$$|R_2| = \frac{1}{|s - r||s - \bar{s}||s|^{n+1}} = \frac{1}{2|s - r||\text{Im } s||s|^{n+1}} < .26/|s|^{n+1} < .2,$$

which along with (8) and (6) implies

$$T_n + R_1 = -R_2 - R_3,$$

so

$$|T_n + R_1| = |R_2 + R_3| \le 2|R_2| < .4;$$

hence

$$T_n - .4 < -R_1 < T_n + .4$$

or, equivalently,

$$T_n < -R_1 + .4 < T_n + 1.$$

Substituting the value of R_1 from (4) into (9) we may rewrite (9) in terms of the greatest integer function and obtain the desired formula:

$$T_n = \left[\frac{1}{|r - s|^2 r^{n+1}} + .4\right].$$

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POLYNOMIALS ASSOCIATED WITH GEGENBAUER POLYNOMIALS

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1. INTRODUCTION

Chebyshev polynomials $T_n(x)$ of the first kind and $U_n(x)$ of the second kind are, respectively, defined as follows:

$$T_n(x) = \cos(n \cos^{-1} x) \qquad (|x| \le 1),$$

$$U_n(x) = \frac{\sin[(n + 1)\cos^{-1} x]}{\sin(\cos^{-1} x)} \qquad (|x| \le 1).$$

In 1974 Jaiswal [6] investigated polynomials $p_n(x)$ related to $U_n(x)$. In 1977 Horadam [5] obtained similar results for polynomials $q_n(x)$, associated with $T_n(x)$. The polynomials $p_n(x)$ and $q_n(x)$ are defined as follows:

$$\text{(1)} \quad \begin{cases} p_n(x) = 2xp_{n-1}(x) - p_{n-3}(x) & (n \ge 3) \text{ with} \\ p_0(x) = 0,\ p_1(x) = 1,\ p_2(x) = 2x \end{cases}$$

and

$$\text{(2)} \quad \begin{cases} q_n(x) = 2xq_{n-1}(x) - q_{n-3}(x) & (n \ge 3) \text{ with} \\ q_0(x) = 0,\ q_1(x) = 2,\ q_2(x) = 2x. \end{cases}$$

Chebyshev's polynomials of both kinds are special cases of Gegenbauer polynomials ([1], [2], [3], [8], [9]) $C_n^\lambda(x)$ $(\lambda > -\frac{1}{2},\ |x| \leq 1)$ defined by

$$C_0^\lambda(x) = 1,\ C_1^\lambda(x) = 2\lambda x,$$

with the recurrence relation

$$nC_n^\lambda(x) = 2(\lambda + n - 1)xC_{n-1}^\lambda(x) - (2\lambda + n - 2)C_{n-2}^\lambda(x),\quad n \geq 2.$$

Polynomials $C_n^\lambda(x)$ are related to $T_n(x)$ and $U_n(x)$ by the relations

$$T_n(x) = \frac{n}{2}\lim_{\lambda\to 0}\frac{C_n^\lambda(x)}{\lambda}\qquad (n \geq 1)$$

and

$$U_n(x) = C_n^1(x).$$

In Jaiswal [6] and Horadam [5], it was established that $x = 1$ in (1) and (2) yields simple relationships with the Fibonacci numbers F_n defined by

$$F_0 = 0,\ F_1 = 1,\ \text{and } F_n = F_{n-1} + F_{n-2}\qquad (n \geq 2),$$

namely,

$$p_n(1) = F_{n+2} - 1$$

(3)

$$q_n(1) = 2F_n.$$

These results prompt the thought that some generalized Fibonacci connection might exist for $C_n^\lambda(x)$.

In the following sections, we define the polynomials $p_n^\lambda(x)$ related to $C_n^\lambda(x)$, determine their generating function, investigate a few properties, and exhibit the connection between these polynomials and Fibonacci numbers.

