6. CONCLUSION

We have proven a number-theoretical problem about a sequence, which is a computer-oriented type, but cannot be solved by any computer approach.

REFERENCE

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WEIGHTED STIRLING NUMBERS OF THE FIRST AND SECOND KIND—II

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1. INTRODUCTION

The Stirling numbers of the first and second kind can be defined by

$$(x)_n \equiv x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} S_1(n, k)x^k, \tag{1.1}$$

and

$$x^n = \sum_{k=0}^{n} S(n, k)x\cdot(x-1)\cdots(x-k+1), \tag{1.2}$$

respectively. In [6], the writer has defined *weighted* Stirling numbers of the first and second kind, $\overline{S}_1(n, k, \lambda)$ and $\overline{S}(n, k, \lambda)$, by making use of certain combinatorial properties of $S_1(n, k)$ and $S(n, k)$. Numerous properties of the generalized quantities were obtained.

The results are somewhat simpler for the related functions:

$$\begin{cases} R_1(n, k, \lambda) = \overline{S}_1(n, k+1, \lambda) + S_1(n, k) \\ R(n, k, \lambda) = \overline{S}(n, k+1, \lambda) + S(n, k). \end{cases} \tag{1.3}$$

In particular, the latter satisfy the recurrences,

$$\begin{cases} R_1(n, k, \lambda) = R_1(n, k-1, \lambda) + (n+\lambda)R_1(n, k, \lambda) \\ R(n, k, \lambda) = R(n, k-1, \lambda) + (k+\lambda)R(n, k, \lambda), \end{cases} \tag{1.4}$$

and the orthogonality relations

$$\begin{aligned} &\sum_{j=0}^{n} R(n, j, \lambda)\cdot(-1)^{j-k}R_1(j, k, \lambda) \\ &= \sum_{j=0}^{n} (-1)^{n-j}R_1(n, j, \lambda)R(j, k, \lambda) = \begin{cases} 1 & (n = k) \\ 0 & (n \neq k). \end{cases} \end{aligned} \tag{1.5}$$

We have also the generating functions

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} R_1(n, k, \lambda)y^k = (1-x)^{-\lambda-y}, \tag{1.6}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} R(n, k, \lambda) y^k = e^{\lambda x} \exp\{y(e^x - 1)\}, \tag{1.7}$$

and the explicit formula

$$R(n, k, \lambda) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (j + \lambda)^n. \tag{1.8}$$

Moreover, corresponding to (1.1) and (1.2), we have

$$(\lambda + y)^n = \sum_{k=0}^{n} R_1(n, k, \lambda) y^k \tag{1.9}$$

and

$$y^n = \sum_{k=0}^{n} (-1)^{n-k} R(n, k, \lambda)(y + \lambda)_k. \tag{1.10}$$

It is well known that the numbers $S_1(n, n-k)$, $S(n, n-k)$ are polynomials in n of degree $2k$. In [4] it is proved that

$$\begin{cases} S_1(n, n-k) = \displaystyle\sum_{j=1}^{k} B_1(k, j) \binom{n+j-1}{2k} \\ S(n, n-k) = \displaystyle\sum_{j=1}^{n} B(k, j) \binom{n+j-1}{2k} \end{cases} \quad (k \geq 1), \tag{1.11}$$

where $B_1(k, j)$, $B(k, j)$ are independent of n, and

$$B_1(k, j) = B(k, k-j+1), \quad (1 \leq j \leq k). \tag{1.12}$$

The representations (1.11) are applied in [4] to give new proofs of the known relations

$$\begin{cases} S(n, n-k) = \displaystyle\sum_{t=0}^{k} \binom{k+n}{k-t} \binom{k-n}{k+t} S_1(k+t, t) \\ S_1(n, n-k) = \displaystyle\sum_{t=0}^{k} \binom{k+n}{k-t} \binom{k-n}{k+t} S(k+t, t). \end{cases} \tag{1.13}$$

For references to (1.13), see [2], [7].

One of the principal objectives of the present paper is to generalize (1.11). The generalized functions $R_1(n, n-k, \lambda)$, $R(n, n-k, \lambda)$ are also polynomials in n of degree $2k$. We show that

$$\begin{cases} R_1(n, n-k, \lambda) = \displaystyle\sum_{j=0}^{k} B_1(k, j, \lambda) \binom{n+j}{2k} \\ R(n, n-k, \lambda) = \displaystyle\sum_{j=0}^{k} B(k, j, \lambda) \binom{n+j}{2k} \end{cases} \tag{1.14}$$

where $B_1(k, j, \lambda)$, $B(k, j, \lambda)$ are independent of n, and

$$B_1(k, j, \lambda) = B(k, k-j, 1-\lambda), \quad (0 \leq j \leq k). \tag{1.15}$$

As an application of (1.14) and (1.15), it is proved that

$$(1.16)\quad \begin{cases} R(n,\ n-k,\ \lambda) = \displaystyle\sum_{t=0}^{k} \binom{k+n+1}{k-t}\binom{k-n-1}{k+t} R_1(k+t,\ t,\ 1-\lambda) \\ R_1(n,\ n-k,\ \lambda) = \displaystyle\sum_{t=0}^{k} \binom{k+n+1}{k-t}\binom{k-n-1}{k+t} R(k+t,\ t,\ 1-\lambda). \end{cases}$$

For $\lambda = 1$, (1.16) reduces to (1.13) with n replaced by $n+1$; for $\lambda = 0$, we apparently get new results.

