REFERENCES

1. L. E. Dickson. *History of the Theory of Numbers*. New York, 1934.
2. J.-C. Bermong, A. Kotzig, & J. Turgeon. "On a Combinatorial Problem of Antennas in Radioastronomy." *Coll. Math. Soc. J. Bolyai* 18. Combinatorics, Keszthely (Hungary), 1976.

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MEANS, CIRCLES, RIGHT TRIANGLES, AND THE FIBONACCI RATIO

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In looking for a convenient way to graph the arithmetic mean (AM), the geometric mean (GM), and the harmonic mean (HM) of two positive numbers, I came across a connection between Kepler's "two great treasures" of geometry, the Pythagorean Theorem and the Golden Ratio, as well as several attractive geometric patterns.

Let us take a and b as the two positive numbers to be averaged and let

$$a + b = k. \tag{1}$$

The three means are defined as

$$\mathrm{AM}(a,\ b) = \frac{a + b}{2} = \frac{k}{2} \tag{2}$$

$$\mathrm{GM}(a,\ b) = \sqrt{ab} \tag{3}$$

$$\mathrm{HM}(a,\ b) = \frac{2ab}{a + b} = \frac{2ab}{k}. \tag{4}$$

To graph the three means, recall that a perpendicular line from a point on a circle to a diameter of the circle is the mean proportional (i.e., geometric mean) of the two segments of the diameter created by the line. In Figure 1, diameter AB, of length k, is composed of line segment $AD = a$ and line segment $DB = b$. The perpendicular DE is the geometric mean. When O is the center of the circle, the AM is equal to any radius, e.g., AO and OB. To find the harmonic mean, we proceed in the following manner. Construct a perpendicular to the diameter at the center O of height equal to DE, say line OP. Next, construct the perpendicular bisector of AP that meets diameter AB at C. Let Q be the center of a circle passing through A, B, and point C on AB. Since OP is the geometric mean of AO and OC, we have $OC = 2ab/k$, and thus the desired HM is line segment OC.

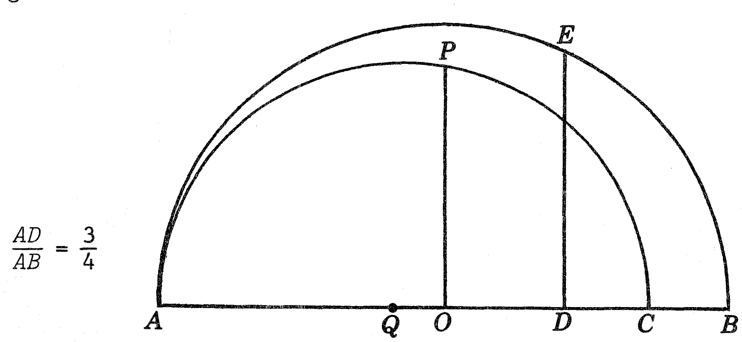

Fig. 1 Constructing the Arithmetic, Geometric, and Harmonic Means

Now, what if points D and C in Figure 1 were the same point? The graph is shown in Figure 2, and it can be seen that

$$AC = a = \text{AM} + \text{HM} = \frac{k}{2} + \frac{2ab}{k}. \tag{5}$$

Replacing b by $k - a$ and solving for a in terms of k, we have

$$a = \frac{1 + \sqrt{5}}{4}k = \frac{\phi}{2}k \tag{6}$$

and

$$b = \frac{k}{2\phi^2} \tag{7}$$

where $\phi = \frac{1}{2}(1 + \sqrt{5})$ is the Golden Ratio. The difference between a and b is k/ϕ. In Figure 2, the AM remains $\frac{1}{2}k$, but

$$\text{HM} = \frac{k}{2\phi} \tag{8}$$

and

$$\text{GM} = \frac{k}{2\sqrt{\phi}}. \tag{9}$$

In right triangle POC,

$$\overline{PC}^2 = \overline{OP}^2 + \overline{OC}^2 = \text{GM}^2 + \text{HM}^2 = \frac{k^2}{4\phi^2} + \frac{k^2}{4\phi} = k^2\frac{(\phi + 1)}{4\phi^2} = \frac{k^2}{4} = \text{AM}^2, \tag{10}$$

since $\phi^2 - \phi - 1 = 0$. Hence, PC is equal to the AM and right triangle POC has sides whose lengths can be expressed in terms of the Golden Ratio.