2. THE POLYNOMIALS $p_n^\lambda(x)$

Letting

$$(\lambda)_0 = 1 \quad\text{and}\quad (\lambda)_n = \lambda(\lambda+1)\ \ldots\ (\lambda+n-1),\ n = 1, 2, \ldots,$$

we find that the first few Gegenbauer polynomials are

(4) $$C_0^\lambda(x) = 1,\ C_1^\lambda(x) = 2\lambda x,\ C_2^\lambda(x) = \frac{(\lambda)_2}{2!}(2x)^2 - \lambda.$$

Listing the polynomials of (4) horizontally and taking sums along the rising diagonals, we get the resulting polynomials denoted by $p_n^\lambda(x)$. The first few polynomials $p_n^\lambda(x)$ are given by

(5) $$p_1^\lambda(x) = 1,\ p_2^\lambda(x) = 2\lambda x,\ p_3^\lambda(x) = \frac{(\lambda)_2}{2!}(2x)^2,\ p_4^\lambda(x) = \frac{(\lambda)_3}{3!}(2x)^3 - \lambda.$$

We define $p_0^\lambda(x) = 0$.

3. GENERATING FUNCTION

Theorem 1: The generating function $G^\lambda(x, t)$ of $p_n^\lambda(x)$ is given by

$$G^\lambda(x, t) = \sum_{n=1}^{\infty} p_n^\lambda(x)t^{n-1} = (1 - 2xt + t^3)^{-\lambda}.$$

Proof: Putting $2x = y$ in (4) we obtain the following figure.

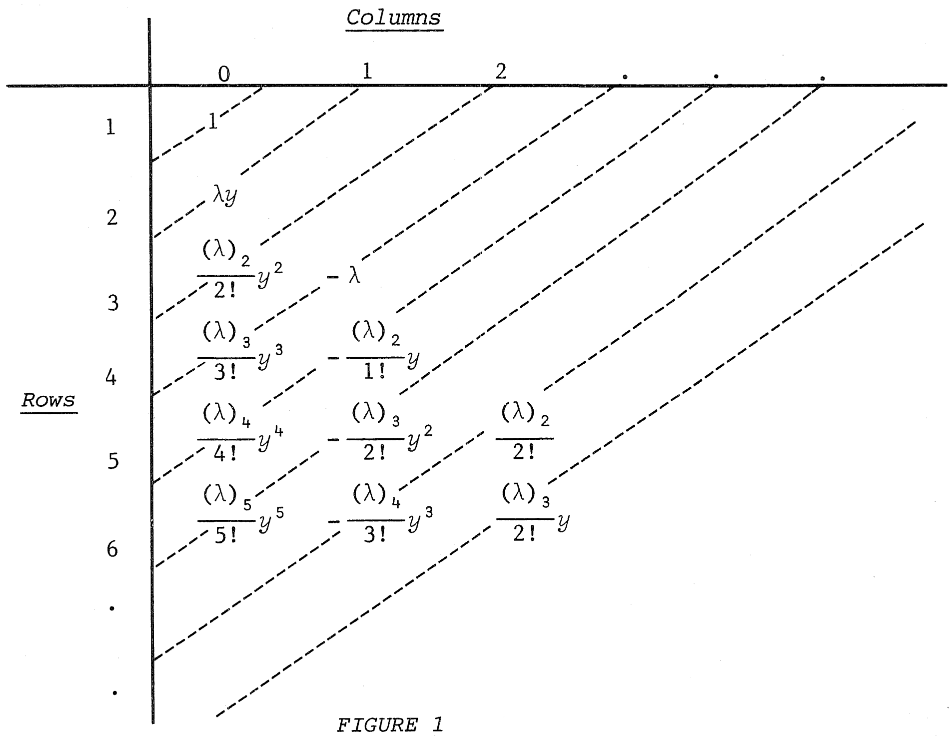

FIGURE 1

It is clear from Figure 1 that the generating function for the kth column is

$$\frac{(-1)^k(\lambda)_k}{k!}(1 - ty)^{-(\lambda+k)}.$$

Since $p_n^\lambda(x)$ are obtained by summing along the rising diagonals of Figure 1, the row-adjusted generating function for the kth column becomes

$$h_k^\lambda(y) = \frac{(-1)^k(\lambda)_k}{k!}(1 - ty)^{-(\lambda+k)}t^{3k}.$$

Since

$$\sum_{k=0}^{\infty} h_k^\lambda(y) = \sum_{k=0}^{\infty} \frac{(-1)^k(\lambda)_k}{k!}\left(\frac{t^3}{1 - ty}\right)^k(1 - ty)^{-\lambda} = (1 - ty + t^3)^{-\lambda},$$

the generating function of $p_n^\lambda(x)$ is given by

$$G^\lambda(x, t) = \sum_{n=1}^{\infty} p_n^\lambda(x)t^{n-1} = (1 - 2tx + t^3)^{-\lambda}. \tag{6}$$

Expanding the right-hand side of (6), we obtain

$$p_{n+1}^\lambda(x) = \sum_{k=0}^{[n/3]} \frac{(-1)^k(\lambda)_{n-2k}}{(n - 2k)!}\binom{n - 2k}{k}(2x)^{n-3k}. \tag{7}$$

Observe from (1), (5), (6), and (7) that $p_n^1(x) = p_n(x)$, $n = 0, 1, \ldots$.

