or

(6) $$a_{-k+i} - b_i = 2c_i + \cdots + 2^{m-2}r_i \text{ for all } i.$$

Possibilities for $a_{-k+i} - b_i$ are 0, 1, and -1. But the right-hand side of (6) is divisible by 2. Hence, we must have that $a_{-k+i} - b_i = 0$ for all i. Since $a_{-k+i} = 0$ for all i, this implies that $b_i = 0$ for all i and hence that $c_i = 0, \ldots, r_i = 0$ for all i. But since this contradicts Theorem 1.8, it follows that the m-tuple $2^{mk+1} - 1, -1, -1, \ldots, -1$ is basic as claimed.

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PYTHAGOREAN TRIPLES AND TRIANGULAR NUMBERS

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1. INTRODUCTION

In [4] W. Sierpiński proves that there are an infinite number of Pythagorean triples in which two members are triangular and the hypotenuse is an integer. [A number T_n is triangular if T_n is of the form $T_n = n(n+1)/2$ for some integer n. A Pythagorean triple is a set of three integers x, y, z such that $x^2 + y^2 = z^2$.] Further, Sierpinski gives an example due to Zarankiewicz,

$$T_{132} = 8778, \quad T_{143} = 10296, \quad \text{and} \quad T_{164} = 13530,$$

in which every member of the Pythagorean triple is triangular. He states that this is the only known nontrivial example of this phenomenon, and that it is not known whether the number of such triples is finite or infinite.

This paper will give some partial results related to the above problem. In particular, we will give necessary and sufficient conditions for the existence of Pythagorean triples in which all members are triangular. We will extend these conditions to discuss the problem of triangulars being represented as sums of powers.

2. PYTHAGOREAN TRIPLES WITH TRIANGULAR SOLUTIONS

By a triangular solution to a Diophantine equation $f(x, \ldots, x_n) = 0$, we mean a solution in which every variable is triangular.

Theorem 1: The Pythagorean equation $x^2 + y^2 = z^2$ has a triangular solution $x = T_a$, $y = T_b$, $z = T_c$ if and only if there exist integers m and k such that

$$T_b^2 = m^3 + (m+1)^3 + \cdots + (m+k)^3;$$

that is, T_b^2 is a sum of $k + 1$ consecutive cubes.

Proof: It is a known formula that

$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} = T_n^2.$$

So if

$$T_a^2 + T_b^2 = T_c^2$$

with $a \le b$, then

$$T_b^2 = T_c^2 - T_a^2 = \sum_{k=a+1}^{c} k^3.$$

To show the converse, we need only reverse the steps. Q.E.D.

Using Zarankiewicz's example, we can note that T_{143}^2 is a sum of 31 cubes; i.e.,

$$T_{143}^2 = \sum_{k=133}^{164} k^3.$$

3. TRIANGULARS AS CUBES AND SUMS OF CUBES

We first show that a triangular cannot be a cube. This is an old result, first proved by Euler in 1738 [2]. However, it is so closely related to our work that we will include a proof here.

Lemma 2: The triangular T_n is a kth power if and only if T_n^2 is a kth power.

Proof: This is an easy exercise using the fact that every integer has a unique decomposition into primes.

Lemma 3: The equality $T_n = m^k$ holds nontrivially if and only if the equations $x^k - 2y^k = \pm 1$ have nontrivial solutions. Take the plus sign if n is even and the minus sign if n is odd.

Proof: Let

$$T_n = \frac{n(n+1)}{2} = m_k.$$

Clearly $(n, n+1) = 1$. Let $n = 2j$; then

$$(2j)(2j+1)/2 = m^k.$$

Thus there are integers x and y such that $j = y$ and $2j + 1 = x^k$; whence

$$x^k - 2y^k = 1.$$

Now let $n = 2j - 1$. In the same way as above, there are integers y, x such that $j = y^k$, $2j - 1 = x^k$, and $x^k - 2y^k = -1$.

Since the steps are reversible, the converse is easily proved. Q.E.D.

Theorem 4: There is no triangular number greater than 1 which is a cube.

Proof: If $T_n = m^3$, then by Lemma 3, $x^3 - 2y^3 = -1$ has a solution. However, by [1, p. 72], $x^3 - 2y^3 = 1$ has only $x = -1$, $y = 0$ as solutions. Hence, by the construction in Lemma 3, $n = 1$ or 0. Q.E.D.

We will now state, without proof, a theorem due to Siegel which will be of utmost importance in that which follows.

Theorem 5: (Siegel [3, p. 264]) The equation

$$y^2 = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

has only a finite number of integer solutions if the right-hand side has at least three different linear factors.

We can immediately apply this theorem in the proof of the following result.

Theorem 6: For a fixed k, there are only a finite number of sums of k consecutive cubes which can be the square of a triangular number. For every k, there is at least one such sum which is the square of a triangular.

Proof: The last statement follows from the identity

$$\sum_{n=0}^{k} n^3 = T_k^2.$$

To prove the first statement we consider two cases. Assume $k = 2l + 1$. Consider the equation

$$T_n^2 = \sum_{j=-l}^{l} (m + j)^3. \tag{1}$$

We want to show that this equation has only a finite number of solutions in n and m. We have

$$\begin{aligned} T_n^2 = \sum_{j=-l}^{l} (m + j)^3 &= Am^3 + Bm \qquad AB \neq 0 \\ &= m(Am^2 + B). \end{aligned} \tag{2}$$

Now $Am^2 + B$ is never a square since $(am + b)^2$ always has a first-degree term. Thus, equation (2) has no squared linear factors on its right-hand side, and by Theorem 5 it has only a finite number of solutions.

