even. This is true when $p = 2^m q$ where m is a positive integer and q is odd and $2^{m-1} < n \leq 2^m$. Now we conclude that $\binom{p+r}{r}$ is odd and $\binom{p+r+1}{r}$ is alternately odd and even for $r = 0, 1, 2, \ldots, n - 1$ where $p = 2^m q$ and $2^{m-1} < n \leq 2^m$.

Remark 1: Care must be taken not to apply the results of Theorem 2 directly in order to obtain the results of Theorem 5. Similarly, the properties of the derivatives and the integrals of a BMS should not be applied directly to H in (7.3).

Remark 2: The authors earnestly hope that the reader will be able to find further applications of the binary sequences of BMSs.

ACKNOWLEDGMENT

Conversations with C. S. Karuppan Chetty were helpful in the preparation of this paper.

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RESTRICTED MULTIPARTITE COMPOSITIONS

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1. INTRODUCTION

In [1] the writer discussed the number of compositions

(1.1) $$n = a_1 + a_2 + \cdots + a_k$$

in positive (or nonnegative) integers subject to the restriction

(1.2) $$a_i \neq a_{i+1} \quad (i = 1, 2, \ldots, k - 1).$$

In [2] he considered the number of compositions (1.1) in nonnegative integers such that

(1.3) $$a_i \not\equiv a_{i+1} \pmod{m} \quad (i = 1, 2, \ldots, k - 1),$$

where m is a fixed positive integer.

In the present paper we consider the number of *multipartite compositions*

(1.4) $$n_j = a_{j1} + a_{j2} + \cdots + a_{jk} \quad (j = 1, 2, \ldots, t)$$

in nonnegative a_{js} subject to

(1.5) $$\mathbf{a}_i \neq \mathbf{a}_{i+1} \quad (i = 1, 2, \ldots, k - 1)$$

or

(1.6) $$\mathbf{a}_i \not\equiv \mathbf{a}_{i+1} \pmod{m} \quad (i = 1, 2, \ldots, k - 1)$$

where $\mathbf{a}_i$ denotes the vector $(a_{1i}, a_{2i}, \ldots, a_{ri})$ and m is a fixed positive integer.

Let $c(\mathbf{n},k)$ denote the number of solutions of (1.4) and (1.5) and let $f(\mathbf{n},k)$ denote the number of solutions of (1.4) and (1.6), where $\mathbf{n} = (n_1, n_2, n_t)$. We show in particular that

$$\sum_{\mathbf{n}} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} \sum_k c(\mathbf{n},k) z^k = \left\{ 1 + \sum_{j=1}^{\infty} \frac{(-1)^j z^j}{(1 - x_1^j)(1 - x_2^j) \cdots (1 - x_t^j)} \right\}^{-1} \tag{1.7}$$

and

$$\sum_{\mathbf{n}} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} \sum_k f(\mathbf{n},k) z^k = \left\{ 1 - \sum_{i_1, \ldots, i_t = 0}^{\infty} \frac{x_1^{i_1} x_2^{i_2} \cdots x_t^{i_t} \lambda}{1 + x_1^{i_1} x_2^{i_2} \cdots x_t^{i_t} \lambda} \right\}^{-1} \tag{1.8}$$

where

$$\lambda = \frac{z}{(1 - x_1^m)(1 - x_2^m) \cdots (1 - x_t^m)}.$$

For simplicity, proofs are given for the case $t = 2$, but the method applies to the general case.

SECTION 2

To simplify the notation, we consider the case $t = 2$ of (1.4); however, the method applies equally well to the general case. Thus, let $c(n,p,k)$ denote the number of solutions of

$$\begin{cases} n = a_1 + a_2 + \cdots + a_k \\ p = b_1 + b_2 + \cdots + b_k \end{cases} \tag{2.1}$$

in nonnegative a_i, b_i such that

$$(a_i, b_i) \neq (a_{i+1}, b_{i+1}) \quad (i = 1, 2, \ldots, k - 1); \tag{2.2}$$

let $c(n,p)$ denote the corresponding enumerant when k is unrestricted. For given nonnegative a, b, let $c_{a,b}(n, p, k)$ denote the number of solutions of (2.1) and (2.2) with $a_1 = a$, $b_1 = b$.

