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A NOTE ON FIBONACCI CUBATURE

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Zaremba [3] considered the two-dimensional cubature formula

$$\int_0^1\int_0^1 f(x,\, y)\,dx\,dy = \frac{1}{F_N}\sum_{k=1}^{F_N} f(x_k,\, y_k),$$

where F_N is the Nth Fibonacci number and the nodes $(x_k,\, y_k)$ are defined as follows: $x_k = k/F_N$ and $y_k = \{F_{N-1}x_k\}$, where { } denotes the fractional part. Thus, $y_k = F_{N-1}x_k - [F_{N-1}x_k]$, where [ ] denotes the greatest integer function. The purpose of this paper is to prove the conjecture stated by Squire in [2]; that is,

Theorem

If $(x_k,\, y_k)$ is a node for $1 \leqslant k \leqslant F_N - 1$ and if N is $\begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix}$, then

$$\begin{pmatrix}(y_k,\, x_k)\\ (y_k,\, 1 - x_k)\end{pmatrix}$$

is also a node.

We will assume throughout that $1 \leqslant k \leqslant F_N - 1$, $N > 2$, and will show:

(i) Each y_k is equal to some x_m, $1 \leqslant m \leqslant F_N - 1$.

(ii) The y_k's are distinct.

By definition, the x_k's are distinct, and so (i) and (ii) imply that for every node $(x_k,\, y_k)$ there is a unique node $(x_m,\, y_m)$ with $x_m = y_k$.

Finally, we show:

(iii) If $(x_m,\, y_m)$ is the node with $x_m = y_k$, then

$$y_m = \begin{cases} x_k & \text{if } N \text{ is even,}\\ 1 - x_k & \text{if } N \text{ is odd.}\end{cases}$$

Proof of (i): We have

$$y_k = \{F_{N-1}x_k\} = \left\{k\frac{F_{N-1}}{F_N}\right\}$$
$$= k\frac{F_{N-1}}{F_N} - \left[k\frac{F_{N-1}}{F_N}\right] \qquad (1)$$
$$= \left(k\,F_{N-1} - F_N\left[k\frac{F_{N-1}}{F_N}\right]\right)\Big/F_N.$$

Now from [1, p. 288], gcd $(F_{N-1}, F_N) = 1$, and so

$$0 < k\frac{F_{N-1}}{F_N} - \left[k\frac{F_{N-1}}{F_N}\right] < 1.$$

Thus

$$0 < k\,F_{N-1} - F_N\left[k\frac{F_{N-1}}{F_N}\right] < F_N,$$

where the middle quantity in this inequality is an integer and is also the numerator of the right-hand side of (1). Hence, y_k is equal to some x_m, $1 \leqslant m \leqslant F_N - 1$.

Proof of (ii): To show the y_k's are distinct, we will prove $y_k = y_m$ if and only if $k = m$. Assume, without loss of generality, that $1 \leqslant m \leqslant k$. If $y_k = y_m$, we have

$$\left\{k\frac{F_{N-1}}{F_N}\right\} = \left\{m\frac{F_{N-1}}{F_N}\right\},$$
$$k\frac{F_{N-1}}{F_N} - \left[k\frac{F_{N-1}}{F_N}\right] = m\frac{F_{N-1}}{F_N} - \left[m\frac{F_{N-1}}{F_N}\right],$$
$$(k - m)\frac{F_{N-1}}{F_N} = \left[k\frac{F_{N-1}}{F_N}\right] - \left[m\frac{F_{N-1}}{F_N}\right]. \qquad (2)$$

Now recalling gcd $(F_{N-1}, F_N) = 1$ and since $0 \leqslant k - m < F_N$, $(k - m)F_{N-1}/F_N$ is never an integer unless $k - m = 0$. However, the right-hand side of (2) is always an integer, and so $y_k = y_m$ if and only if $k = m$.

Proof of (iii): Assume that (x_m, y_m) is the node with $x_m = y_k$. Then

$$y_m = \{F_{N-1}x_m\} = \{F_{N-1}y_k\}$$

$$= \left\{ F_{N-1}\left(k\,\frac{F_{N-1}}{F_N} - \left[k\,\frac{F_{N-1}}{F_N} \right] \right) \right\}$$

$$= \left\{ k\,\frac{F_{N-1}^2}{F_N} - F_{N-1}\left[k\,\frac{F_{N-1}}{F_N} \right] \right\}.$$

From [1, p. 294], we have $F_{N-1}^2 = F_N F_{N-2} + (-1)^{N-2}$ for $N \geqslant 3$, and so

$$y_m = \left\{ k\,F_{N-2} + (-1)^{N-2}k/F_N - F_{N-1}\left[k\,\frac{F_{N-1}}{F_N} \right] \right\}.$$

Now if n is any integer $\{n + x\} = x - [x]$, and since

$$k\,F_{N-2} - F_{N-1}[k\,F_{N-1}/F_N]$$

is an integer, we have

$$y_m = (-1)^{N-2}k/F_N - [(-1)^{N-2}k/F_N]$$

$$= \begin{cases} k/F_N - 0 = x_k & \text{if } N \text{ is even,} \\ -k/F_N - (-1) = 1 - x_k & \text{if } N \text{ is odd.} \end{cases}$$

REFERENCES

1. David M. Burton. *Elementary Number Theory*. Boston: Allyn and Bacon, 1980.
2. William Squire. "Fibonacci Cubature." *The Fibonacci Quarterly* 19, no. 4 (1981):313-14.
3. S. K. Zaremba. "Good Lattice Points, Discrepancy, and Numerical Integration." *Ann. Mat. Pura. Appl.* 73 (1966):293-317.

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