows as before.

Theorem 4.1: If $F_n^2(a) - F_k^2(a) = K^2$, $n > k > 0$, has solutions in positive integers, then $n \neq 4k$. If a is even, n cannot be even. If a is odd, $n \neq 3k$ and $n \neq 4k$.

Proof: Lemmas 4.1 and 4.2 show that $3|F_{4k}(a)$, and since 3 divides one leg in a Pythagorean triple, $n = 4k$ would cause a common factor of 3, preventing a primitive triple. For similar reasons, $n \neq 2k$ if a is even, and $n \neq 3k$ if a is odd.

Conjecture 4.2: Any possible solution for $P_n^2 - P_k^2 = K^2$, $n > k > 0$, occurs only if $n = 2p + 1$ and $k = 4w$, or if P_n is odd and P_k is a multiple of 12.

Proof: Considering (4.1), there is no solution to $P_{n-k} = x^2$, $P_{n+k} = y^2$ if n and k have the same parity, if Conjecture 2.3 holds. Also, n cannot be even, because $2|P_{2m}$ and 4 divides one leg in a Pythagorean triple, precluding a primitive triple. If k is even, then P_k is even, and the even leg is divisible by 4, making P_k have the form P_{4w}. Since $P_4 = 12$, P_{4w} is a multiple of 12.

Theorem 4.3: $F_n^2 - F_k^2 = K^2$ has solutions in positive integers for $n = 7$, $k = 5$, forming the triple 5-12-13, and for $n = 5$, $k = 4$, forming the triple 3-4-5. Any other solutions occur only if n and k have opposite parity, where either $n = 12w \pm 2$ and k is odd, or $n = 6m \pm 1$ and k is even.

Proof: Using (4.1) and Theorem 2.2, the only solution for $F_{n-k} = x^2$ and $F_{n+k} = y^2$ where n and k have the same parity is $n = 7$, $k = 5$, making the triple 5-12-13. If any other solutions exist, n and k have opposite parity. It is known that $n = 5$, $k = 4$ provides a solution, giving the triple 3-4-5. If n is even, $n \neq 3k$, $n \neq 4k$, so $n = 12w \pm 2$, and k is odd. If n is odd, $n \neq 3k$, so $n = 6m \pm 1$ and k is even.

Theorem 4.4: If n and k have different parity, any solutions for $F_n^2 - F_k^2 = K^2$ other than $n = 5$, $k = 4$, or the triple 3-4-5, must have $n \geq k + 5$.

Proof: $F_{n+1}^2 - F_n^2 = F_{n-1}F_{n+2}$, where $(F_{n-1}, F_{n+2}) = 1$ or 2, so that $F_{n-1}F_{n+2} = K^2$ either if $F_{n-1} = x^2$ and $F_{n+2} = y^2$, or if $F_{n-1} = 2x^2$ and $F_{n+2} = 2y^2$. By Theorem 2.2, there are no solutions to $F_{n-1} = x^2$ and $F_{n+2} = y^2$, but $F_{n-1} = 2x^2$ and $F_{n+2} = 2y^2$ is solved by $n = 4$, yielding the 3-4-5 triple. There are no other solutions for subscripts differing by 1. Since n and k have opposite parity, they differ by an odd number.

$$F_{n+3}^2 - F_n^2 = 4F_{n+1}F_{n+2} \neq K^2 \text{ unless } n = 0 \text{ or } -1 \text{ by Theorem 2.2.}$$

Thus, the hypotenuse has a subscript at least five greater than the leg.

Theorem 4.5: $F_n^2(a) - F_k^2(a) = K^2$ has no solution in positive integers if $F_n(a)$ is prime.

Proof: See the discussion at the end of Section 1.

Theorem 4.6: If $L_n^2 - L_k^2 = K^2$, $n > k > 0$, has solutions in positive integers, then either $n = 4m$ and k is odd, or $n = 6p \pm 1$ and k is even.

Proof: We parallel the proof of Theorem 3.6, except here we take n and k with the same parity, so that $n + k$ is even, and subtract:

$$L_{n-k}L_{n+k} - L_n^2 = 5(-1)^{n+k}F_k^2$$

$$(-1)^{n-k}L_{n-k}L_{n+k} - L_k^2 = 5(-1)^{n+k}F_n^2$$

$$L_n^2 - L_k^2 = 5(F_n^2 - F_k^2) = 5F_{n-k}F_{n+k} = K^2$$

if and only if $F_{n-k} = 5x^2$ and $F_{n+k} = y^2$, or $F_{n+k} = 5x^2$ and $F_{n-k} = y^2$. By Theorem 2.5, the only solution for n and k the same parity is $n - k = 0$, which does not solve our equation.

If n and k do not have the same parity, consider n even. Then, $n = 4k$ or $n = 4k + 2$, but $n = 4k + 2$ is impossible because the hypotenuse would have the factor 3 in common with a leg. Thus, $n = 4k$, and k is odd. If n is odd, then $n = 6p \pm 1$ to avoid a factor of $L_2 = 3$, and k is even.

Conjecture: The only solutions to $F_n^2(a) \pm F_k^2(a) = K^2$, $n > k > 0$, in positive integers, are found in the two Pythagorean triples 3 - 4 - 5 and 5 - 12 - 13. If $a \geq 3$ and $a \neq k^2$, the only squares in $\{F_n(a)\}$ are 0 and 1.

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