FIBONACCI POWERS AND PASCAL'S TRIANGLE IN A MATRIX – PART II

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3. THE P MATRIX — RECURSION RELATIONSHIPS FOR PRODUCTS AND POWERS OF u_n.

A convenient technique [2] for generating several basic Fibonacci identities lies in the use of the second order matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3.1}$$

The technique is based upon the fact that the characteristic polynomial of P is the characteristic polynomial of the second-order recurrent relation $u_{n+1} = u_n + u_{n-1}$ defining the Fibonacci sequence, i.e.,

$$|xI - P| = x^2 - x - 1 \ . \tag{3.2}$$

From (3.1) and (3.2) we have at once

$$P^2 = P + I\,, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$P^n = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} u_{n-1} & u_n \\ u_n & u_{n+1} \end{pmatrix}$$

We shall show that the matrix P_n of (1.1) provides a generalization of (3.1) relative to the n-th powers of u_i. Indeed, (3.1) is Q_1 of (1.1), and $\phi_1(x)$ in (2.20) compares with (3.2).

<u>Theorem I</u> (due originally to Jarden [3])

Let

$$b_r = \prod_{j=1}^{n} h_r^j$$

be the element by element product of n (not necessarily distinct) sequences $\{h_r^j\}$ each of which satisfy the relation

$$h_{r+1}^j = h_r^j + h_{r-1}^j \quad . \tag{3.3}$$

Then $\{b_r\}$ satisfies the recurrence relation

$$\phi_n\{b\} = 0$$

for ϕ_n defined in (2.19).

Proof.

By virtue of (2.18) it is sufficient to show that the determinant $D_n\{b\}$ vanishes for $n+1$ consecutive members of the sequence $\{b_r\}$. Examining $D_n\{a\}$ we note that we can express the element in the r-th row and s-th column by

$$a_{r+1} - u_{r+1}^n a_1 - u_r^n a_0 \,, \qquad \text{if } s = 1$$

$$u_{r+1}^{n+1-s}\, u_r^{s-1} \,, \qquad \text{if } s \neq 1 \; .$$

Hence the determinant is zero for the sequence $\{a\}$ if we can find a solution $\{A_s\}$ which is independent of r and satisfies

$$a_{r+1} = u_{r+1}^n a_1 + u_r^n a_0 + \sum_{s=2}^{n} A_s\, u_{r+1}^{n+1-s}\, u_r^{s-1} \; ; \tag{3.4}$$

that is to say, some method of annihilating the first column by adding a linear combination of the remaining columns. We take

$$a_{r+1} = b_{r+1} = \prod_{j=1}^{n} h_{r+1}^j \; . \tag{3.5}$$

Using the well known formula for general sequences of the type (3.3)

$$h_{r+1} = u_{r+1}h_1 + u_r h_0$$

in (3.5) and expanding, we have

$$a_{r+1} = \prod_{j=1}^{n} (u_{r+1}h_1^j + u_r h_0^j) ,$$

$$a_{r+1} = u_{r+1}^n \prod_{j=1}^{n} h_1^j + u_r^n \prod_{j=1}^{n} h_0^j + \sum_{s=2}^{n} H_s u_{r+1}^{n+1-s} u_r^{s-1} .$$

Clearly, H_s is a combination of the h_0^j and h_1^j, and independent of r. We have satisfied (3.4) and the proof is complete.

Theorem I establishes the recurrence formulae $\phi_n\{a\} = 0$ of (2.19) as generators for the n-th powers and n-th order products of the sequence $\{h_r\}$ of (3.3), and in particular, products of the Fibonacci sequence $\{u_r\}$ of (2.2).

There remains to be constructed the link between P_n and these recurrence formulae. We prove

Theorem II

$$\phi_n(P_n) = 0 .$$

Proof.

Since P_n of (1.1) is related to Q_n of (2.6) by $P_n = E Q_n^T E^{-1}$ with $E = E^{-1}$ being a matrix with ones on the counter diagonal and zeros elsewhere, P_n and Q_n are similar and hence satisfy the same polynomial equations. It is sufficient to show that $\phi_n(Q_n) = 0$.

