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ADDENDA TO GEOMETRY OF A GENERALIZED SIMSON'S FORMULA

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1. INTRODUCTION

In [3], the author considers the loci in the Euclidean plane satisfied by points whose Cartesian coordinates are pairs of successive members in recurrence sequences of a special type. The purpose of this paper is to extend that discussion.

We begin as in [1] and [3] by defining the general term of the sequence $\{w_n(a,\ b;\ p,\ q)\}$ as

$$w_{n+2} = pw_{n+1} - qw_n,\ w_0 = a,\ w_1 = b, \tag{1.1}$$

where a, b, p, q belong to some number system, but are generally thought of as integers. In this paper, they will always be integers.

In [1], we find

$$w_n w_{n+2} - w_{n+1}^2 = eq^n, \tag{1.2}$$

where

$$e = pab - qa^2 - b^2. \tag{1.3}$$

Combining (1.1) and (1.2) as in [3], we obtain

$$qw_n^2 - pw_n w_{n+1} + w_{n+1}^2 + eq^n = 0, \tag{1.4}$$

which, with $w_n = x$ and $w_{n+1} = y$, becomes

$$qx^2 - pxy + y^2 + eq^n = 0. \tag{1.5}$$

The graph of (1.5) is a hyperbola if $p^2 - 4q > 0$, an ellipse (or circle) if $p^2 - 4q < 0$, and a parabola if $p^2 - 4q = 0$ (degenerate cases excluded). The xy term can be eliminated by performing a counterclockwise rotation of the axes through the angle θ, where

$$\cot 2\theta = \frac{1 - q}{p}, \tag{1.6}$$

using

$$\begin{aligned} x &= \bar{x}\cos\theta - \bar{y}\sin\theta \\ y &= \bar{x}\sin\theta + \bar{y}\cos\theta. \end{aligned} \tag{1.7}$$

When $q = 1$, $\theta = \pi/4$, and (1.5) becomes

$$(2 - p)\bar{x}^2 + (p + 2)\bar{y}^2 + 2e = 0 \tag{1.8}$$

with

$$e = pab - a^2 - b^2. \tag{1.9}$$

When $q \neq 1$, and therefore $\theta \neq \pi/4$, we let

$$r = \sqrt{p^2 + (q - 1)^2}. \tag{1.10}$$

Substituting (1.7) into (1.5) and using the double angle formulas for $\cos 2\theta$ and $\sin 2\theta$, we find that

$$\bar{x}^2\left(\frac{q + 1 + (q - 1)\cos 2\theta - p \sin 2\theta}{2}\right)$$
$$+ \bar{y}^2\left(\frac{q + 1 - (q - 1)\cos 2\theta + p \sin 2\theta}{2}\right) + eq^n = 0.$$

Now, by (1.1), depending upon the values of q and p, we have

$$\begin{cases} \left(\frac{q + 1 + r}{2}\right)\bar{x}^2 + \left(\frac{q + 1 - r}{2}\right)\bar{y}^2 + eq^n = 0 \text{ if } p < 0 \\ \left(\frac{q + 1 - r}{2}\right)\bar{x}^2 + \left(\frac{q + 1 + r}{2}\right)\bar{y}^2 + eq^n = 0 \text{ if } p > 0. \end{cases} \tag{1.11}$$

We now consider the special cases when $q = 1$ (§2) and when $q = -1$ (§3).

2. THE SPECIAL CASES WHEN $q = 1$

If $p = 2$, we have from (1.8) the degenerate conic $2\bar{y}^2 = -e = (a - b)^2$ which gives rise to the parallel lines $x - y = b - a$ and $x - y = a - b$. The sequence of terms associated with this degenerate conic is

$$a,\ b,\ 2b - a,\ 3b - 2a,\ 4b - 3a,\ 5b - 4a,\ \ldots\ . \tag{2.1}$$

Since none of the successive pairs of (2.1) satisfy $x - y = b - a$, we see that all pairs (w_n, w_{n+1}) of (2.1) lie on the line $x - y = a - b$.

If $p = -2$, the degenerate conic is $2\bar{x}^2 = -e = (a + b)^2$, the sequence is

$$a,\ b,\ -2b - a,\ 3b + 2a,\ -4b - 3a,\ 5b + 4a,\ \ldots, \tag{2.2}$$

and the successive pairs (w_n, w_{n+1}) of (2.2) satisfy $x + y = a + b$ if n is even and $x + y = -(a + b)$ when n is odd.