Clearly

$$c(n,p,k) = \sum_{a,b} c_{a,b}(n,p,k).$$

It is convenient to define $c(n,p,k)$ and $c_{a,b}(n,p,k)$, $k = 0$, as follows:

$$c(n,p,0) = \begin{cases} 1 & (n = p = 0) \\ 0 & (\text{otherwise}) \end{cases} \tag{2.4}$$

and

(2.4)′ $$c_{a,b}(n,p,0) = \begin{cases} 1 & (n = p = a = b = 0) \\ 0 & (\text{otherwise}). \end{cases}$$

It follows at once from the definitions that

(2.5) $$c_{a,b}(n,p,k) = \sum_{(r,s) \neq (a,b)} c_{r,s}(n - a, p - b, k - 1) \qquad (k > 1).$$

Note that (2.5) holds for $k = 1$ except when $n = p = a = b = 0$.

Generating functions $C_{a,b}(x,y,k)$ and $\Phi_k(x,y,u,v)$ are defined by

(2.6) $$C_{a,b}(x,y,k) = \sum_{n,p=0}^{\infty} c_{a,b}(n,p,k)x^n y^p \qquad (k \geq 0)$$

and

(2.7) $$\Phi_k(x,y,u,v) = \sum_{a,b=0}^{\infty} c_{a,b}(x,y,k)u^a v^b \qquad (k \geq 0).$$

It follows from (2.4)′ that

(2.8) $$C_{a,b}(x,y,0) = \begin{cases} 1 & (a = b = 0) \\ 0 & (\text{otherwise}) \end{cases}$$

and

(2.9) $$\Phi_0(x,y,u,v) = 1.$$

In the next place, by (2.5) and (2.6), we have for $k > 1$,

$$\begin{aligned} C_{a,b}(x,y,k) &= \sum_{n,p=0}^{\infty} x^n y^p \sum_{(r,s) \neq (a,b)} c_{r,s}(n - a, p - b, k - 1) \\ &= x^a y^b \sum_{n,p=0}^{\infty} x^n y^p \left\{ \sum_{r,s} c_{r,s}(n,p,k - 1) - c_{a,b}(n,p,k - 1) \right\} \\ &= x^a y^b \sum_{n,p=0}^{\infty} x^n y^p \left\{ c(n,p,k - 1) - c_{a,b}(n,p,k - 1) \right\}. \end{aligned}$$

Hence,

(2.10) $$C_{a,b}(x,y,k) = x^a y^b \left\{ C(x,y,k - 1) - C_{a,b}(x,y,k - 1) \right\} \qquad (k > 1),$$

where

(2.11) $$C(x,y,k) = \sum_{a,b=0}^{\infty} C_{a,b}(x,y,k) = \sum_{n,p=0}^{\infty} c(n,p,k)x^n y^p.$$

Thus, (2.10) yields

$$\sum_{a,b=0}^{\infty} C_{a,b}(x,y,k)u^a v^b = C(x,y,k - 1) \sum_{a,b=0}^{\infty} (xu)^a (yv)^b$$

$$- \sum_{a,b=0}^{\infty} F_{a,b}(x,y,k - 1)(xu)^a (yv)^b$$

so that, by (2.7), for $k > 1$,

(2.12) $$\Phi_k(x,y,u,v) = \frac{1}{1 - xu} \frac{1}{1 - yv} \Phi_{k-1}(x,y,1,1) - \Phi_{k-1}(x,y,xu,yv).$$

