ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate, signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-317 *Proposed by Lawrence Somer, Washington, D.C.*

Let $\{G_n\}_{n=0}^{\infty}$ be any generalized Fibonacci sequence such that

$$G_{n+2} = G_{n+1} + G_n, \ (G_0, G_1) = 1,$$

and $\{G_n\}$ is not a translation of the Fibonacci sequence. Show that there exists at least one prime p such that both

$$G_n + G_{n+1} \equiv G_{n+2} \pmod{p}$$

and

$$G_{n+1} \equiv rG_n \pmod{p}$$

for a fixed $r \not\equiv 0 \pmod{p}$ and for all $n \geq 0$.

H-318 *Proposed by James Propp, Harvard College, Cambridge, Mass.*

Define the sequence operator M so that for any infinite sequence $\{u_i\}$,

$$M(u_n) = M(u_n) - \sum_{i|n} M(u_i)\mu\left(\frac{n}{1}\right),$$

where is the Möbius function. Let the "Möbinacci Sequence" S be defined so that $S_1 = 1$ and

$$S_n = M(S_n) + M(M(S_n)), \text{ for } n > 1.$$

Find a formula for S_n in terms of the prime factorization of n.

H-319 *Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, CA.*

If $F_n < x < F_{n+1} < y < F_{n+2}$, then $x + y$ is never a Fibonacci number.

2 Corrected Problem Proposals

H-294 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Evaluate

$$\Delta = \begin{vmatrix} F_{2r+1} & F_{6r+3} & F_{10r+5} & F_{14r+7} & F_{18r+9} \\ F_{4r+2} & -F_{12r+6} & F_{20r+10} & -F_{28r+14} & F_{36r+18} \\ F_{6r+3} & F_{18r+9} & F_{30r+15} & F_{42r+21} & F_{54r+27} \\ F_{8r+4} & -F_{24r+12} & F_{40r+20} & -F_{56r+28} & F_{72r+36} \\ F_{10r+5} & F_{30r+15} & F_{50r+25} & F_{70r+35} & F_{90r+45} \end{vmatrix}$$

H-295 *Proposed by G. Wulczyn, Bucknell University, Lewisburg, PA.*

Establish the identities:

(a) $F_k F^2_{k+6r+3} - F_{k+8r+4}F^2_{k+2r+1} = (-1)^{k+1}F^3_{2r+1}L_{2r+1}L_{k+4r+2}$;

(b) $F_k F^2_{k+6r} - F_{k+8r}F^2_{k+2r} = (-1)^{k+1}F^3_{2r}L_{2r}L_{k+4r}$.

SOLUTIONS

One or Five

H-285 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA. (Vol. 16, No. 5, October 1978)*

Consider two sequences $\{H_n\}_{n=1}^{\infty}$ and $\{G_n\}_{n=1}^{\infty}$ such that

(a) $(H_n, H_{n+1}) = 1$,
(b) $(G_n, G_{n+1}) = 1$,
(c) $H_{n+2} = H_{n+1} + H_n$ $(n \geq 1)$, and
(d) $H_{n+1} + H_{n-1} = sG_n$ $(n \geq 1)$,
where s is independent of n.

Show $s = 1$ or $s = 5$.

Solution by Lawrence Somer, Washington, D.C.

The following examples from the Fibonacci and Lucas sequences show that s may actually attain both values of 1 and 5:

$$F_{n-1} + F_{n+1} = 1 \cdot L_n,\ L_{n-1} + L_{n+1} = 5F_n.$$

We are also evidently assuming that s is nonnegative. Otherwise, let

$$\{H_n\} = \{-F_n\} \quad \text{and} \quad \{G_n\} = \{L_n\}.$$

Then $H_{n-1} + H_{n+1} = (-1)G_n$. Similarly, if

$$\{H_n\} = \{-L_n\} \quad \text{and} \quad \{G_n\} = \{F_n\},$$

then $H_{n-1} + H_{n+1} = (-5)G_n$.