If $p = 0$, the sequence $\{w_n(a, b; 0, 1)\}$ is

$$a,\ b,\ -a,\ -b,\ a,\ b,\ -a,\ -b,\ \ldots, \tag{2.3}$$

so that for (2.3) the only distinct pairs of successive coordinates on the circle $x^2 + y^2 = -e = a^2 + b^2$ are (a, b), $(b, -a)$, $(-a, -b)$, $(-b, a)$.

If $p = 1$, then equation (1.8) becomes $\bar{x}^2 + 3\bar{y}^2 = 2(a^2 + b^2 - ab)$. But $a^2 + b^2 - ab > 0$ if a and b are not both zero, so the graph of (1.5) is always an ellipse with the equation $x^2 + y^2 - xy = a^2 + b^2 - ab$. The sequence $\{w_n(a, b; 1, 1)\}$ is

$$a,\ b,\ b - a,\ -a,\ -b,\ -b + a,\ a,\ b,\ \ldots\ . \tag{2.4}$$

The only distinct pairs of successive coordinates on the ellipse for (2.4) are (a, b), $(b, b - a)$, $(b - a, -a)$, $(-a, -b)$, $(-b, -b + a)$, $(-b + a, a)$.

When $p = -1$, equation (1.8) becomes $3\bar{x}^2 + \bar{y}^2 = -2e = 2(a^2 + b^2 + ab)$. If a and b are not both zero, then $a^2 + b^2 + ab > 0$, so that the graph of (1.5) with equation $x^2 + xy + y^2 = a^2 + b^2 + ab$ is an ellipse. The sequence $\{w_n(a, b; -1, 1)\}$ is

$$a,\ b,\ -b - a,\ a,\ b,\ -b - a,\ \ldots, \tag{2.5}$$

so that the only pairs of successive coordinates of (2.5) on the ellipse are (a, b), $(b, -b - a)$, $(-b - a, a)$.

One might wonder about the case $e = 0$. Under this condition, since a and p are integers, we have $p = \pm 2$, which has already been discussed, or $a = b = 0$, which is a trivial case.

For all other values of p, the graph of (1.8), and hence of (1.5), is a hyperbola. Thus, there exists an infinte number of distinct pairs of integers (w_n, w_{n+1}) lying on each hyperbola for a given p. The following facts help to characterize the hyperbola for a given p.

If $p > 2$ and $e < 0$ or $p < -2$ and $e > 0$, then the asymptotes for (1.5) are

$$y = \frac{p \pm \sqrt{p^2 - 4}}{2}\, x, \tag{2.6}$$

the vertices are

$$\left(\sqrt{\frac{-e}{p+2}}, -\sqrt{\frac{-e}{p+2}}\right) \quad \text{and} \quad \left(-\sqrt{\frac{-e}{p+2}}, \sqrt{\frac{-e}{p+2}}\right) \tag{2.7}$$

the eccentricity is

$$\sqrt{\frac{2p}{p-2}}, \tag{2.8}$$

the foci are

$$\left(\sqrt{\frac{-2pe}{p^2-4}}, -\sqrt{\frac{-2pe}{p^2-4}}\right) \quad \text{and} \quad \left(-\sqrt{\frac{-2pe}{p^2-4}}, \sqrt{\frac{-2pe}{p^2-4}}\right) \tag{2.9}$$

and the endpoints of the latera recta are

$$\left(\frac{s-t}{p^2-4}, \frac{s+t}{p^2-4}\right), \left(\frac{s+t}{p^2-4}, \frac{s-t}{p^2-4}\right), \left(\frac{s+t}{4-p^2}, \frac{s-t}{4-p^2}\right), \left(\frac{s-t}{4-p^2}, \frac{s+t}{4-p^2}\right), \tag{2.10}$$

where $s = (p+2)\sqrt{-e(p+2)}$ and $t = -\sqrt{2pe(p^2-4)}$.

If $p > 2$ and $e > 0$ or $p < -2$ and $e < 0$, then the asymptotes and eccentricity are found by using (2.6) and (2.8). The vertices are

$$\left(\sqrt{\frac{e}{p-2}}, \sqrt{\frac{e}{p-2}}\right), \left(-\sqrt{\frac{e}{p-2}}, -\sqrt{\frac{e}{p-2}}\right), \tag{2.11}$$

the foci are

$$\left(\sqrt{\frac{2pe}{p^2-4}}, \sqrt{\frac{2pe}{p^2-4}}\right), \left(-\sqrt{\frac{2pe}{p^2-4}}, -\sqrt{\frac{2pe}{p^2-4}}\right), \tag{2.12}$$