Iteration gives

$$\Phi_k(x,y,u,v) = \frac{1}{1-xu}\frac{1}{1-yv}\Phi_{k-1}(x,y,1,1) - \frac{1}{1-x^2u}\frac{1}{1-y^2v}\Phi_{k-2}(x,y,1,1) + \Phi_{k-2}(x,y,x^2u,y^2v) \quad (k > 2),$$

and generally

$$\Phi_k(x,y,u,v) = \sum_{j=1}^{s}\frac{(-1)^{j-1}}{(1-x^ju)(1-y^jv)}\Phi_{k-j}(x,y,1,1) + (-1)^s\Phi_{k-s}(x,y,x^su,y^sv) \quad (k > s).$$

In particular, for $s = k - 1$, this becomes

$$\Phi_k(x,y,u,v) = \sum_{j=1}^{k-1}\frac{(-1)^{j-1}}{(1-x^ju)(1-y^jv)}\Phi_{k-j}(x,y,1,1) + (-1)^{k-1}\Phi_1(x,y,x^{k-1}u,y^{k-1}v). \tag{2.13}$$

We have

$$\Phi_1(x,y,u,v) = \sum_{n,p=0}^{\infty}\sum_{a,b} c_{a,b}(n,p,1)x^ny^pu^av^b = \frac{1}{(1-xu)(1-yv)}$$

and (2.13) becomes

$$\Phi_k(x,y,u,v) = \sum_{j=1}^{k}\frac{(-1)^{j-1}}{(1-x^ju)(1-y^jv)}\Phi_{k-j}(x,y,1,1) \quad (k \geq 1). \tag{2.14}$$

In particular, for $u = v = 1$, (2.14) reduces to

$$\Phi_k(x,y,1,1) + \sum_{j=1}^{k}\frac{(-1)^j}{(1-x^j)(1-y^j)}\Phi_{k-j}(x,y,1,1) = \delta_{k,0}. \tag{2.15}$$

It follows from (2.15) that

$$C(x,y,z) = \left\{1 + \sum_{j=1}^{\infty}\frac{(-1)^jz^j}{(1-x^j)(1-y^j)}\right\}^{-1} \tag{2.16}$$

where $C(x,y,z)$ is defined by (2.11).

Returning to (2.14), we have

$$\sum_{k=1}^{\infty}\Phi_k(x,y,u,v)z^k = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}z^j}{(1-x^ju)(1-y^jv)}\sum_{k=0}^{\infty}\Phi_k(x,y,1,1)z^k,$$

and therefore

$$\sum_{k=1}^{\infty}\Phi_k(x,y,u,v)z^k = \frac{\displaystyle\sum_{j=1}^{\infty}\frac{(-1)^{j-1}z^j}{(1-x^ju)(1-y^jv)}}{\displaystyle 1 + \sum_{j=1}^{\infty}\frac{(-1)^jz^j}{(1-x^j)(1-y^j)}} \tag{2.17}$$

Note that the L.H.S. of (2.16) is

$$(2.16)' \qquad \sum_{n,p,k=0}^{\infty} c(n,p,k)x^n y^p z^k;$$

the L.H.S. of (2.17) is

$$(2.17)' \qquad \sum_{n,p,a,b,k=0}^{\infty} c_{a,b}\,(n,p,k)x^n y^p u^a v^b z^k.$$

Also, it can be shown (compare [1, §5]) that

$$(2.18) \qquad \sum_{n,p=0}^{\infty} c(n,p)x^n y^p = \left\{1 - \sum_{j=1}^{\infty} \frac{x^{2j-1}(1-x) + y^{2j-1}(1-y) - (xy)^{2p-1}(1-xy)}{(1-x^{2j-1})(1-x^{2j})(1-y^{2j-1})(1-y^{2j})}\right\}^{-1}$$

for $|x| < A$, $|y| < A$, where $A \geq \frac{1}{8}$.

