A PROPERTY OF THE FIBONACCI SEQUENCE (F_m), $m = 0, 1, \ldots$

L. KUIPERS
Rue d'Orzival 4, 3960 SIERRE, Switzerland
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It is well known that the sequence of the (natural) logarithms reduced mod 1 of the terms F_m of the Fibonacci sequence are dense in the unit interval. See [1], [2]. This is also the case when the logarithms are taken with respect to a base b, where b is a positive integer ≥ 2. In order to see this, we start from the fact that

$$\log F_{n+1} - \log F_n \to \log \frac{1+\sqrt{5}}{2} \text{ as } n \to \infty.$$

Now $\log \frac{1+\sqrt{5}}{2} / \log b$ is an irrational number, for if we suppose that

$$\log \frac{1+\sqrt{5}}{2} / \log b = r/s,$$

where r and s are natural numbers, then we would have

$$b^r = ((1+\sqrt{5})/2)^s,$$

obviously a contradiction. Hence, $\log_b F_{n+1} - \log_b F_n$ tends to an irrational number as $n \to \infty$. This implies that the fractional parts of the sequence

$$(\log_b F_m),\ m = 1, 2, \ldots$$

is dense in the unit interval.

We assume that the Fibonacci numbers F_m, $m \geq 1$, are written in base b, that is,

$$F_m = a_0 b^n + a_1 b^{n-1} + \cdots + a_n,$$

where $a_0 \geq 1$, $0 \leq a_j \leq b - 1$, $j = 0, 1, \ldots, n$, $m = 1, 2, \ldots$, or to any m a set of digits $\{a_0, a_1, \ldots, a_n\}$ is associated.

Now, given an arbitrary sequence of digits $\{a_0, a_1, \ldots, a_r\}$, one may ask whether there exists an F_m which possesses this set as *initial digits*. The question can be answered in the affirmative.

We associate to the sequence $\{a_0, a_1, \ldots, a_r\}$ the value

$$a_0 + \frac{a_1}{b} + \cdots + \frac{a_r}{b^r},$$

which is a point on the interval $[1, b)$. This value is the left endpoint of the interval

$$T = T(r) = \left[a_0 + \frac{a_1}{b} + \cdots + \frac{a_r}{b^r},\ a_0 + \frac{a_1}{b} + \cdots + \frac{a_r}{b^r} + \frac{a_r + 1}{b^r}\right).$$

The function $\log_b x$, mapping $[1, b)$ onto $[0, 1)$, maps this interval $T(r)$ onto the interval

$$T^* = T^*(r) = \left[\log_b\left(a_0 + \frac{a_1}{b} + \cdots + \frac{a_r}{b^r}\right),\ \log_b\left(a_0 + \frac{a_1}{b} + \cdots + \frac{a_r}{b^r} + \frac{1}{b^r}\right)\right),$$

a subinterval of $[0, 1)$.

Since the fractional parts of the logarithms to base b of the numbers F_m are dense in the unit interval, there is an m such that $\log_b F_m \pmod 1 \in T^*$. It follows that there exists a positive integer $n \geq r$ such that

$$\log_b F_m \pmod 1 = \log_b\left(a_0 + \frac{a_1}{b} + \frac{a_2}{b^2} + \cdots + \frac{a_r}{b^r} + \cdots + \frac{a_n}{b^n}\right).$$

Hence, there exists an integer $k \geq n$ such that

$$\log_b F_m = k + \log_b\left(a_0 + \frac{a_1}{b_1} + \cdots + \frac{a_r}{b^r} + \cdots + \frac{a_n}{b^n}\right),$$

or

$$F_m = b^k\left(a_0 + \frac{a_1}{b} + \cdots + \frac{a_r}{b^r} + \cdots + \frac{a_n}{b^n}\right)$$

$$= a_0 b^k + a_1 b^{k-1} + \cdots + a_r b^{k-r} + \cdots + a_n^{k-n}.$$

References

1. R. L. Duncan. "An Application of Uniform Distributions to the Fibonacci Numbers." *The Fibonacci Quarterly* 5, no. 2 (1967):137-140.
2. L. Kuipers. "Remark on a paper by R. L. Duncan Concerning the Uniform Distribution Mod 1 of the Sequence of the Logarithms of the Fibonacci Numbers." *The Fibonacci Quarterly* 7, no. 5 (1969):465, 466, 473.

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