4. RECURRENCE RELATION

Theorem 2: The recurrence relation is given by

$$p_n^\lambda(x) = \frac{(2x)(\lambda + n - 2)}{n - 1} p_{n-1}^\lambda(x) - \frac{3\lambda + n - 4}{n - 1} p_{n-3}^\lambda(x), \quad (n \ge 3). \tag{8}$$

Proof: From (7), the kth term on the right-hand side of (8) is

$$(-1)^k \frac{(\lambda + n - 2)}{n - 1} \frac{(\lambda)_{n-2-2k}}{(n - 2 - 2k)!} \binom{n - 2 - 2k}{k} (2x)^{n-3k-1}$$

$$- (-1)^{k-1} \frac{(3\lambda + n - 4)}{n - 1} \frac{(\lambda)_{n-4-2(k-1)}}{(n - 4 - 2(k - 1))!} \binom{n - 4 - 2(k - 1)}{k - 1} (2x)^{n-3k-1}.$$

After simplification, this becomes

$$\frac{(-1)^k (\lambda)_{n-1-2k} (2x)^{n-3k-1}}{k!(n - 1 - 3k)!},$$

which is the kth term on the left-hand side of (8).

Ordinary Fibonacci numbers F_n are expressible in two equivalent forms:

$$\begin{cases} F_n = F_{n-1} + F_{n-2} \cdots & (\alpha) \\ F_n = 2F_{n-1} - F_{n-3} \cdots & (\beta). \end{cases} \tag{9}$$

Observe that expression (8) in Theorem 2 is of the form (β) in $p_n^\lambda(x)$. An attempt to obtain the recurrence relation in the corresponding form (α), namely,

$$p_n^\lambda(x) = Ap_{n-1}^\lambda(x) + Bp_{n-2}^\lambda(x),$$

where A and B are functions of λ, leads to an intractable cubic. Perhaps the form (8) that follows the patterns of the forms for $p_n(x)$ and $q_n(x)$ is the best available.

The following recurrence relation involving the derivatives of $p_n^\lambda(x)$ is easily proved.

Theorem 3:

$$2x(p_{n+2}^\lambda(x))' - 3(p_n^\lambda(x))' = 2(n + 1)p_{n+2}^\lambda(x). \tag{10}$$

Equation (10) corresponds to the similar results satisfied by $p_n(x)$ and $q_n(x)$.

5. THE POLYNOMIALS $S_n(x)$

Define

$$\begin{cases} S_0(x) = 0, \ S_1(x) = 3, \text{ and} \\ S_n^\lambda(x) \equiv S_n(x) = (n - 1) \lim_{\lambda \to 0} \left[\dfrac{p_n^\lambda(x)}{\lambda} \right] \\ \qquad = \displaystyle\sum_{k=0}^{\left[\frac{n-1}{3}\right]} \frac{(-1)^k (n - 1)}{n - 2k - 1} \binom{n - 2k - 1}{k} y^{n-1-3k}, \\ \qquad (y = 2x), \ n \ge 2. \end{cases} \tag{11}$$

From (5) and (11) we obtain

(12) $$\begin{cases} S_2(x) = 2x,\ S_3(x) = (2x)^2,\ S_4(x) = (2x)^3 - 3, \\ S_5(x) = (2x)^4 - 4(2x),\ S_6(x) = (2x)^5 - 5(2x)^2,\ \ldots . \end{cases}$$

Using (7) and (11) and following the argument of Theorem 2, we have

Theorem 4: $S_n(x) = 2xS_{n-1}(x) - S_{n-3}(x) \qquad (n \geq 3).$

We readily observe the similarity of the form for $S_n(x)$ in Theorem 4 with the forms for $p_n(x)$ and $q_n(x)$ in (1) and (2).