Now suppose that $s \neq 1$ or 5. Since $(H_n, H_{n+1}) = 1$ and $(G_n, G_{n+1}) = 1$, clearly $s \neq 0$. I claim that the period (mod s) of $\{H_n\}$ divides 4. This follows, since $H_1 + H_3 \equiv 0 \pmod{s}$ and $H_3 + H_5 \equiv 0 \pmod{s}$ together imply that $H_1 \equiv H_5 \pmod{s}$. Similarly, $H_2 \equiv H_6 \pmod{s}$.

Now, $H_1 + H_3 \equiv 0 \pmod{s}$ and $H_1 + H_2 \equiv H_3 \pmod{s}$ imply that $H_2 \equiv -2H_1 \pmod{s}$. Thus, using the recursion relation for $\{H_n\}$, the first five terms of $\{H_n\}$ (mod s) are

$$H_1,\ H_2 \equiv -2H_1,\ H_3 \equiv -H_1,\ H_4 \equiv -3H_1, \text{ and } H_5 \equiv -4H_1.$$

Thus, $-4H_1 \equiv H_1$ or $5H_1 \equiv 0 \pmod{s}$. If $(5, s) = 1$, then $5H_1 \equiv 0 \pmod{s}$ implies that $H_1 \equiv 0 \pmod{s}$. But then $H_2 \equiv -2H_1 \equiv 0 \pmod{s}$ and $(H_1, H_2) \neq 1$. Hence, $s > 5$ and $(5, s) = 5$. However, then $5H_1 \equiv 0 \pmod{s}$ implies that $(s/5)|(H_1, s)$. But then since $H_2 \equiv -2H_1 \pmod{s}$ and a fortiori $H_2 \equiv -2H_1 \equiv 0 \pmod{s/5}$, $(s/5)|H_2$ also. Therefore, $(s/5)|(H_1, H_2)$ and $(H_1, H_2) \neq 1$ as we assumed. Thus, $s = 1$ or 5.

Also solved by P. Bruckman and G. Lord.

Power Mod

H-286 *Proposed by P. Bruckman, Concord, CA. (Vol. 16, No. 5, October 1978)*

Prove the following congruences:

(1) $F_{5^n} \equiv 5^n \pmod{5^{n+3}}$;

(2) $F_{5^n} \equiv L_{5^{n+1}} \pmod{5^{2n+1}}$, $n = 0, 1, 2, \ldots$.

Solution by the proposer.

Proof of (1): We will use the following identity,

(3) $$F_{5m} = 25F_m^5 + 25(-1)^m F_m^3 + 5F_m, \; m = 0, 1, 2, \ldots .$$

Let S be the set of nonnegative integers n for which (1) holds. Since $F_5 = 5$, clearly $1 \in S$. Even more obviously, $F_1 = 1 = 5^0$, so $0 \in S$. Suppose $k \in S$, and let $m = 5^k$. Then, for some integer a, $F_m = m(1 + 125a)$. Hence, by (3),

$$F_{5m} = 5^2m^5(1 + 5^3a)^5 - 5^2m^3(1 + 5^3a)^3 + 5m(1 + 5^3a)$$
$$\equiv 5^2m^5 - 5^2m^3 + 5m \pmod{5^4m}.$$

But $5^2|m^2$, assuming k is positive. Hence, $5^4m|5^2m^3|5^2m^5$. Thus, $F_{5m} \equiv 5r$ $(\bmod\ 5^4m)$, i.e.,

$$F_{5^{k+1}} \equiv 5^{k+1} \pmod{5^{k+4}}.$$

Therefore, $k \in S \Rightarrow (k + 1) \in S$. The result of (1) now follows by induction.

Proof of (2): We will use the following identities,

$$\left.\begin{array}{lll} (4) & L_{5m} = L_m^5 - 5(-1)^m L_m^3 + 5L_m, & \\ (5) & L_m^2 = 5F_m^2 + 4(-1)^m, & \end{array}\right\} m = 0, 1, 2, \ldots .$$

Let $m = 5^n$. Then $L_{5m} - L_m = (L_m^3 + L_m)(L_m^2 + 4) = 5F_m^2(L_m^3 + L_m)$. But, by (1), $m|F_m$, which implies $5m^2 | 5F_m^2$. Therefore, $L_{5m} \equiv L_m (\bmod\ 5m^2)$, i.e.,

$$L_{5^{n+1}} \equiv L_{5^n} \pmod{5^{2n+1}},$$

which proves (2).