In the next place, we show that

$$(1.17)\quad \begin{cases} R(n,\ n-k,\ \lambda) = \dbinom{n}{k} B_k^{(-n+k)}(\lambda) \\ R(n,\ n-k,\ \lambda) = \dbinom{k-n-1}{k} B_k^{(n+1)}(1-\lambda), \end{cases}$$

where $B_k^{(k)}(\lambda)$ is the Bernoulli polynomial of higher order defined by [8, Ch. 6]:

$$\sum_{n=0}^{\infty} B_k^{(k)}(\lambda)\frac{u^k}{k!} = \left(\frac{u}{e^u - 1}\right)^z e^{\lambda u}.$$

We remark that (1.17) can be used to give a simple proof of (1.16). For the special case of Stirling numbers, see [2].

It is easily verified that, for $\lambda = 0$ and 1, (1.17) reduces to well-known representations [8, Ch. 6] of $S(n,\ n-k)$ and $S_1(n,\ n-k)$.

In view of the formulas (for notation and references see [3]),

$$(1.18)\quad \begin{cases} S(n,\ n-k) = \displaystyle\sum_{j=0}^{k-1} S'(k,\ j)\binom{n}{2k-j} \\ S_1(n,\ n-k) = \displaystyle\sum_{j=0}^{k} S'(k,\ j)\binom{n}{2k-j}, \end{cases}$$

it is of interest to define coefficients $R'(k, j, \lambda)$ and $R_1'(k, j, \lambda)$ by means of

$$(1.19)\quad \begin{cases} R(n,\ n-k,\ \lambda) = \displaystyle\sum_{j=0}^{\lambda} R'(k,\ j,\ \lambda)\binom{n}{2k-j} \\ R_1(n,\ n-k,\ \lambda) = \displaystyle\sum_{j=0}^{\lambda} R_1'(k,\ j,\ \lambda)\binom{n}{2k-j}. \end{cases}$$

Each coefficient is a polynomial in λ of degree $2k$ and has properties generalizing those of $S'(k,\ j)$ and $S_1'(k,\ j)$.

Finally (§9), we derive a number of relations similar to (1.16), connecting the various functions defined above. For example, we have

$$(1.20)\quad \begin{cases} R_1(n,\ n-k,\ \lambda) = \displaystyle\sum_{j=0}^{k} (-1)^{k-j}\binom{n+j}{k+j} R'(k,\ k-j,\ 1-\lambda) \\ R(n,\ n-k,\ \lambda) = \displaystyle\sum_{j=0}^{k} (-1)^{k-j}\binom{n+j}{k+j} R_1'(k,\ k-j,\ 1-\lambda) \end{cases}$$

and

$$(1.21)\qquad \begin{cases} R_1'(n, k, \lambda) = \displaystyle\sum_{t=0}^{k} (-1)^t \binom{n-t}{k-t} R'(n, t, 1-\lambda) \\ R'(n, k, \lambda) = \displaystyle\sum_{t=0}^{k} (-1)^t \binom{n-t}{k-t} R_1'(n, t, 1-\lambda). \end{cases}$$

In the proofs, we make use of the relations (1.15).

2. REPRESENTATIONS OF $R(n, n-k, \lambda)$

As a special case of a more general result proved in [5], if $f(x)$ is an arbitrary polynomial of degree $\leq m$, then there is a *unique* representation in the form

$$(2.1)\qquad f(x) = \sum_{j=0}^{m-1} a_j \binom{x+j}{m},$$

where the a are independent of x. Thus, since $R(n, n-k, \lambda)$ is a polynomial in n of degree $2k$, we may put, for $k \geq 1$,

$$(2.2)\qquad R(n, n-k, \lambda) = \sum_{j=0}^{2k} B(k, j, \lambda)\binom{n+j}{2k},$$

where the coefficients $B(k, j, \lambda)$ are independent of n.

By (1.4), we have, for $k > 1$,

$$(2.3)\qquad R(n+1, n-k+1, \lambda) = (n-k+1+\lambda)R(n, n-k+1, \lambda) + R(n, n-k, \lambda).$$

Thus, (2.2) yields

$$\sum_{j=0}^{2k} B(k, j, \lambda)\binom{n+j}{2k-1} = (n-k+1+\lambda)\sum_{j=0}^{2k-2} B(k-1, j, \lambda)\binom{n+j}{2k-2}.$$

Since

$$n-k+1+\lambda = (n+j-2k+2) + (k-j-1+\lambda),$$

we get

$$\sum_j B(k, j, \lambda)\binom{n+j}{2k-1} = \sum_j (2k-1)B(k-1, j, \lambda)\binom{n+j}{2k-1} + \sum_j (k-j-1+\lambda)B(k-1, j, \lambda)\left\{\binom{n+j+1}{2k-1}\binom{n+j}{2k-1}\right\}.$$

It follows that

$$(2.4)\qquad B(k, j, \lambda) = (k+j-\lambda)B(k-1, j, \lambda) + (k-j+\lambda)B(k-1, j-1, \lambda).$$

We shall now compute the first few values of $B(k, j, \lambda)$. To begin with we have the following values of $R(n, n-k, \lambda)$. Clearly, $R(n, n, \lambda) = 1$. Then, by (2.3), with $k = 1$, we have

$$R(n+1, n, \lambda) - R(n, n-1, \lambda) = n + \lambda.$$

It follows that

$$(2.5)\qquad R(n, n-1, \lambda) = \binom{n}{2} + n\lambda.$$

Next, taking $k = 2$ in (2.3),

$R(n + 1, n - 1, \lambda) - R(n, n - 2, \lambda) = (n - 1 + \lambda)R(n, n - 1, \lambda)$,

we find that

(2.6) $$R(n, n - 2, \lambda) = 3\binom{n}{4} + \binom{n}{3} + 3\binom{n}{3}\lambda + \binom{n}{2}\lambda^2, \quad (n \geq 2).$$