SECTION 3

We shall now discuss the problem of enumerating the multipartite compositions that satisfy (1.6). We again take $t = 2$. Let $f(n,p,k)$ denote the number of solutions of

$$(3.1) \qquad \begin{cases} n = a_1 + a_2 + \cdots + a_k \\ p = b_1 + b_2 + \cdots + b_k \end{cases}$$

in nonnegative a_s, b_s such that

$$(3.2) \qquad (a_s, b_s) \not\equiv (a_{s+1}, b_{s+1}) \pmod{m} \quad (s = 1, 2, \ldots, k-1).$$

Let $f_{i,j}\,(n,p,k)$, for $0 \leq i < m$, $0 \leq j < m$, denote the number of solutions of (3.1) and (3.2) that also satisfy

$$(3.3) \qquad a_1 \equiv i, \ b_1 \equiv j \pmod{m}.$$

Finally, let $f_{i,j}\,(n,p,k,a,b)$ denote the number of solutions of (3.1), (3.2), and (3.3) with $a_1 = a$, $b_1 = b$. Thus $f_{i,j}\,(n,p,k,a,b) = 0$ unless $a \equiv i$, $b \equiv j \pmod{m}$.

It is convenient to extend the definitions to include the case $k = 0$. We define

$$(3.4) \qquad f(n,p,0) = \delta_{n0}\delta_{p0}, \ f_{i,j}(n,p,0) = \delta_{i0}\delta_{j0}f(n,p,0)$$

and

$$(3.5) \qquad f_{i,j}\,(n,p,0,a,b) = \delta_{a0}\delta_{b0}f_{i,j}\,(n,p,0).$$

Thus $f(n,p,0) = 0$ unless $n = p = 0$, $f_{i,j}(n,p,0) = 0$ unless $n = p = i = j = 0$, $f_{i,j}\,(n,p,0,a,b) = 0$ unless $n = p = i = j = a = b = 0$.

It follows from the definition that

$$(3.6) \qquad f(n,p,k) = \sum_{i,j=0}^{m-1} f_{i,j}\,(n,p,k)$$

$$= \sum_{i,j=0}^{m-1} \sum_{a=0}^{n} \sum_{b=0}^{p} f_{i,j}\ (n,p,k,a,b) \quad (n \geq 0,\ p \geq 0,\ k \geq 0).$$

Moreover, we have the recurrence

$$f_{i,j}\ (n,p,k,a,b) = \sum_{\substack{i',j'=0 \\ (i',j') \neq (i,j)}}^{m-1} \sum_{a=0}^{n} \sum_{b=0}^{p} f_{i',j'}\ (n,p,k,a,b)$$

$$[k > 0,\ a \equiv i,\ b \equiv j \pmod{m}].$$

This reduces to

$$f_{i,j}\ (n,p,k,a,b) = \sum_{\substack{i',j'=0 \\ (i',j') \neq (i,j)}}^{m-1} f_{i',j'}\ (n-a,\ p-b,\ k-1) \tag{3.7}$$

$$[k > 0,\ a \equiv i,\ b \equiv j \pmod{m}].$$

Corresponding to the enumerants, we define a number of generating functions:

$$\begin{cases} F_{i,j}\ (x,y,z) = \displaystyle\sum_{n,p,k=0}^{\infty} f_{i,j}\ (n,p,k)x^n y^p z^k \\ F(x,y,z) = \displaystyle\sum_{n,p,k=0}^{\infty} f(n,p,k)x^n y^p z^k \\ F_{i,j}\ (x,y,z,a,b) = \displaystyle\sum_{n,p,k=0}^{\infty} f_{i,j}\ (n,p,k,a,b)x^n y^p z^k. \end{cases}$$

Since

$$\begin{cases} f_{0,0}(n,p,1,a,b) = \delta_{na}\delta_{pb} \qquad [a \equiv b \equiv 0 \pmod{m}] \\ f_{0,0}(n,p,0,a,b) = \delta_{na}\delta_{pb}\delta_{n0}\delta_{p0}, \end{cases}$$

it follows that

$$F_{0,0}(x,y,z,a,b) = \delta_{a0}\delta_{b0} + x^a y^b z + x^a y^b z \sum_{(i,j) \neq (0,0)} F_{i,j}\ (x,y,z)$$

$$[a \equiv b \equiv 0 \pmod{m}].$$

Summing over a and b, we get

$$F_{0,0}(x,y,z) \tag{3.8}$$

$$= 1 + \frac{z}{(1-x^m)(1-y^m)} + \frac{z}{(1-x^m)(1-y^m)} \sum_{(i,j) \neq (0,0)} F_{i,j}\ (x,y,z)$$