More Identities

H-288 *Proposed by G. Wulczyn, Bucknell University, Lewisburg, PA.*
(Vol. 16, No. 5, October 1978)

Establish the identities:

(a) $F_k L_{k+6r+3}^2 - F_{k+8r+4} L_{k+2r+1}^2 = (-1)^{k+1} L_{2r+1}^3 F_{2r+1} L_{k+4r+2}$;

(b) $F_k L_{k+6r}^2 - F_{k+8r} L_{k+2r}^2 = (-1)^{k+1} L_{2r}^3 F_{2r} L_{k+4r}$.

Solution by the Proposer

(a) $F_k L_{k+6r+3}^2 - F_{k+8r+4} L_{k+2r+1}^2 =$

$$= \frac{1}{\sqrt{5}}\{(\alpha^k - \beta^k)[\alpha^{2k+12r+6} + \beta^{2k+12r+6} + 2(-1)^{k+1}]$$
$$- (\alpha^{k+8r+4} - \beta^{k+8r+4})[\alpha^{2k+4r+2} + \beta^{2k+4r+2} + 2(-1)^{k+1}]\}$$

$$= \frac{(-1)^{k+1}}{\sqrt{5}}\{\alpha^{k-4r-2}(\alpha^{16r+8} - 2\alpha^{12r+6} + 2\alpha^{4r+2} - 1)$$
$$- \beta^{k-4r-2}(\beta^{16r+8} - 2\beta^{12r+6} + 2\beta^{4r+2} - 1)\}$$

$$= \frac{(-1)^{k+1}}{\sqrt{5}}\{\alpha^{k-4r-2}(\alpha^{4r+2} - 1)(\alpha^{4r+2} + 1) - \beta^{k-4r-2}(\beta^{4r+2} - 1)(\beta^{4r+2} + 1)\}$$

$$= \frac{(-1)^{k+1}}{\sqrt{5}}\{\alpha^{k+4r+2}(\alpha^{2r+1} + \beta^{2r+1})^3(\alpha^{2r+1} - \beta^{2r+1})$$
$$+ \beta^{k+4r+2}(\alpha^{2r+1} + \beta^{2r+1})^3(\alpha^{2r+1} - \beta^{2r+1})\}$$

$$= (-1)^{k+1} L_{2r+1}^3 F_{2r+1} L_{k+4r+2}.$$

(b) $F_k L^2_{k+6r} - F_{k+8r} L^2_{k+2r}$

$= F_k [L_{2k+12r} + 2(-1)^k] - F_{k+8r} [L_{2k+4r} + 2(-1)^k]$

$$= \frac{(-1)^{k+1}}{\sqrt{5}}\{\alpha^{k-4r}(\alpha^{16r} + 2\alpha^{12r} - 2\alpha^{4r} - 1) - \beta^{k-4r}(\beta^{16r} + 2\beta^{12r} - 2\beta^{4r} - 1)\}$$

$$= \frac{(-1)^{k+1}}{\sqrt{5}}\{\alpha^{k-4r}(\alpha^{4r} - 1)(\alpha^{4r} + 1)^3 - \beta^{k-4r}(\beta^{4r} - 1)(\beta^{4r} + 1)^3\}$$

$$= \frac{(-1)^{k+1}}{\sqrt{5}}\{\alpha^{k+4r}(\alpha^{2r} - \beta^{2r})(\alpha^{2r} + \beta^{2r})^3 + \beta^{k+4r}(\alpha^{2r} - \beta^{2r})(\alpha^{2r} + \beta^{2r})^3\}$$

$= (-1)^{k+1} F_{2r} L^3_{2r} L_{k+4r}$.

Also solved by P. Bruckman.