A little computation gives the following table of values:

$B(k, j, \lambda)$

$k \backslash j$	0	1	2	3
0	1			
1	$1 - \lambda$	λ		
2	$(1 - \lambda)_2$	$1 + 3\lambda - 2\lambda^2$	λ^2	
3	$(1 - \lambda)_3$	$8 + 7\lambda - 12\lambda^2 + 3\lambda^3$	$1 + 4\lambda + 6\lambda^2 - 3\lambda^3$	λ^3

The last line was computed by using the recurrence (2.4).

Note that the sum of the entries in each row above is independent of λ. This is in fact true generally. By (2.2), this is equivalent to saying that the coefficient of the highest power of λ in $R(n, n - k, \lambda)$ is independent of λ. To prove this, put

$$R(n, n - k, \lambda) = a\, n^{2k} + a' n^{2k-1} + \cdots .$$

Then

$$\begin{aligned} R(n + 1, n - k + 1, \lambda) - R(n, n - k, \lambda) &= a_k((n + 1)^{2k} - n^{2k}) + a_k'((n + 1)^{2k-1} - n^{2k-1}) + \cdots \\ &= 2ka_k n^{2k-1} + \cdots . \end{aligned}$$

Thus, by (2.3), $2ka_k = a_{k-1}$. Since $a_1 = \frac{1}{2}$, we get

$$a_k = \frac{1}{2k(2k - 2) \ldots 2} = \frac{1}{2^k k!} .$$

Therefore,

(2.7) $$\sum_{j=0}^{k} B(k, j, \lambda) = \frac{(2k)!}{2^k k!} = 1.3.5 \ldots (2k - 1).$$

This can also be proved by induction using (2.4).

However, the significant result implied by the table together with the recurrence (2.4) is that

(2.8) $$B(k, j, \lambda) = 0, \quad (j > k).$$

Hence, (2.2) reduces to

(2.9) $$R(n, n - k, \lambda) = \sum_{j=0}^{k} B(k, j, \lambda)\binom{n + j}{2k}.$$

It follows from (2.9) that the *polynomial* $R(n, n - k, \lambda)$ vanishes for $0 \leq n < k$.

Incidentally, we have anticipated (2.9) in the upper limit of summation in (2.7).

3. REPRESENTATION OF $R_1(n, n-k, \lambda)$

Since $R_1(n, n-k, \lambda)$ is a polynomial in n of degree $2k$, we may put, for $k \geq 1$,

$$R_1(n, n-k, \lambda) = \sum_{j=0}^{2k} B_1(k, j, \lambda)\binom{n+j}{2k}, \tag{3.1}$$

where $B_1(k, j, \lambda)$ is independent of n.

By (1.4) we have, for $k > 1$,

$$R_1(n+1, n-k+1, \lambda) = (n+\lambda)R_1(n, n-k+1, \lambda) + R_1(n, n-k, \lambda). \tag{3.2}$$

Thus, by (3.1), we get

$$\begin{aligned}\sum_{j=0}^{2k} B_1(k, j, \lambda)\binom{n+j}{2k-1} &= (n+\lambda)\sum_{j=0}^{2k-2} B_1(k-1, j, \lambda)\binom{n+j}{2k-2}\\ &= \sum_j (2k-1)B_1(k-1, j, \lambda)\binom{n+j}{2k-1}\\ &\quad + \sum_j (2k-j-2+\lambda)B_1(k-1, j, \lambda)\left\{\binom{n+j+1}{2k-1} - \binom{n+j}{2k-1}\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}B_1(k, j, \lambda) = {}&(j+1-\lambda)B_1(k-1, j, \lambda)\\ &+ (2k-j-1+\lambda)B_1(k-1, j-1, \lambda).\end{aligned} \tag{3.3}$$

As in the previous section, we shall compute the first few values of $B_1(k, j, \lambda)$.

To begin with, we have $R_1(n, n, \lambda) = 1$. Then by (3.2), with $k = 1$, we have

$$R_1(n+1, n, \lambda) - R_1(n, n-1, \lambda) = n + \lambda,$$

so that

$$R_1(n, n-1, \lambda) = \binom{n}{2} + n. \tag{3.4}$$

Next, taking $k = 2$ in (3.2),

$$R_1(n+1, n-1, \lambda) - R_1(n, n-2, \lambda) = (n+\lambda)R_1(n, n-1, \lambda).$$

It follows that

$$R_1(n, n-2, \lambda) = 3\binom{n}{4} + 2\binom{n}{3} + \left\{3\binom{n}{3} + \binom{n}{2}\right\}\lambda + \binom{n}{2}\lambda^2, \quad (n \geq 2). \tag{3.5}$$

A little computation gives the following table of values:

$B_1(k, j, \lambda)$

$k \backslash j$	0	1	2	3
0	1			
1	$1 - \lambda$	λ		
2	$(1 - \lambda)^2$	$2 + \lambda - 2\lambda^2$	$(\lambda)_2$	
3	$(1 - \lambda)^3$	$8 - 7\lambda - 3\lambda^2 + 3\lambda^3$	$6 + 8\lambda - 3\lambda^2 - 3\lambda^3$	$(\lambda)_3$

Exactly as above, we find that

$$\sum_{j=0}^{k} B_1(k, j, \lambda) = \frac{(2k)!}{2^k k!} = 1.3.4 \ldots (2k - 1). \tag{3.6}$$

This can also be proved by induction using (3.3). Moreover,

$$B_1(k, j, \lambda) = 0, \quad (j > k), \tag{3.7}$$

so that (3.1) becomes

$$R_1(n, n - k, \lambda) = \sum_{j=0}^{k} B_1(k, j, \lambda)\binom{n + j}{2k}. \tag{3.8}$$

Thus, the *polynomial* $R_1(n, n - k, \lambda)$ vanishes for $0 \leq n < k$.