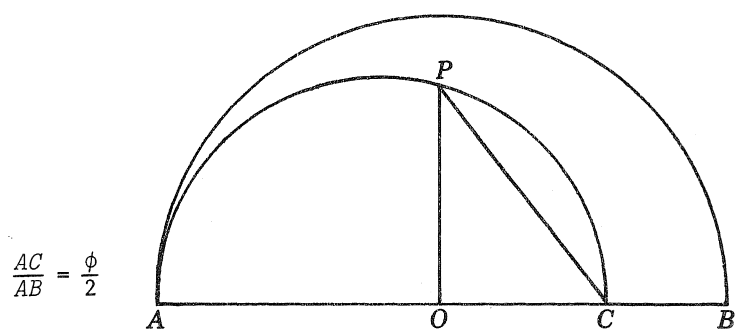

Fig. 2 The Arithmetic, Geometric, and Harmonic Means of Fibonacci Related Numbers Forming a Right Triangle

Since AM is larger than HM and GM, the AM must be the hypotenuse of a right triangle whose sides are AM, HM, and GM. Using the Pythagorean Theorem for that right triangle, we have $\text{AM}^2 = \text{HM}^2 + \text{GM}^2$ or

$$\left(\frac{a + b}{2}\right)^2 = \frac{4a^2b^2}{(a + b)^2} + ab. \tag{11}$$

Clearing of fractions and solving for a in terms of b, we obtain

$$a = b\sqrt{9 + 4\sqrt{5}}.$$

But $9 + 4\sqrt{5} = (2 + \sqrt{5})^2 = (\phi^3)^2$, hence

$$a = b\phi^3. \tag{12}$$

Therefore, the arithmetic, harmonic, and geometric means of positive numbers a and b can form the sides of a right triangle if and only if $a = b\phi^3$. When $b = 1$, the hypotenuse of that right triangle is ϕ^2, and the legs are $\phi^{3/2}$ and ϕ. Sequences of such triangles and a discussion of their relationships to Fibonacci sequences can be found in [1].

Expanding upon Figure 2, using the same values for a and b (i.e., from Eqs. (6) and (7)], we have the elegant picture of Figure 3. The diameters of both inner circles lie on AB and are of length a = AM + HM = $\frac{1}{2}\phi k$. Line segment FC is twice the harmonic mean (or k/ϕ), PR is twice the geometric mean, and FP, FR, CP, and CR are equal to the arithmetic mean. The ratio of the area of each inner circle to the area of the outer circle is $\phi^2/4$. The ratio of the area of the overlap between the two inner circles to the area of each inner circle is $[2w/\pi + 4/\pi\phi^{4.5}]$, while the ratio of the area of the overlap to the area of the outer circle is $[w\phi^2/2\pi - 1/\pi\phi^{2.5}]$, where $\tan w = 2\phi^{1.5}$, with w measured in radians. While those latter ratios are a bit complex, the image of Figure 3 remains one of unity and harmony.

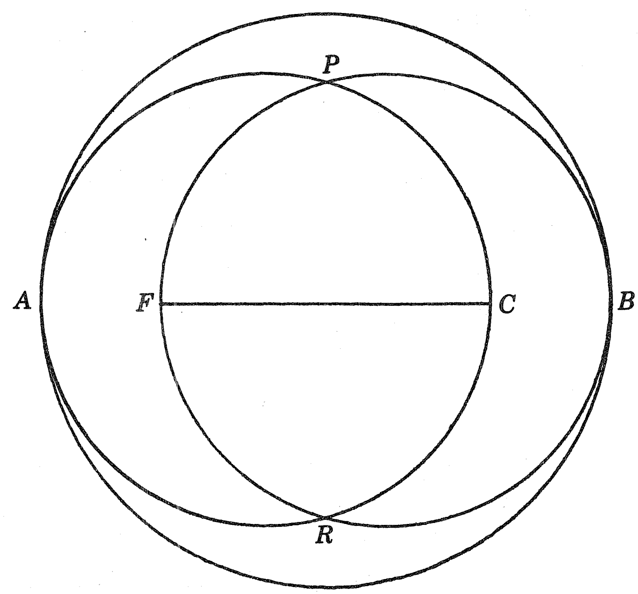

Fig. 3 A Harmonious Blending of Means

REFERENCE

1. Joseph L. Ercolano. "A Geometric Treatment of Some of the Algebraic Properties of the Golden Section." *The Fibonacci Quarterly* 11 (1973):204-208.

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