On the other hand, for $(i,j) \neq (0,1)$ and $a \equiv i,\ b \equiv j \pmod{m}$, it follows from (3.7) that

$$F_{i,j}(x,y,z,a,b) = \sum_{n,\ ,k} x^n y^p z^k \sum_{(i',j') \neq (i,j)} f_{i',j'}\ (n-a,\ p-b,\ k-1)$$

$$= x^a y^b z \sum_{(i',j') \neq (i,j)} F_{i',j'}(x,y,z).$$

Hence, summing over a and b, we get

$$F_{i,j}(x,y,z) = \frac{x^i y^j z}{(1-x^m)(1-y^m)} \sum_{(i',j') \neq (i,j)} F_{i,j}(x,y,z) \tag{3.9}$$
$$[(i,j) \neq (0,0)].$$

Since

$$\sum_{(i',j') \neq (i,j)} F_{i',j'}(x,y,z) = F(x,y,z) - F_{i,j}(x,y,z),$$

(3.8) and (3.9) become

$$\left(1 + \frac{z}{(1-x^m)(1-y^m)}\right) F_{0,0}(x,y,z)$$
$$= 1 + \frac{z}{(1-x^m)(1-y^m)} + \frac{z}{(1-x^m)(1-y^m)} F(x,y,z)$$

and

$$\left(1 + \frac{x^i y^j z}{(1-x^m)(1-y^m)}\right) F_{i,j}(x,y,z)$$
$$= \frac{x^i y^j z}{(1-x^m)(1-y^m)} F(x,y,z) \qquad (i,j) \neq (0,0),$$

respectively. Hence,

$$\left\{\begin{aligned} F_{0,0}(x,y,z) &= 1 + \frac{\dfrac{z}{(1-x^m)(1-y^m)}}{1 + \dfrac{z}{(1-x^m)(1-y^m)}} F(x,y,z) \\ F_{i,j}(x,y,z) &= \frac{\dfrac{x^i y^j z}{(1-x^m)(1-y^m)}}{1 + \dfrac{x^i y^j z}{(1-x^m)(1-y^m)}} F(x,y,z) \end{aligned}\right. \qquad [(i,j) \neq (0,0)]. \tag{3.10}$$

Summing over the m^2 equations in (3.10), we get

$$\left\{1 - \sum_{i,j=0}^{m-1} \frac{\dfrac{x^i y^j z}{(1-x^m)(1-y^m)}}{1 + \dfrac{x^i y^j z}{(1-x^m)(1-y^m)}}\right\} F(x,y,z) = 1. \tag{3.11}$$

For brevity, put

$$\lambda = \frac{z}{(1-x^m)(1-y^m)}$$

so that (3.11) becomes

$$\left\{1 - \sum_{i,j=0}^{m-1} \frac{x^i y^j \lambda}{1 + x^i y^j \lambda}\right\} F(x,y,z) = 1. \tag{3.12}$$

Let

$$P_m(\lambda) = P_m(\lambda, x, y) = \prod_{i,j=0}^{m-1} (1 + x^i y^j \lambda); \tag{3.13}$$

clearly $P_m(\lambda)$ is a polynomial in λ of degree m^2. By logarithmic differentiation

$$\frac{\lambda P_m'(\lambda)}{P_m(\lambda)} = \sum_{i,j=0}^{m-1} \frac{x^i y^j \lambda}{1 + x^i y^j \lambda}.$$