4. RELATION OF $B_1(k, j, \lambda)$ TO $B(k, j, \lambda)$

In (2.4) replace j by $k - j$ and we get

$$B(k, k - j, \lambda) = (2k - j - \lambda)B(k - 1, k - j, \lambda) + (j + \lambda)B(k - 1, k - j - 1, \lambda). \tag{4.1}$$

Put

$$\overline{B}(k, j, \lambda) = B(k - j, \lambda).$$

Then (4.1) becomes

$$\overline{B}(k, j, \lambda) = (2k - j - \lambda)\overline{B}(k - 1, j - 1, \lambda) + (j + \lambda)\overline{B}(k - 1, j, \lambda). \tag{4.2}$$

Comparison of (4.2) with (3.3) gives

$$B_1(k, j, \lambda) = \overline{B}(k, j, 1 - \lambda),$$

and therefore

$$B_1(k, j, \lambda) = B(k, k - j, 1 - \lambda). \tag{4.3}$$

In particular,

$$\begin{cases} B_1(k, 0, \lambda) = B(k, k, 1 - \lambda) = (1 - \lambda)^k \\ B_1(k, k, \lambda) = B(k, 0, 1 - \lambda) = (\lambda)_k. \end{cases} \tag{4.4}$$

We recall that

$$R(n, k, 0) = S(n, k), \ R(n, k, 1) = S(n + 1, k + 1) \tag{4.5}$$

and

(4.6) $$R_1(n, k, 0) = S_1(n, k), \; R_1(n, k, 1) = S_1(n + 1, k + 1).$$

In (2.9), take $\lambda = 0$. Then, by (1.11) and (4.5) with k replaced by $n - k$,

$$\sum_{j=0}^{k} B(k, j, 0)\binom{n + j}{2k} = \sum_{j=1}^{k} B(k, j)\binom{n + j - 1}{2k}.$$

It follows that

(4.7) $$B(k, j, 0) = B(k, j + 1), \quad (0 \le j < k); \; B(k, k, 0) = 0.$$

Similarly, taking $\lambda = 1$ in (2.9), we get

$$\sum_{j=0}^{k} B(k, j, 1)\binom{n + j}{2k} = \sum_{j=1}^{k} B(k, j)\binom{n + j}{2k}.$$

Thus

(4.8) $$B(k, j, 1) = B(k, j), \quad (1 \le j \le k); \; B(k, 0, 1) = 0.$$

Next, take $\lambda = 0$ in (3.8), and we get

$$\sum_{j=0}^{k} B_1(k, j, 0)\binom{n + j}{2k} = \sum_{j=1}^{k} B(k, j)\binom{n + j - 1}{2k}.$$

This gives

(4.9) $$B_1(k, j, 0) = B_1(k, j + 1), \quad (0 \le j < k); \; B_1(k, k, 0) = 0.$$

Similarly, we find that

(4.10) $$B_1(k, j, 1) = B_1(k, j), \quad (1 \le j \le k); \; B_1(k, 0, 1) = 0.$$

It is easily verified that (4.9) and (4.10) are in agreement with (4.4). Moreover, for $\lambda = 0$, (4.3) reduces to

$$B_1(k, j, 0) = B(k, k - j, 1);$$

by (4.8) and (4.9), this becomes

$$B_1(k, j + 1) = B(k, k - j),$$

which is correct. For $\lambda = 1$, (4.3) reduces to

$$B_1(k, j, 1) = B(k, k - j, 0);$$

by (4.7) and (4.10), this becomes

$$B_1(k, j) = B(k, k - j + 1)$$

as expected.

5. THE COEFFICIENTS $B(k, j, \lambda)$; $B_1(k, j, \lambda)$

It is evident from the recurrences (2.4) and (3.3) that $B(k, j, \lambda)$ and $B_1(k, j, \lambda)$ are polynomials of degree $\le k$ in λ with integral coefficients. Moreover, they are related by (4.3). Put

(5.1) $$f_k(\lambda, x) = \sum_{j=0}^{k} B(k, j, \lambda)x^j$$

and

(5.2) $$f_{1,k}(\lambda,\ x) = \sum_{j=0}^{k} B_1(k,\ j,\ \lambda)x^j.$$

By (4.3), we have

(5.3) $$f_{1,k}(\lambda,\ x) = x^k f_k\left(1 - \lambda,\ \frac{1}{x}\right).$$

By (2.7) and (3.6),

(5.4) $$f_k(\lambda,\ 1) = f_{1,k}(\lambda,\ 1) = \frac{(2k)!}{2^k k!}.$$

In the next place, by (2.4), (5.1) becomes

$$f_k(\lambda,\ x) = \sum_{j=0}^{k} \{(k + j - \lambda)B(k - 1,\ j,\ \lambda) + (k - j + \lambda)B(k - 1,\ j - 1,\ \lambda)\}x^j.$$