First, each element of the matrix $B_{n+1,i}$ (2.4) is an element of a sequence of the type

$$b_r = \prod_{j=1}^{n} h_r^j , \text{ defined in (2.8).}$$

Construct the sequence $b_r, b_{r+1}, \ldots, b_{r+n+1}$ by choosing the corresponding elements from the matrix sequence $B_{n+1,i}$, $B_{n+1,i+1}, \ldots, B_{n+1,i+n+1}$. By Theorem I, $\phi_n\{b\} = 0$. Since this is true for any element of $B_{n+1,i}$ it is true for the entire matrix. We have

$$\phi_n\{B\} = 0 \quad \text{identically.} \tag{3.6}$$

Writing out the summation in (3.6),

$$\sum_{r=0}^{n+1} (-1)^r S_r \begin{bmatrix} n+1 \\ r \end{bmatrix} B_{n+1,\, n+1-r-i} = 0 . \tag{3.7}$$

The matrix Q_n may be used (as in (2.5)) to shift the index of B so that

$$B_{n+1,\, n+1-r-i} = B_{n+1,\, 0}\, Q_n^{n+1-r-i} = B_{n+1,\, 0} Q^{-i} Q^{n+1-r} . \tag{3.8}$$

Using (3.8) in (3.7) we have

$$B_{n+1,\, 0}\, Q_n^{-i} \sum_{r=0}^{n+1} (-1)^r S_r \begin{bmatrix} n+1 \\ r \end{bmatrix} Q^{n+1-r} = 0 .$$

Now B is never singular, (2.9), nor is Q, (2.7), so that

$$\sum_{r=0}^{n+1} (-1)^r S_r \begin{bmatrix} n+1 \\ r \end{bmatrix} Q^{n+1-r} = 0$$

which is to say, by (2.20),

$$\phi_n (Q_n) = 0 .$$

Theorem II is implied more directly by Theorem I after having established the following representations for Q_n^r:

$$Q_1^r = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^r = \begin{bmatrix} u_{r+1} & u_r \\ u_r & u_{r-1} \end{bmatrix}$$

$$Q_2^r = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^r = \begin{bmatrix} u_{r+1}^2 & u_{r+1}u_r & u_r^2 \\ 2u_{r+1}u_r & u_{r+1}u_{r-1} + u_r^2 & 2u_r u_{r-1} \\ u_r^2 & u_r u_{r-1} & u_{r-1}^2 \end{bmatrix}$$

$$Q_3^r = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^r = \begin{bmatrix} u_{r+1}^3 & u_{r+1}^2 u_r & u_{r+1}u_r^2 & u_r^3 \\ 3u_{r+1}^2 u_r & \cdots & \cdots & 3u_r u_{r-1} \\ 3u_{r+1}u_r^2 & \cdots & \cdots & 3u_r u_{r-1}^2 \\ u_r^3 & u_r^2 u_{r-1} & u_r u_{r-1}^2 & u_{r-1}^3 \end{bmatrix}$$

etc., where the bordering elements of Q_n^r build up in the manner suggested by these cases and the internal elements, while being more complicated in structure, nevertheless are sums of n-th order products of u's.

Before stating the final theorem we will examine the special case used earlier in terms of what we now know. We have the two matrices

$$B_1 = \begin{bmatrix} u_2^4 & u_2^3 u_1 & u_2^2 u_1^2 & u_2 u_1^3 & u_1^4 \\ u_3^4 & u_3^3 u_2 & u_3^2 u_2^2 & u_3 u_2^3 & u_2^4 \\ u_4^4 & u_4^3 u_3 & u_4^2 u_3^2 & u_4 u_3^3 & u_3^4 \\ u_5^4 & u_5^3 u_4 & u_5^2 u_4^2 & u_5 u_4^3 & u_4^4 \\ u_6^4 & u_6^3 u_5 & u_6^2 u_5^2 & u_6 u_5 & u_5^4 \end{bmatrix}$$

(where the index 1 on B indicates the indices of the first row) and

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

We have the polynomial

$$\phi(x) = x^5 - (5x^4 + 15x^3 - 15x^2 - 5x + 1) \tag{3.9}$$

from (2.21) with $n = 4$ and the corresponding recursion relation

$$b_{n+5} = 5b_{n+4} + 15b_{n+3} - 15b_{n+2} - 5b_{n+1} + b_n \tag{3.10}$$

which is satisfied by any sequence whose members are the element by element product of four Fibonacci sequences — in particular it is satisfied by the sequences formed by extending each <u>column</u> of B_1 ad infinitum, the index of each sequence increasing downward. In view of this fact we construct the matrix