If $k = 2l$, we consider

$$\begin{aligned} T^2 &= \sum_{-l}^{l+1} (m + j)^3 \\ &= (2L + 1)m^3 + L(L - 1)(2L + 1)m + (m + L - 1)^3 \\ &= (L + 1)\big(2m^3 + 3m^2 + (2L^2 + 4L + 3)m + (L + 1)^2\big). \end{aligned} \tag{3}$$

To show that the right-hand side does not have a square linear factor, we show that it and its derivative,

$$6m^2 + 6m + (2L^2 + 4L + 3)$$

have a greatest common divisor of 1. This is an easy application of the Euclidean algorithm. Hence, using Theorem 5, equation (3) has only a finite number of integral solutions. Q.E.D.

Combining Theorems 1 and 6, we have a type of finiteness condition for all members of a Pythagorean triple to be triangular. Of course, the k can vary, so we do not have the condition that only a finite number of such triples exist, but that for a fixed k, only a finite number exist.

4. TRIANGULARS AND SUMS OF HIGHER POWERS

We can prove theorems similar to Theorems 4 and 6 for higher powers.

Theorem 7: The equations $T_n = m^4$ and $T_n = m^5$ are impossible for $n > 1$.

Proof: This follows from Lemma 3 and the fact that the equations

$$x^4 - 2y^4 = \pm 1 \quad \text{and} \quad x^5 - 2y^5 = \pm 1$$

have no nontrivial solutions [1]. Q.E.D.

Theorem 7 was first stated by Fermat in 1658, but he apparently gave no proof; at least none has been found. The first proof was given by Euler [3].

Theorem 8: For a fixed k, the equations

$$T_n^2 = \sum_{i=0}^{k} (m + i)^4$$

and

$$T_n^2 = \sum_{i=0}^{k} (m + i)^5$$

have only a finite number of solutions.

Proof: These statements are proven using techniques completely similar to the proof of Theorem 6. Greatest common divisor calculations are extremely complicated and are therefore omitted. Q.E.D.

The techniques of Theorem 6 appear to apply to even higher powers. However, there does not appear to be a general method of handling all such cases simultaneously because of the differences of the equations and the derivatives.

5. THE EQUATION $T_{(n+1)^2} = k^2$

The theorems of this section digress from the main topics of this paper, but they are included as nice illustrations of the use of Theorem 5.

Theorem 9: The equation $T_{(n+1)^2} = k^2$ has only a finite number of solutions.

Proof: If $(n + 1)^2((n + 1)^2 + 1)/2 = k^2$, then

$$2k^2 = n^4 + 4n^3 + 7n^2 + 6n + 2. \tag{4}$$

The derivative of the right-hand side is

$$4n^3 + 12n^2 + 14n + 6 = 2(n + 1)(2n^2 + 4n + 3).$$

It is easy to check that no root of the derivative is a root of equation (4), so equation (4) has no squared factor. Hence, by Theorem 5, there are only a finite number of solutions to the equation of the theorem. Q.E.D.

Note that $T_{(1)} = (1)^2$ and $T_{(7)^2} = (35)^2$.

In [4] Sierpiński shows that the equation

$$(T_{2u})^2 + (T_{2u+1})^2 = [(2u + 1)v]^2$$

with $v^2 = u^2 + (u + 1)^2$ has only a finite number of solutions. Since we have that the identity

$$(T_{2u+1})^2 + (T_{2u})^2 = T_{(2u+1)^2}$$

holds, we have the following theorem.

Theorem 10: The equation

$$T_{(2u+1)^2} = [(2u + 1)v]^2$$

with $v^2 = u^2 + (u + 1)^2$ has only a finite number of solutions.

Proof: Use Theorem 9.

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EXTENSIONS OF THE W. MNICH PROBLEM

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ABSTRACT

W. Sierpiński publicized the following problem proposed by Werner Mnich in 1956: *Are there three rational numbers whose sum and product are both one*? In 1960, J. W. S. Cassels proved that there are no rationals that meet the Mnich condition. This paper extends the Mnich problem to k-tuples of rationals whose sum and product are one by providing infinite solutions for all $k > 3$. It also provides generating forms that yield infinite solutions to the original Mnich problem in real and complex numbers, as well as providing infinite solutions for rational sums and products other than one.

HISTORICAL OVERVIEW

Sierpiński [6] cited a question posed by Werner Mnich as a most interesting problem, and one that at that time was unsolved. The Mnich question concerned the existence of three rational numbers whose sum and product are both one:

(1) $$x + y + z = xyz = 1 \quad (x,\ y,\ z \text{ rational}).$$

Cassels [1] proved that there are no rationals that satisfy the conditions of (1). Cassels also shows that this problem was expressed by Mordell [3], in equivalent, if not exact form. Additionally, Cassels has compiled an excellent bibliography that demonstrates that the "Mnich" problem has its roots in the work of Sylvester [13] who in turn obtained some results from the 1870 work of the Reverend Father Pépin. Sierpiński [9] provides a more elementary proof of the impossibility of a weaker version of (1), along with an excellent summary of some of the equivalent forms of the "Mnich" problem. Later, Sansone and Cassels [4] provided another proof of the impossibility of (1).