Since

$$\sum_{j=0}^{k} (k + j - \lambda)B(k - 1,\ j,\ \lambda)x^j = (k - \lambda + xD)f_{k-1}(\lambda,\ x)$$

and

$$\sum_{j=0}^{k} (k - j + \lambda)B(k - 1,\ j - 1,\ \lambda)x^j = x\sum_{j=0}^{k-1} (k - j - 1 + \lambda)B(k - 1,\ j,\ \lambda)x^j$$

$$= x(k - 1 + \lambda - xD)f_{k-1}(\lambda,\ x),$$

where $D \equiv d/dx$, it follows that

(5.5) $$f_k(\lambda,\ x) = \{k - \lambda + (k - 1 + \lambda)x + x(1 - x)D\}f_{k-1}(\lambda,\ x).$$

The corresponding formula for $f_{1,k}(\lambda,\ x)$ is

(5.6) $$f_{1,k}(\lambda,\ x) = \{1 - \lambda + (2k - 2 + \lambda)x + x(1 - x)D\}f_{1,k-1}(\lambda,\ x).$$

Let E denote the familiar operator defined by $Ef(n) = f(n + 1)$. Then, by (2.9) and (5.1), we have

(5.7) $$R(n,\ n - k,\ \lambda) = f_k(\lambda,\ E)\binom{n}{2k}.$$

Similarly, by (3.8) and (5.2),

(5.8) $$R_1(n,\ n - k,\ \lambda) = f_{1,k}(\lambda,\ E)\binom{n}{2k}.$$

Thus, the recurrence

$$R(n + 1,\ n - k + 1,\ \lambda) - R(n,\ n - k,\ \lambda) = (\lambda + n - k + 1)R(n,\ n - k + 1,\ \lambda)$$

becomes

$$f_k(\lambda,\ E)\binom{n + 1}{2k} - f_k(\lambda,\ E)\binom{n}{2k} = (\lambda + n - k + 1)f_{k-1}(\lambda,\ x)\binom{n}{2k - 2}.$$

Since

$$\binom{n + 1}{2k} - \binom{n}{2k} = \binom{n}{2k - 1},$$

we have

(5.9) $$f_k(\lambda,\ E)\binom{n}{2k - 1} = (\lambda + n - k + 1)f_{k-1}(\lambda,\ x)\binom{n}{2k - 2}.$$

Applying the finite difference operator Δ, we get

(5.10) $$f_k(\lambda,\ E)\binom{n}{2k - 1} = (\lambda + n - k + 2)f_{k-1}(\lambda,\ x)\binom{n}{2k - 3} + f_{k-1}(\lambda,\ x)\binom{n}{2k - 2}$$

Similarly, the recurrence

$$R_1(n+1,\ n-k+1,\ \lambda) - R_1(n,\ n-k,\ \lambda) = (\lambda+n)R_1(n,\ n-k+1,\ \lambda)$$

yields

$$f_{1,k}(\lambda,\ E)\binom{n}{2k-1} = (\lambda+n)f_{1,k-1}(\lambda,\ E)\binom{n}{2k-2} \tag{5.11}$$

and

$$f_{1,k}(\lambda,\ E)\binom{n}{2k-2} = (\lambda+n+1)f_{1,k-1}(\lambda,\ E)\binom{n}{2k-3} + f_{1,k-1}\binom{n}{2k-2}. \tag{5.12}$$

6. AN APPLICATION

We shall prove the following two formulas:

$$R(n,\ n-k,\ 1-\lambda) = \sum_{t=0}^{k}\binom{k+n+1}{k-t}\binom{k-n-1}{k+t}R_1(k+t,\ t,\ \lambda), \tag{6.1}$$

and

$$R_1(n,\ n-k,\ 1-\lambda) = \sum_{t=0}^{k}\binom{k+n+1}{k-t}\binom{k-n-1}{k+t}R(k+t,\ t,\ \lambda). \tag{6.2}$$

Note that the coefficients on the right of (6.1) and (6.2) are the same.

To begin with, we invert (2.9) and (3.8). It follows from (2.9) that

$$\begin{aligned}\sum_{n=k}^{\infty}R(n,\ n-k,\ \lambda)x^{n-k} &= \sum_{j=0}^{k}B(k,\ j,\ \lambda)x^{k-j}\sum\binom{n+j}{2k}x^{n-2k+j}\\ &= \sum_{j=0}^{k}B(k,\ j,\ \lambda)x^{k-j}\sum_{m=0}^{\infty}\binom{m+2k}{2k}x^{m}\\ &= (1-x)^{-2k-1}\sum_{j=0}^{k}B(k,\ j,\ \lambda)x^{k-j},\end{aligned}$$

so that

$$\begin{aligned}\sum_{j=0}^{k}B(k,\ k-j,\ \lambda)x^{j} &= (1-x)^{2k+1}\sum_{n=k}^{\infty}R(n,\ n-k,\ \lambda)x^{n-k}\\ &= \sum_{m=0}^{2k+t}(-1)^{m}\binom{2k+1}{m}x^{m}\sum_{t=0}^{\infty}R(k+t,\ t)x^{t}.\end{aligned}$$

It follows that

$$B(k,\ k-j,\ \lambda) = \sum_{t=0}^{j}(-1)^{j-t}\binom{2k+1}{j-t}R(k+t,\ t,\ \lambda). \tag{6.3}$$