and the endpoints of the latera recta are

$$\begin{gathered}\left(\frac{s_1-t_1}{p^2-4}, \frac{s_1+t_1}{p^2-4}\right), \left(\frac{s_1+t_1}{p^2-4}, \frac{s_1-t_1}{p^2-4}\right), \\ \left(\frac{s_1+t_1}{4-p^2}, \frac{s_1-t_1}{4-p^2}\right), \left(\frac{s_1-t_1}{4-p^2}, \frac{s_1+t_1}{4-p^2}\right),\end{gathered} \tag{2.13}$$

where $s_1 = \sqrt{2pe(p^2-4)}$ and $t_1 = (p-2)\sqrt{e(p-2)}$.

3. THE SPECIAL CASES WHEN $q = -1$

Letting $q = -1$ in (1.11) and simplifying, we have

$$\overline{x}^2 - \overline{y}^2 = \frac{2e(-1)^n}{r}, \; p > 0, \tag{3.1}$$

and

$$\bar{y}^2 - \bar{x}^2 = \frac{2e(-1)^n}{r}, \; p < 0, \tag{3.2}$$

where the values of e and r from (1.3) and (1.10) are now

$$e = pab + a^2 - b^2 \quad \text{and} \quad r = \sqrt{p^2 + 4}. \tag{3.3}$$

The case $p = 0$ is trivial and, therefore, omitted.

Since (3.1) and (3.2) are always the equations of a hyperbola, unless $e = 0$, which is a trivial case, $a = b = 0$, there are always an infinite number of distinct pairs of integers (w_n, w_{n+1}) which lie on the original hyperbola of (1.5) for any value of p, unless the sequence is cyclic. The following facts characterize the hyperbola for different values of p, e, and n.

The asymptotes of (1.5) which are perpendicular are always given by

$$y = \frac{p \pm r}{2}x, \tag{3.4}$$

and the eccentricity is always 2, giving a rectangular hyperbola. These cases are in accord with the cases $p = 1$ and $p = 2$ given in [3].

If $p > 0$ and $e(-1)^n > 0$, then the vertices are

$$(u, v), \; (-u, -v), \tag{3.5}$$

the foci are

$$(u\sqrt{2}, v\sqrt{2}), \; (-u\sqrt{2}, -v\sqrt{2}), \tag{3.6}$$

and the endpoints of the latera recta are

$$\begin{gathered}(u\sqrt{2} - v, u + v\sqrt{2}), \; (u\sqrt{2} + v, -u + v\sqrt{2}), \\ (-u\sqrt{2} - v, u - v\sqrt{2}), \; (-u\sqrt{2} + v, -u - v\sqrt{2}),\end{gathered} \tag{3.7}$$

where $u = \frac{1}{r}\sqrt{e(-1)^n(r + 2)}$ and $v = \frac{1}{r}\sqrt{e(-1)^n(r - 2)}$.

If $p > 0$ and $e(-1)^n < 0$, then the vertices are

$$(v_1, -u_1), \; (-v_1, u_1), \tag{3.8}$$

the foci are

$$(-v_1\sqrt{2}, u_1\sqrt{2}), \; (v_1\sqrt{2}, -u_1\sqrt{2}), \tag{3.9}$$

and the endpoints of the latera recta are

$$\begin{aligned}&(u_1 - v_1\sqrt{2},\ u_1\sqrt{2} + v_1),\ (u_1 + v_1\sqrt{2},\ -u_1\sqrt{2} + v_1),\\&(-u_1 - v_1\sqrt{2},\ u_1\sqrt{2} - v_1),\ (-u_1 + v_1\sqrt{2},\ -u_1\sqrt{2} - v_1),\end{aligned} \tag{3.10}$$

where $u_1 = \frac{1}{r}\sqrt{e(-1)^{n+1}(r + 2)}$ and $v_1 = \frac{1}{r}\sqrt{e(-1)^{n+1}(r - 2)}$.

If $p < 0$ and $e(-1)^n > 0$, then the vertices are

$$(-u,\ v),\ (u,\ -v), \tag{3.11}$$

the foci are

$$(-u\sqrt{2},\ v\sqrt{2}),\ (u\sqrt{2},\ -v\sqrt{2}), \tag{3.12}$$

and the endpoints of the latera recta are

$$\begin{aligned}&(v - u\sqrt{2},\ v\sqrt{2} + u),\ (v + u\sqrt{2},\ - v\sqrt{2} + u),\\&(-v - u\sqrt{2},\ v\sqrt{2} - u),\ (-v + u\sqrt{2},\ -v\sqrt{2} - u),\end{aligned} \tag{3.13}$$

where u and v are as before.