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -5 & -15 & 15 & 5 \end{bmatrix} \tag{3.11}$$

whose obvious property is that of transforming any column vector

$$\begin{bmatrix} b_n \\ b_{n+1} \\ b_{n+2} \\ b_{n+3} \\ b_{n+4} \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} b_{n+1} \\ b_{n+2} \\ b_{n+3} \\ b_{n+4} \\ b_{n+5} \end{bmatrix}$$

if the elements of the vector satisfy the relationship (3.10). E has the property, then, that

$$E B_1 = B_2 . \tag{3.12}$$

It is not difficult to show that the characteristic polynomial of (3.11) is

$$|xI - E| = \phi_4(x)$$

for $\phi_4(x)$ defined in (3.9). Combining (3.12) with the property (2.5) of Q

$$B_2 = E B_1 = B_1 Q ,$$

and B_1 is not singular, hence Q, and therefore P, is similar to, and has the same characteristic polynomial as E.

The preceding example illustrates the proof of the final

Theorem III

The $(n+1) \times (n+1)$ matrix P_n of (1.1), formed by imbedding Pascal's triangle in a square matrix, has the characteristic polynomial

$$(3.13) \qquad |xI - Q_n| = \sum_{r=0}^{n+1} (-1)^r (-1)^{r(r-1)/2} \begin{bmatrix} n+1 \\ r \end{bmatrix} x^{n+1-r}$$

where $\begin{bmatrix} n \\ r \end{bmatrix}$ is a generalized "binomial coefficient" defined by

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{u_n \cdot u_{n-1} \cdots u_{n-r+1}}{u_r \cdot u_{r-1} \cdots u_1} , \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = 1 .$$

Furthermore, the polynomial (3.13) is the same polynomial which characterizes the recursion relation for the element by element product sequence of any n sequences each of which satisfies the Fibonacci recurrence relation $u_{n+1} = u_n + u_{n-1}$.

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2. S. L. Basin and V. E. Hoggatt, "A Primer on the Fibonacci Sequence — Part II," Fibonacci Quarterly, 1(1963), No. 2, pp. 61-68.

3. Dov Jarden, Recurring Sequences, Riveon Lematematika, 1958, pp. 42-44.

Additional Reading

4. R. Bellman, Introduction to Matrix Analysis, McGraw-Hill, 1960, pp. 228-229 (Kronecker Powers of Matrices.)

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THE GOLDEN CUBOID

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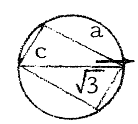
The problem of finding the dimensions of a cuboid (rectangular parallelopiped) of unit volume, having a diagonal 2 units in length leads to an interesting result.

Suppose the lengths of the edges are a, b and c. Then

$$(1) \quad a \cdot b \cdot c = 1 \qquad \text{and} \qquad (2) \sqrt{(a^2 + b^2 + c^2)} = 2$$

If only the ratios of these lengths are required, we may, without loss of generality, write $b = 1$, provided that $a \cdot c$ can have the value unity and that $a^2 + c^2 = 3$. Now it is evident from Fig. 1, which represents the base of the cuboid, that the maximum value of $a \cdot c$ occurs when $a = c = \sqrt{3/2}$, so that $a \cdot c$ may have any value from zero to 3/2.



base

Fig. 1

Substituting $c = 1/a$ from (1) in (2), we have

$$a^2 + \frac{1}{a^2} = 3 \quad \text{i.e.,} \quad a^4 - 3a^2 + 1 = 0, \quad \text{whence}$$

$$a^2 = \frac{3 + \sqrt{5}}{2} = 1 + \varphi = \varphi^2,$$

so that $a = \varphi$, the Golden Section. The positive solution of the equation $x^2 - x - 1 = 0$ and the value of u_n/u_{n-1} as $n \to \infty$, where u_n is a member of the Fibonacci Series.

From (1) it follows that $c = \varphi^{-1}$, so that the required ratios are $a:b:c = \varphi:1:\varphi^{-1}$. It is easily verified that $\varphi^2 + 1 + \varphi^{-2} = 4$.

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