Similarly,

$$B_1(k-k-j,\ \lambda) = \sum_{t=0}^{k}(-1)^{j-t}\binom{2k+1}{j-t}R_1(k+t,\ t,\ \lambda). \tag{6.4}$$

By (2.9), (4.3), and (6.4), we have

$$\begin{aligned}R(n,\ n-k,\ 1-\lambda) &= \sum_{j=0}^{k}B_1(k,\ k-j,\ \lambda)\binom{n+j}{2k}\\ &= \sum_{j=0}^{k}\binom{n+j}{2k}\sum_{t=0}^{j}(-1)^{j-t}\binom{2k+1}{j-t}R_1(k+t,\ t,\ \lambda)\end{aligned}$$

$$(6.5)\qquad = \sum_{t=0}^{k} R_1(k+t,\ t,\ \lambda) \sum_{j=t}^{k} (-1)^{j-t} \binom{2k+1}{j-1} \binom{n+j}{2k}.$$

The inner sum is equal to

$$\sum_{j=0}^{k-t} (-1)^j \binom{2k+1}{j} \binom{n+t+j}{2k} = \binom{n+t}{2k} \sum_{j=0}^{k-t} \frac{(-2k-1)_j\,(n+t+1)_j\,(-k+t)_j}{j!\,(n+t-2k+1)_j\,(-k+t)_j}$$

$$= \binom{n+t}{2k}\,{}_3F_2\left[\begin{matrix} -2k-1,\ n+t+1,\ -k+t \\ n+t-2k+1,\ -k+t \end{matrix}\right].$$

The ${}_3F_2$ is Saalschützian [1, p. 9], and we find, after some manipulation, that

$$\sum_{j=0}^{k-t} (-1)^j \binom{2k+1}{j} \binom{n+t+j}{2k} = \binom{k+n+1}{k-t} \binom{k-n-1}{k+t}.$$

Thus, (6.5) becomes

$$R(n,\ n-k,\ 1-\lambda) = \sum_{t=0}^{k} \binom{k+n+1}{k-t} \binom{k-n-1}{k+t} R_1(k+t,\ t,\ \lambda).$$

This proves (6.1). The proof of (6.2) is exactly the same.

7. BERNOULLI POLYNOMIALS OF HIGHER ORDER

Nörlund [9, Ch. 6] defined the Bernoulli function of order z by means of

$$(7.1)\qquad \sum_{n=0}^{\infty} B_n^{(z)}(\lambda) \frac{u^n}{n!} = \left(\frac{u}{e^u - 1}\right)^z e^{\lambda u}.$$

It follows from (7.1) that $B_n^{(z)}(\lambda)$ is a polynomial of degree n in each of the parameters z, λ.

Consider

$$(7.2)\qquad Q(n,\ n-k,\ \lambda) = \binom{n}{k} B^{(-n+k)}(\lambda)$$

and

$$(7.3)\qquad Q_1(n,\ n-k,\ \lambda) = \binom{k-n-1}{k} B^{(n+1)}(1-\lambda).$$

It follows from (7.2) that

$$\sum_{n=k}^{\infty} Q(n,\ k,\ \lambda) \frac{u^n}{n!} = \sum_{n=k}^{\infty} \binom{u}{n-k} B_{n-k}^{(-k)}(\lambda) \frac{u^n}{n!} = \frac{u^k}{k!} \sum_{n=0}^{\infty} B_n^{(-k)}(\lambda) \frac{u^n}{n!}.$$

Hence, by (7.1), we have

$$(7.4)\qquad \sum_{n=k}^{\infty} Q(n,\ k,\ \lambda) \frac{u^n}{n!} = \frac{1}{k!}(e^u - 1)^k e^{\lambda u}.$$

Comparison of (7.4) with (1.7) gives $Q(n, k, \lambda) = R(n, k, \lambda)$, so that

$$(7.5)\qquad R(n,\ n-k,\ \lambda) = \binom{n}{k} B^{(-n+k)}(\lambda).$$

Next, by (7.3),

$$\sum_{n=k}^{\infty} Q_1(n,\ k,\ \lambda)\frac{u^n}{n!} = \sum_{n=k}^{\infty}\binom{-k-1}{n-k}B_{n-k}^{(n+1)}(1-\lambda)\frac{u^n}{n!}$$

$$= \sum_{n=k}^{\infty}(-1)^{n-k}\binom{n}{n-k}B_{n-k}^{(n+1)}(1-\lambda)\frac{u^n}{n!}$$

$$= \frac{u^k}{k!}\sum_{n=0}^{\infty}(-1)^n B_{n-k}^{(n+1)}(1-\lambda)\frac{u^n}{n!}.$$

It is known [8, p. 134] that

$$(1+t)^{x-1}\left(\log(1+t)\right)^k = \sum_{n=k}^{\infty}\frac{t^n}{(n-k)!}B_{n-k}^{(n+1)}(x).$$