Thus (3.12) becomes

$$F(x,y,z) = \frac{P_m(\lambda)}{Q_m(\lambda)} \qquad \lambda = \frac{z}{(1 - x^m)(1 - y^m)}, \tag{3.14}$$

where

$$Q_m(\lambda) = P_m(\lambda) - P_m'(\lambda). \tag{3.15}$$

For example, for $m = 2$,

$$\begin{cases} P_2(\lambda) = 1 + (1 + x)(1 + y)\lambda + (x + y + 2xy + x^2y + xy^2)\lambda^2 \\ \qquad\qquad + xy(1 + x)(1 + y)\lambda^3 + x^2y^2\lambda^4 \\ Q_2(\lambda) = 1 - (x + y + 2xy + x^2y + xy^2)\lambda^2 \\ \qquad\qquad - 2xy(1 + x)(1 + y)\lambda^3 - 3\lambda^4. \end{cases}$$

SECTION 4

As in [2], the limiting case, $m = \infty$ of $f(n,p,k)$, is closely related to $c(n,p,k)$. We assume $|x| < 1$, $|y| < 1$, so that

$$\lambda = \frac{z}{(1 - x^m)(1 - y^m)} \to z \quad (m \to \infty).$$

Thus, (3.12) becomes

$$\left\{1 - \sum_{i,j=0}^{\infty} \frac{x^i y^j z}{1 + x^i y^j z}\right\} F^*(x,y,z) = 1, \tag{4.1}$$

where

$$F^*(x,y,z) = \lim F(x,y,z).$$

Now

$$\sum_{i,j=0}^{\infty} \frac{x^i y^j z}{1 + x^i y^j z} = \sum_{i,j=0}^{\infty} \sum_{s=1}^{\infty} (-1)^{s-1} x^{is} y^{js} z^s$$

$$= \sum_{s=1}^{\infty} (-1)^{s-1} \frac{z^s}{(1 - x^s)(1 - y^s)}.$$

Hence, we may replace (4.1) by

$$\left\{1 + \sum_{s-1}^{\infty} (-1)^s \frac{z^s}{(1 - x^s)(1 - y^s)}\right\} F^*(x,y,z) = 1. \tag{4.2}$$

Comparing (4.2) with (2.16) and (2.16)′, it follows at once that

$$f^*(n,p,k) = c(n,p,k), \tag{4.3}$$

where $f^*(n,p,k)$ is the limiting case $(m = \infty)$ of $f(n,p,k)$; (4.3) is of course to be expected from the definitions.

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THE RECURRENCE RELATION $(r + 1)f_{r+1} = xf_r' + (K - r + 1)x^2 f_{r-1}$

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1. INTRODUCTION

In a recent note, in [3], Worster conjectured, on the basis of computer calculations, that for each positive integer k there exists an odd polynomial $Q_{2k-1}(x)$ of degree $2k - 1$ such that, for every zero a of the Bessel function $J_0(x)$

$$\int_0^a Q_{2k-1}(x)[J_0(x)]^{2k}\,dx = [aJ_1(a)]^{2k}.$$

The conjecture was extended and proved in [1] the extended result being: for each positive k there exists an odd polynomial $Q(x)$, with nonnegative integer coefficients and of degree k or $k - 1$ according to whether k is odd or even, such that for every zero a of $J_0(x)$

$$\int_0^a Q(x)[J_0(x)]^k\,dx = (k - 1)![aJ_1(a)]^k. \tag{1.1}$$

If the factor $(k - 1)!$ on the right-hand side is omitted, then the coefficients in $Q(x)$ are no longer integers. In addition, [1] also contained the following generalization due to Hammersley: if F_0, F_1, G_0, and G_1 are four functions of x such that

$$G_0\frac{dF_0}{dx} = -F_1, \qquad \frac{dF_1}{dx} = G_1F_0,$$

and $F_0(a) = G_0(0) = 0$, so that $F_1(0) = 0$,

then there exists $Q(x)$ depending only on G_0, G_1, and K with the property