Letting $\lambda = 1$ in (7), using (11), and comparing kth terms, we have

Theorem 5: $S_n(x) = p_n(x) - 2p_{n-3}(x) \qquad (n \geq 3).$

Theorem 6: $S_n(x) = 2q_n(x) - p_n(x) \qquad (n \geq 0).$

Proof: From Horadam [5, Eq. 6],

$$p_n(x) = q_n(x) + p_{n-3}(x) \tag{i}$$

Therefore,

$$\begin{aligned} S_n(x) &= p_n(x) - 2(p_n(x) - q_n(x)) \quad \text{from Theorem 5 and (i)} \\ &= 2q_n(x) - p_n(x), \end{aligned}$$

which proves the Theorem.

Letting $x = 1$, we have by (3)

$$S_n(1) = 2q_n(1) - p_n(1) = 2F_n - F_{n-1} + 1.$$

Using the known generating functions for $p_n(x)$ and $q_n(x)$ given in [6] and [5], respectively, we can readily deduce the generating function for $S_n(x)$ from Theorem 6.

Theorem 2 is valid for all x. Hence Theorem 4 also follows from Theorem 2 on dividing throughout by λ and letting $\lambda \to 0$.

6. THE POLYNOMIALS $q_n^\lambda(x)$

Instead of examining $p_n^\lambda(x)$ as obtained in (7), suppose one investigates the rising diagonal functions $q_n^\lambda(x)$ of

(13) $$n \lim_{\lambda \to 0} \frac{c_n^\lambda(x)}{\lambda} \qquad (n \geq 1).$$

An explicit formulation of $q_n^\lambda(x)$ is

(14) $$q_n^\lambda(x) = \sum_{k=0}^{[n/3]} \frac{(-1)^k (n-k)(\lambda)'_{n-2k}}{(n-2k)!} \binom{n-2k}{k} y^{n-3k} \qquad (y = 2x),$$

where

(15) $$(\lambda)'_{n-2k} = \lambda(\lambda)_{n-2k}.$$

Writing

(16) $$r_n^\lambda(x) = p_{n+1}^\lambda(x) - q_n^\lambda(x)$$

and using (7) and (14), we obtain

(17) $$r_n^\lambda(x) = \sum_{k=0}^{[n/3]} \frac{(-1)^k(\lambda^{-1} - n + k)}{k!(n-3k)!} (\lambda)'_{n-2k} y^{n-3k}.$$

Results similar to those obtained for $p_n^\lambda(x)$ may be obtained for $q_n^\lambda(x)$. At this stage, it is not certain just how useful a study of $q_n^\lambda(x)$ and $r_n^\lambda(x)$ might be.

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ENUMERATION OF PERMUTATIONS BY SEQUENCES—II

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1. André [1] discussed the enumeration of permutations by number of sequences; his results are reproduced in Netto's book [5, pp. 105-12]. Let $P(n, s)$ denote the number of permutations of $Z_n = \{1, 2, \ldots, n\}$ with s ascending or descending sequences. It is convenient to put

$$P(0, s) = P(1, s) = \delta_{0,s}. \tag{1.1}$$

André proved that $P(n, s)$ satisfies

$$P(n+1, s) = sP(n, s) + 2P(n, s-1) + (n-s+1)P(n, s-2), \quad (n \geq 1). \tag{1.2}$$

The following generating function for $P(n, s)$ was obtained in [2]:

$$\sum_{s=0}^{\infty}(1-x^2)^{-n/2}\frac{z^n}{n!}\sum_{s=0}^{n}P(n+1, s)x^{n-s} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2}+\sin z}{x-\cos z}\right)^2. \tag{1.3}$$

However, an explicit formula for $P(n, s)$ was not found.

In the present note, we shall show how an explicit formula for $P(n, s)$ can be obtained. We show first that the polynomial

$$p_n(x) = \sum_{s=0}^{n} P(n+1, x)(-x)^{n-s} \tag{1.4}$$

satisfies

$$p_{2n}(x) = \frac{1}{2^{n-1}}(1-x)^{n-1}\left\{2\sum_{k=1}^{n}(-1)^{n+k}A_{2n+1,k}T_{n-k+1}(x) - A_{2n+1,n+1}\right\} \tag{1.5}$$