Series Consideration

H-289 *Proposed by L. Carlitz, Duke University, Durham, N.C.*
(Vol. 16, No. 5, October 1978)

Put the multinomial coefficient

$$(m_1, m_2, \ldots, m_k) = \frac{(m_1 + m_2 + \cdots + m_k)!}{m_1!m_2! \ldots m_k!}.$$

Show that

$$(*) \quad \sum_{r+s+t=\lambda} (r, s, t)(m-2r, n-2s, p-2t)$$

$$= \sum_{i+j+k+u=\lambda} (-2)^{i+j+k}(i, j, k, u)(m-j-k, n-k-i, p-i-j) \quad (m+n+p \geq 2\lambda).$$

Solution by Paul Bruckman, Concord, CA.

Let

$$(1) \qquad A(m, n, p) = \sum_{r+s+t=\lambda} (r, s, t)(m-2r, n-2s, p-2t),$$

$$(2) \qquad B(m, n, p) = \sum_{i+j+k+u=\lambda} (-2)^{i+j+k}(i, j, k, u)(m-j-k, n-k-i, p-i-j).$$

Also, let

$$(3) \qquad F(x, y, z) = \sum_{m+n+p\geq 2\lambda} A(m, n, p)x^m y^n z^p,$$

$$(4) \qquad G(x, y, z) = \sum_{m+n+p\geq 2\lambda} B(m, n, p)x^m y^n z^p,$$

assuming λ is fixed. It will suffice to show that F and G are identical functions, for then the desired result would follow by comparing coefficients.

Now

$$F(x, y, z) = \sum_{r+s+t=\lambda} (r, s, t) \sum_{m+n+p \geq 2\lambda} (m - 2r, n - 2s, p - 2t)x^m y^n z^p$$

$$= \sum_{r+s+t=\lambda} (r, s, t) \sum_{m \geq 2r, n \geq 2s, p \geq 2t} (m - 2r, n - 2s, p - 2t)x^m y^n z^p$$

$$= \sum_{r+s+t=\lambda} (r, s, t)x^{2r} y^{2s} z^{2t} \sum_{m, n, p \geq 0} (m, n, p)x^m y^n z^p.$$

Now

$$\sum_{m,n,p \geq 0} (m, n, p)x^m y^n z^p = \sum_{k=0}^{\infty} \sum_{m+n+p=k} (m, n, p)x^m y^n z^p$$

$$= \sum_{k=0}^{\infty} (x + y + z)^k = (1 - x - y - z)^{-1}.$$

Hence,

$$F(x, y, z) = (1 - x - y - z)^{-1} \sum_{r+s+t=\lambda} (r, s, t)x^{2r} y^{2s} z^{2t},$$

or

(5) $$F(x, y, z) = (x^2 + y^2 + z^2)^{\lambda}(1 - x - y - z)^{-1}.$$

Also,

$$G(x, y, z) = \sum_{i+j+k+u=\lambda} (-2)^{i+j+k}(i, j, k, u)$$

$$\cdot \sum_{m+n+p \geq 2\lambda} (m - j - k, n - k - i, p - i - j)x^m y^n z^p.$$

The condition $m + n + p \geq 2\lambda$ is equivalent to

$$(m - j - k) + (n - k - i) + (p - i - j) \geq 2(\lambda - i - j - k) = 2u.$$

Hence,

$$G(x, y\ \ z) = \sum_{i+j+k+u=\lambda} (-2)^{i+j+k}(i, j, k, u)\, x^{j+k} y^{k+i} z^{i+j}$$

$$\cdot \sum_{m+n+p \geq 2u} (m, n, p)x^m y^n z^p$$

$$= \sum_{i+j+k+u=\lambda} (-2)^{i+j+k}(i, j, k, u)x^{j+k} y^{k+i} z^{i+j}$$

$$\cdot \sum_{h=2u}^{\infty} \sum_{m+n+p=h} (m, n, p)x^m y^n z^p$$

$$= \sum_{i+j+k+u=\lambda} (-2)^{i+j+k}(i, j, k, u)x^{j+k} y^{k+i} z^{i+j} \sum_{h=2u}^{\infty} (x+y+z)^h$$