If $p < 0$ and $e(-1)^n < 0$, then the vertices are

$$(v_1,\ u_1),\ (-v_1,\ -u_1), \tag{3.14}$$

the foci are

$$(v_1\sqrt{2},\ u_1\sqrt{2}),\ (-v_1\sqrt{2},\ -u_1\sqrt{2}), \tag{3.15}$$

and the endpoints of the latera recta are

$$\begin{aligned}&(v_1\sqrt{2} - u_1,\ v_1 + u_1\sqrt{2}),\ (v_1\sqrt{2} + u_1,\ -v_1 + u_1\sqrt{2}),\\&(-v_1\sqrt{2} - u_1,\ v_1 - u_1\sqrt{2}),\ (-v_1\sqrt{2} + u_1,\ -v_1 - u_1\sqrt{2}),\end{aligned} \tag{3.16}$$

where u_1 and v_1 are as before.

4. CONCLUDING REMARKS

Consider $p > 0$. Note that the hyperbola for $e < 0$ and n odd is the same as the hyperbola for $e > 0$ and n even for any pairs (a, b) giving the same value of e, while the hyperbola for $e < 0$ and n even is the same as the hyperbola for $e > 0$ and n odd for any pair $(a,\ b)$ giving the same e. A similar statement holds if $p < 0$.

For the sequence $Q = \{w_n(a,\ b;\ p,\ -1)\}$, we know from (1.4) that

$$pw_nw_{n+1} + w_n^2 - w_{n+1}^2 = \pm e,$$

depending on whether n is even or odd. Let $e < 0$ for $n = 0$ and

$$R = \{w_n(w_{2m},\ w_{2m+1};\ p,\ -1)\}.$$

The successive pairs of Q and R lie on $\overline{y}^2 - \overline{x}^2 = \frac{-2e}{r}$ if n is even and on $\overline{x}^2 - \overline{y}^2 = \frac{-2e}{r}$ if n is odd. Let

$$S = \{w_n(w_{2m+1},\ w_{2m+2};\ p,\ -1)\}.$$

then the successive pairs of S with n even lie on the same hyperbola as the successive pairs of Q with n odd. That is, they lie on

$$\overline{x}^2 - \overline{y}^2 = \frac{2e(-1)^n}{r}.$$

Furthermore, the successive pairs of S with n odd lie on the same hyperbola as the successive pairs of Q with n even. That is, they lie on

$$\overline{y}^2 - \overline{x}^2 = \frac{2e(-1)^{n+1}}{r}.$$

We close by mentioning that the vertices for the Fibonacci sequence $\{w_n(0,\ 1;\ 1,\ -1)\}$ are

$$\left(\sqrt{\frac{\sqrt{5}+2}{5}},\ \sqrt{\frac{\sqrt{5}-2}{5}}\right),\ \left(-\sqrt{\frac{\sqrt{5}+2}{5}},\ -\sqrt{\frac{\sqrt{5}-2}{5}}\right)$$

when n is odd and

$$\left(\sqrt{\frac{\sqrt{5}-2}{5}},\ -\sqrt{\frac{\sqrt{5}+2}{5}}\right),\ \left(-\sqrt{\frac{\sqrt{5}-2}{5}},\ \sqrt{\frac{\sqrt{5}+2}{5}}\right)$$

when n is even. Furthermore, all the pairs $(F_{2n},\ F_{2n+1})$ lie on the right half of the positive branch of $\overline{y}^2 - \overline{x}^2 = 2/r$ when $n > 0$, and on the left half of the positive branch of $\overline{y}^2 - \overline{x}^2 = 2/r$ when $n < 0$, so that no points $(F_n,\ F_{n+1})$ lie on the negative branch of the hyperbola. A similar remark holds for $(F_{2n+1},\ F_{2n+2})$ and $\overline{x}^2 - \overline{y}^2 = 2/r$.

REFERENCES

1. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* 3, No. 3 (1965):161-76.
2. A. F. Horadam. "Special Properties of the Sequence $w_n(a,\ b;\ p,\ q)$." *The Fibonacci Quarterly* 5, No. 5 (1967):424-34.
3. A. F. Horadam. "Geometry of a Generalized Simson's Formula." *The Fibonacci Quarterly* 20, No. 2 (1982):164-68.
4. Edwards & Penny. "Calculus and Analytic Geometry." Prentice-Hall, Inc., 1982, pp. 416-42.

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