Thus,

$$\sum_{k=0}^{\infty} y^k \sum_{n=k}^{\infty} Q_1(n,\ k,\ \lambda)\frac{u^n}{n!}y^k = \sum_{k=0}^{\infty}\frac{y^k}{k!}(1-u)^{-\lambda}\left(\log\frac{1}{1-u}\right)^k$$

$$= (1-u)^{-\lambda}(1-u)^{-y}.$$

Therefore, $Q_1(n,\ k,\ \lambda) = R_1(n,\ k,\ \lambda)$, so that

$$R_1(n,\ n-k,\ \lambda) = \binom{k-n-1}{k}B_k^{(n+1)}(1-\lambda). \tag{7.6}$$

For $\lambda = 0$, (7.5) reduces to

$$S(n,\ n-k) = \binom{n}{k}B_k^{(-n+k)};$$

for $\lambda = 1$, we get

$$S(n+1,\ n-k+1) = \binom{n}{k}B_k^{(-n+k)}(1) = \binom{n}{k}\left(1 - \frac{k}{-n+k-1}\right)B_k^{(-n+k-1)}$$

$$= \binom{n+1}{k}B_k^{(-n+k-1)}.$$

For $\lambda = 1$, (7.6) reduces to

$$S_1(n+1,\ n-k+1) = \binom{k-n-1}{k}B_k^{(n+1)};$$

for $\lambda = 0$, we get

$$S_1(n,\ n-k) = \binom{k-n-1}{k}\left(1-\frac{k}{n}\right)B_k^{(n)} = \binom{k-n}{k}B_k^{(n)}.$$

Thus, in all four special cases, (7.5) and (7.6) are in agreement with the corresponding formulas for $S(n,\ n-k)$ and $S_1(n,\ n-k)$.

8. THE FUNCTIONS $R'(n,\ k,\ \lambda)$ AND $R_1'(n,\ k,\ \lambda)$

We may put

$$R(n,\ n-k,\ \lambda) = \sum_{j=0}^{k} R'(k,\ j,\ \lambda)\binom{n}{2k-j} \tag{8.1}$$

and

$$R_1(n,\ n-k,\ \lambda) = \sum_{j=0}^{k} R'(k,\ j,\ \lambda)\binom{n}{2k-j}. \tag{8.2}$$

The upper limit of summation is justified by (2.9) and (3.8).

Using the recurrence (2.3), we get

$$\begin{aligned}
&R(n + 1, n - k + 1, \lambda) - R(n, n - k, \lambda)\\
&\quad = (n - k + 1 + \lambda)\sum_{j=0}^{k-1} R'(k - 1, j, \lambda)\binom{n}{2k - j - 2}\\
&\quad = \sum_{j=0}^{k-1} (2k - j - 1)R'(k - 1, j, \lambda)\binom{n}{2k - j - 1}\\
&\qquad + \sum_{j=0}^{k-1} (k - j - 1 + \quad)R'(k - 1, j, \quad)\binom{n}{2k - j - 2}.
\end{aligned}$$

Since

$$R(n + 1, n - k + 1, \lambda) - R(n, n - k, \lambda) = \sum_{j=0}^{k-1} R'(k, j, \lambda)\binom{n}{2k - j - 1},$$

we get

(8.3) $$R'(k, j, \lambda) = (2k - j - 1)R'(k - 1, j, \lambda) + (k - j + \lambda)R'(k - 1, j - 1, \lambda).$$

For $k = 0$, (8.1) gives

(8.4) $$R'(0, 0, \lambda) = 1, R'(0, j, \lambda) = 0, \quad (j > 0).$$

The following values are easily computed using the recurrence (8.3).

$R'(k, j, \lambda)$

$k \backslash j$	0	1	2	3	4
0	1				
1	1	λ			
2	3	$1 + 3\lambda$	λ^2		
3	15	$10 + 15\lambda$	$1 + 4\lambda + 6\lambda^2$	λ^3	
4	105	$105 + 105\lambda$	$25 + 60\lambda + 45\lambda^2$	$1 + 5\lambda + 10\lambda^2 + 4\lambda^3$	λ^4

It is easily proved, using (8.3), that

(8.5) $$R'(k, 0, \lambda) = 1.3.5 \ldots (2k - 1)$$

and

(8.6) $$R'(k, k, \lambda) = \lambda^k.$$

Also,

(8.7) $$\sum_{j=0}^{k} (-1)^j R'(k, j, \lambda) = (1 - \lambda)_k.$$

Moreover, it is clear that $R'(k, j, \lambda)$ is a polynomial in λ of degree j.

To invert (8.1), multiply both sides by $(-1)^{m-n}\binom{m}{n}$ and sum over n. Changing the notation slightly, we get

(8.8) $$R'(k, k - j, \lambda) = \sum_{t=0}^{j} (-1)^{j+t}\binom{k + j}{k + t}R(k + t, t, \lambda).$$

Turning next to (8.2) and employing (3.2), we get

$$R_1(n+1,\ n-k+1,\ \lambda) - R_1(n,\ n-k,\ \lambda)$$

$$= (n+\lambda)\sum_{j=0}^{k-1} R_1'(k-1,\ j,\ \lambda)\binom{n}{2k-j-2}$$

$$= \sum_{j=0}^{k-1}(2k-j-1)R_1'(k-1,\ j,\ \lambda)\binom{n}{2k-j-1}$$

$$+ \sum_{j=0}^{k-1}(2k-j-2+\lambda)R_1'(k-1,\ j,\ \lambda)\binom{n}{2k-j-2}.$$

It follows that

(8.9) $R_1'(k,\ j,\ \lambda) = (2k-j-1)R'(k-1,\ j,\ \lambda) + (2k-j-1+\lambda)R_1'(k-1,\ j-1,\lambda).$

For $k = 0$, we have

(8.10) $$R_1'(0,\ 0,\ \lambda) = 1,\ R_1'(0,\ j,\ \lambda) = 0,\quad (j > 0).$$

The following values are readily computed by means of (8.9) and (8.10).