$$= (1 - x - y - z)^{-1} \sum_{i+j+k+u=\lambda} (-2)^{i+j+k}$$

$$\cdot (i, j, k, u)x^{j+k} y^{k+i} z^{i+j}(x + y + z)^{2u}$$

$$= (1 - x - y - z)^{-1} \sum_{i+j+k+u=\lambda} (-2yz)^i (-2xz)^j (-2xy)^k (x + y + z)^{2u} (i, j, k, u)$$

$$= (1 - x - y - z)^{-1}\{-2yz - 2xz - 2xy + (x + y + z)^2\}^\lambda$$

$$= (1 - x - y - z)^{-1}(x^2 + y^2 + z^2)^\lambda = F(x, y, z). \quad \text{Q.E.D.}$$

Also solved by the proposer.

Identical

H-290 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA. (Vol. 16, No. 6, December 1978)*

Show that

(a) $F_k F^2_{k+6r+3} - F^3_{k+4r+2} = (-1)^{k+1} F^2_{2r+1} (F_{k+8r+4} - 2F_{k+4r+2})$;

(b) $F_k F^2_{k+6r} - F^3_{k+4r} = (-1)^{k+1} F^2_{2r} (F_{k+8r} + 2F_{k+4r})$.

Solution by the proposer.

(a) $F_k F^2_{k+6r+3} - F^3_{k+4r+2}$

$$= \frac{1}{5\sqrt{5}}\{(\alpha^k - \beta^k)[\alpha^{2k+2r+6} + \beta^{2k+2r+6} + 2(-1)^k] - [\alpha^{3k+2r+6} - \beta^{3k+2r+6} + 3(-1)^{k+1}]\}$$

$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\{\alpha^k(\alpha^{12r+6} - 3\alpha^{4r+2} - 2) - \beta^k(\beta^{12r+6} - 3\beta^{4r+2} - 2)\}$$

$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\{\alpha^k(\alpha^{4r+2} + 1)^2(\alpha^{4r+2} - 2) - \beta^k(\beta^{4r+2} + 1)(\alpha^{4r+2} - 2)\}$$

$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\{\alpha^{k+4r+2}(\alpha^{2r+1} - \beta^{2r+1})^2(\alpha^{4r+2} - 2) - \beta^{k+4r+2}(\alpha^{2r+1} - \beta^{2r+1})^2(\beta^{4r+2} - 2)\}$$

$$= (-1)^{k+1}F^2_{2r+1}(F_{k+8r+4} - 2F_{k+4r+2}).$$

(b) $F_k F^2_{k+6r} - F^3_{k+4r}$

$$= \frac{1}{5\sqrt{5}}\{(\alpha^k - \beta^k)[\alpha^{2k+12r} + \beta^{2k+12r} + 2(-1)^{k+1}]$$
$$- [\alpha^{3k+12r} - \beta^{3k+12r} + 3(-1)^{k+1}(\alpha^{k+4r} - \beta^{k+4r})]\}$$

$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\{\alpha^k(\alpha^{12r} - 3\alpha^{4r} + 2) - \beta^k(\beta^{12r} - 3\beta^{4r} + 2)\}$$

$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\{\alpha^k(\alpha^{4r} - 1)^2(\alpha^{4r} + 2) - \beta^k(\beta^{4r} - 1)(\beta^{4r} + 2)\}$$

$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\{\alpha^{k+4r}(\alpha^{2r} - \beta^{2r})^2(\alpha^{4r} + 2) - \beta^{k+4r}(\alpha^{2r} - \beta^{2r})^2(\beta^{4r} + 2)\}$$

$$= (-1)^{k+1}F^2_{2r}(F_{k+8r} + 2F_{k+4r})$$

Also solved by P. Bruckman.

Square Your Cubes

H-291 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX. (Vol. 16, No. 6, December 1978)*

Prove that there are infinitely many squares which are differences of consecutive cubes.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI.

Clearly, it suffices to show that the equation $(x + 1)^3 - x^3 = y^2$ has infinitely many solutions (x, y) where x and y are positive integers. But the preceding equation is equivalent to $(2y)^2 - 3(2x + 1)^2 = 1$. Hence, we need only determine the solutions of the Pell's equation $u^2 - 3v^2 = 1$ in positive integers u, v such that u is even and v is odd. Its least solution in positive integers is $u_0 = 2$, $v_0 = 1$. Thus, all of its positive integer solutions are contained in the infinite sequence (u_k, v_k), $k = 1, 2, \ldots$, where

$$u_{k+1} = 2u_k + 3v_k \quad \text{and} \quad v_{k+1} = u_k + 2v_k,\ k = 0, 1, 2, \ldots .$$

[The preceding is an immediate consequence of the following result which is generally established as part of the theory involving Pell's equation: All of the solutions of the equation $u^2 - Dv^2 = 1$ in positive integers are contained in the infinite sequence

$$(u_0, v_0), (u_1, v_1), (u_2, v_2), \ldots,$$

where (u_0, v_0) is the least positive integer solution and (u_k, v_k) is defined inductively by $u_{k+1} = u_0u_k + Dv_0v_k$, $v_{k+1} = v_0u_k + u_0v_k$, $k = 1, 2, \ldots$.]

It is easily seen that, if u_k is even and v_k is odd, then u_{k+1} is odd and v_{k+1} is even. Also, if u_k is odd and v_k is even, then u_{k+1} is even and v_{k+1} is odd. This implies that all of the solutions of the equation

$$u^2 - 3v^2 = 1$$

in positive integers u, v with u even and v odd are (u_{2k}, v_{2k}) where $k = 0, 1, 2, \ldots$. Therefore, the equation $(x + 1)^3 - x^3 = y^2$ has infinitely many positive integer solutions.

Also solved by H. Klauser, P. Bruckman, E. Starke, L. Somer, G. Wulczyn, W. Brady, S. Singh, G. Chainbus, and the proposer.

Get the Point

H-292 *Proposed by F. S. Cater and J. Daily, Portland State University, Portland, OR. (Vol. 16, No. 6, December 1978).*

Find all real numbers $r \in (0, 1)$ for which there exists a one-to-one function f_r mapping $(0, 1)$ onto $(0, 1)$ such that

(1) f_r and f_r^{-1} are infinitely many times differentiable on $(0, 1)$, and

(2) the sequence of functions

$$f_r,\ f_r \circ f_r,\ f_r \circ f_r \circ f_r,\ f_r \circ f_r \circ f_r \circ f_r,\ \ldots$$

converges pointwise to r on $(0, 1)$.

Solution by the proposers.

Let q denote the golden ratio $\frac{1}{2}(-1 + \sqrt{5})$, let $f(x) = 1 - (1 - x^2)^2$ and $g(x) = f(x) - x$. Then $f(q) - q = g(q) = 0$ by inspection and $g''(x) = -12x^2 + 4$

changes sign once in $(0, 1)$, from positive to negative. Since $g(0) = g(1) = 0$, it follows that $g(x) < 0$ for $0 < x < q$ and $g(x) > 0$ for $q < x < 1$. Also f and f^{-1} are evidently increasing on $(0, 1)$, so for any $x \in (0, q)$,

$$x < f^{-1}(x) < (f^{-1} \circ f^{-1})(x) < (f^{-1} \circ f^{-1} \circ f^{-1})(x) < \cdots < q,$$

and for $x \in (q, 1)$,

$$x > f^{-1}(x) > (f^{-1} \circ f^{-1})(x) > (f^{-1} \circ f^{-1} \circ f^{-1})(x) > \cdots > q.$$

In either case, this sequence converges to some point $w \in (0, 1)$. Since f^{-1} is continuous at w, $f^{-1}(w) = w$. But q is the only fixed point of f and of f^{-1} in $(0, 1)$, so $w = q$. Thus,

$$f^{-1},\ f^{-1} \circ f^{-1},\ f^{-1} \circ f^{-1} \circ f^{-1},\ \ldots$$

converges pointwise to q on $(0, 1)$. Also,

$$f^{-1}(x) = (1 - (1 - x)^{1/2})^{1/2},$$

so f and f^{-1} are both infinitely many times differentiable on $(0, 1)$. More generally, put $t = (\log q)/(\log r)$. Then, $f_r(x) = (f^{-1}(x^t))^{1/t}$ satisfies (1) and (2). Thus, all numbers $r \in (0, 1)$ satisfy the requirements of the problem.