$R_1'(k,\ j,\ \lambda)$

$k \backslash j$	0	1	2	3	4
0	1				
1	1	λ			
2	3	$2 + 3\lambda$	$(\lambda)_2$		
3	15	$20 + 15\lambda$	$6 + 14\lambda + 6\lambda^2$	$(\lambda)_3$	
4	105	$210 + 105\lambda$	$130 + 165\lambda + 45\lambda^2$	$24 + 70\lambda + 50\lambda^2 + 10\lambda^3$	$(\lambda)_4$

We have

(8.11) $$R_1'(k,\ 0,\ \lambda) = 1.3.5\ \ldots\ (2k-1)$$

and

(8.12) $$R_1'(k,\ k,\ \lambda) = (\lambda)_k.$$

Also

(8.13) $$\sum_{j=0}^{k}(-1)^j R_1'(k,\ j,\ \lambda) = (1-\lambda)^k.$$

Clearly, $R_1'(k,\ j,\ \lambda)$ is a polynomial in λ of degree j.
Parallel to (8.8), we have

(8.14) $$R_1'(k,\ k-j,\ \lambda) = \sum_{t=0}^{j}(-1)^{j+t}\binom{k+j}{k+t}R_1(k+t,\ t,\ \lambda).$$

9. ADDITIONAL RELATIONS

(Compare [3, 4].) By (8.14) and (3.1), we have

$$\begin{aligned}R_1'(k,\ k-j,\ \lambda) &= \sum_{t=0}^{j}(-1)^t\binom{k+j}{t}R_1'(k+j-t,\ j-t,\ \lambda)\\ &= \sum_{t=0}^{j}(-1)^t\binom{k+j}{t}\sum_{s=0}^{k}B_1(k,\ s,\ \lambda)\binom{k+j-t+s}{2k}\\ &= \sum_{s=0}^{k}B_1(k,\ s,\ \lambda)\sum_{t=0}^{j}(-1)^t\binom{k+j}{t}\binom{k+j-t+s}{2k}.\end{aligned}$$

It can be verified that the inner sum is equal to $\binom{s}{k-j}$. Thus,

$$R_1'(k,\ j,\ \lambda) = \sum_{s=j}^{k}\binom{s}{j}B_1(k,\ s,\ \lambda). \tag{9.1}$$

Similarly,

$$R'(k,\ k-j,\ \lambda) = \sum_{s=k-j}^{k}\binom{s}{k-j}B(k,\ s,\ \lambda). \tag{9.2}$$

The inverse formulas are

$$B_1(k,\ t,\ \lambda) = \sum_{j=t}^{k}(-1)^{j-t}\binom{j}{t}R_1'(k,\ j,\ \lambda) \tag{9.3}$$

and

$$B(k,\ t,\ \lambda) = \sum_{j=t}^{k}(-1)^{j-t}\binom{j}{t}R'(k,\ j,\ \lambda). \tag{9.4}$$

In the next place, by (9.4) and (3.1),

$$\begin{aligned}R_1(n,\ n-k,\ \lambda) &= \sum_{t=0}^{k}B_1(k,\ t,\ \lambda)\binom{n+t}{2k} = \sum_{t=0}^{k}B(k,\ k-t,\ 1-\lambda)\binom{n+t}{2k}\\ &= \sum_{t=0}^{k}B(k,\ t,\ 1-\lambda)\binom{n+k-t}{2k}\\ &= \sum_{t=0}^{k}\binom{n+k-t}{2k}\sum_{j=t}^{k}(-1)^{j-t}\binom{j}{t}R'(k,\ j,\ 1-\lambda)\\ &= \sum_{j=0}^{k}R'(k,\ j,\ 1-\lambda)\sum_{t=0}^{k}(-1)^{j-t}\binom{j}{t}\binom{n+k-t}{2k}.\end{aligned}$$

The inner sum is equal to $(-1)^j\binom{n+k-j}{2k-j}$, and therefore

$$R_1(n,\ n-k,\ \lambda) = \sum_{j=0}^{k}(-1)^{k-j}\binom{n+j}{k+j}R'(k,\ k-j,\ 1-\lambda). \tag{9.5}$$

Similarly,

$$R(n,\ n-k,\ \lambda) = \sum_{j=0}^{k}(-1)^{k-j}\binom{n+j}{k+j}R_1'(k,\ k-j,\ 1-\lambda). \tag{9.6}$$

The inverse formulas are less simple. We find that

$$R_1'(n, k, \lambda) = \sum_{j=0}^{n} (-1)^{n-j} C_n(k, j) R(n + j, j, 1 - \lambda) \tag{9.7}$$

and

$$R'(n, k, \lambda) = \sum_{j=0}^{n} (-1)^{n-j} C_n(k, j) R_1(n + j, j, 1 - \lambda), \tag{9.8}$$

where

$$C_n(k, j) = \sum_{t=0}^{n-j} \binom{n - t}{k - t} \binom{2n - t}{n + j}. \tag{9.9}$$

It does not seem possible to simplify $C_n(k, j)$.

We omit the proof of (9.7) and (9.8).

Finally, we state the pair

$$R_1'(n, k, \lambda) = \sum_{t=0}^{k} (-1)^t \binom{n - t}{k - t} R'(n, t, 1 - \lambda), \tag{9.10}$$

$$R'(n, k, \lambda) = \sum_{t=0}^{k} (-1)^t \binom{n - t}{k - t} R_1'(n, t, 1 - \lambda). \tag{9.11}$$

The proof is like the proof of (8.8) and (8.14).

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