Remark: Functions similar to f_r given here were studies by R. I. Jewett, in "A Variation on the Weierstrass Theorem," *PAMS* 14 (1963):690.

The Old Hermite

H-293 *Proposed by Leonard Carlitz, Duke University, Durham, N.C. (Vol. 16, No. 6, December 1978)*

It is known that the Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$ defined by

$$\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!} = e^{2xz - z^2}$$

satisfy the relation

$$\sum_{n=0}^{\infty} H_{n+k}(x)\frac{z^n}{n!} = e^{2xz - z^2} H_k(x - z),\ (k = 0, 1, 2, \ldots).$$

Show that, conversely, if a set of polynomials $\{f_n(x)\}_{n=0}^{\infty}$ satisfy

(1) $$\sum_{n=0}^{\infty} f_{n+k}(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty} f_n(x)\frac{z^n}{n!} f_k(x - z),\ (k = 0, 1, 2, \ldots),$$

where $f_0(x) = 1$, $f_1(x) = 2x$, then

$$f_n(x) = H_n(x),\ (n = 0, 1, 2, \ldots).$$

Solution by Paul F. Byrd, San Jose State University, San Jose, CA.

Let

(1) $$G(x, z) = \sum_{n=0}^{\infty} f_n(x)\frac{z^n}{n!},$$

$$\left[G(x, 0) = f_0(x) = 1,\ \left.\frac{\partial G}{\partial z}\right|_{z=0} = f_1(x) = 2x\right],$$

denote the generating function for the set of polynomials $\{f_n(x)\}$. Then the given relation can be written as

(2) $$\sum_{n=0}^{\infty} f_{n+k}(x)\frac{z^n}{n!} = G(x, z)f_k(x - z), \quad (k = 0, 1, 2, \ldots).$$

Multiplying this by $u^k/k!$ and summing yields

(3) $$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f_{n+k}(x)\frac{z^n u^k}{n!k!} = G(x, z)\sum_{k=0}^{\infty} f_k(x - z)\frac{u^k}{k!}.$$

Now with the use of Cauchy's product rule, the lefthand side of (3) becomes

(4) $$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f_{n+k}(x)\frac{z^n u^k}{n!k!} = \sum_{n=0}^{\infty} f_n(x) \sum_{k=0}^{n} \frac{z^{n-k}u^k}{k!(n-k)!}$$

$$= \sum_{n=0}^{\infty} f_n(x)\frac{(z+u)^n}{n!} = G(x, z + u).$$

But the righthand side of (3) is clearly equal to $G(x, z)G(x - z, u)$. Thus, from (3) and (4), we have the functional equation

(5) $$G(x, z + u) = G(x, z)G(x - z, u)$$

whose unique solution is

(6) $$G(x, z) = e^{2xz - z^2}, \text{ (for any value of } u).$$

But, this is precisely the same well-known generating function for the Hermite polynomials $H_n(x)$. Hence,

(7) $$e^{2xz - z^2} = \sum_{n=0}^{\infty} f_n(x)\frac{z^n}{n!},$$

and it follows from Taylor's theorem that

(8) $$f_n(x) = e^{x^2}\left[\frac{\partial^n}{\partial z^n} e^{-(x-z)^2}\right]_{z=0} = (-1)^n e^{x^2} \cdot \frac{d^n}{dx^n}(e^{x^2}) = H_n(x),$$

with $f_0(x) = 1 = H_0(x)$, $f_1(x) = 2x = H_1(x)$.

Also solved by P. Bruckman, T. Shannon, and the proposer.

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