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RULES FOR CONSTRUCTING HYPERPERFECT NUMBERS

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1. INTRODUCTION

As usual, let $\sigma(n)$ denote the sum of all the divisors of n [with $\sigma(1) = 1$] and let $\omega(n)$ denote the number of different prime factors of n [with $\omega(1) := 0$]. The set of prime numbers will be denoted by $\mathscr{P}$. The set of hyperperfect numbers (HP's) is the set $M := \bigcup_{n=1}^{\infty} M_n$, where

$$M_n := \{m \in \mathbf{N} \mid m = 1 + n[\sigma(m) - m - 1]\}. \tag{1}$$

We also define the sets

$$_kM_n := \{m \in M_n \mid \omega(m) = k\},\ k,\ n \in \mathbf{N}, \tag{2}$$

and $_kM := \bigcup_{n=1}^{\infty} {}_kM_n$; clearly, we have $M_n = \bigcup_{k=1}^{\infty} {}_kM_n$. We will also use the related set $M^* := \bigcup_{n=1}^{\infty} M_n^*$, where

$$M_n^* := \{m \in \mathbf{N} \mid m = 1 + n[\sigma(m) - m]\}, \tag{3}$$

and the sets

$$_kM_n^* := \{m \in M_n^* \mid \omega(m) = k\},\ k \in \mathbf{N} \cup \{0\},\ n \in \mathbf{N}, \tag{4}$$

and $_kM^* := \bigcup_{n=1}^{\infty} {}_kM_n^*$, so that also $M_n^* = \bigcup_{k=0}^{\infty} {}_kM_n^*$.

It is not difficult to verify that $_1M_n = \emptyset$, $\forall n \in \mathbf{N}$, and that

$$\begin{cases} {}_0M_n^* = \{1\},\ \forall n \in \mathbf{N} \text{ and} \\ {}_1M_n^* = \begin{cases} \{(n+1)^{\alpha},\ \alpha \in \mathbf{N}\}, & \text{if } n+1 \in \mathscr{P}, \\ \emptyset, & \text{if } n+1 \notin \mathscr{P}. \end{cases} \end{cases} \tag{5}$$

M_1 is the set of perfect numbers [for which $\sigma(m) = 2m$]. The n-hyperperfect numbers M_n, introduced by Minoli and Bear [1], are a meaningful generalization of the even perfect numbers because of the following rule.

RULE 0 (from [2]): If $p \in \mathscr{P}$, $\alpha \in \mathbf{N}$, and if $q := p^{\alpha+1} - p + 1 \in \mathscr{P}$, then $p^{\alpha}q \in M_{p-1}$.

There are 71 hyperperfect numbers below 10^7 (see [2], [3], [4], [5]). Only one of them belongs to ${}_3M$, all others are in ${}_2M$. In [6] and [7] the present author has constructively computed several elements of ${}_3M$ and two of ${}_4M$.

In Section 2 of this paper, we shall give rules by which one may find (with enough computer time) an element of ${}_{(k+2)}M_n$ and of ${}_{(k+1)}M_n$ from an element of ${}_kM_n^*$ ($k \geqslant 0$), and an element of ${}_kM_n^*$ from an element of ${}_{(k-2)}M_n^*$ ($k \geqslant 2$). Because of (5), this suggests the possibility to construct HP's with k different prime factors for any positive integer $k \geqslant 2$. By actually applying the rules, we have found many elements of ${}_3M$, seven elements of ${}_4M$, and one element of ${}_5M$.[1]

In Section 3, necessary and sufficient conditions are given for numbers of the form $p^{\alpha}q$, $\alpha \in \mathbf{N}$, to be hyperperfect. For example, for $\alpha \geqslant 3$, these conditions imply that there are no other HP's of the form $p^{\alpha}q$ than those characterized by Rule 0. The results of this section enable us to compute very cheaply *all* HP's of the form $p^{\alpha}q$ below a given bound. Unfortunately, we have not been able to extend these results to more complicated HP's like those of the form $p^{\alpha}q^{\beta}$, $\alpha \geqslant 2$ and $\beta \geqslant 2$, or $p^{\alpha}q^{\beta}r^{\gamma}$ with $\alpha \geqslant 1$, $\beta \geqslant 1$ and $\gamma \geqslant 1$, etc. (However, these numbers are extremely scarce compared to HP's of the form $p^{\alpha}q$, and no HP's of the form $p^{\alpha}q^{\beta}$ and $p^{\alpha}q^{\beta}r^{\gamma}$ with $\alpha \geqslant 2$ and $\beta \geqslant 2$ have been found to date.)

Because of the importance of the set M^* for the construction of hyperperfect numbers, we given in Section 4 the results of an exhaustive search for all $m \in M^*$ with $m \leqslant 10^8$ and $\omega(m) \geqslant 2$. It turned out that elements of ${}_3M^*$ are very rare compared with ${}_2M^*$, in analogy with the sets ${}_3M$ and ${}_2M$. This search also gave all elements $\leqslant 10^8$ of M, at very low cost, because of the similarity of the equations defining M^* and M. See note 1 below.

The paper concludes with a few remarks, in Section 5, on a possible generalization of hyperfect numbers to so-called hypercycles, special cases of which are the ordinary perfect numbers and the amicable number pairs.

[1] *Lists of these numbers may be obtained from the author on request.*

Remark: After completing this paper, the author computed, with the rules given in Section 2, 860 HP's below the bound 10^{10}. See note 1 above.

2. RULES FOR CONSTRUCTING HYPERPERFECT NUMBERS

We have found the following rules [we write $\overline{a}$ for $\sigma(a)$]:

RULE 1: Let $k \in \mathbf{N}$, $n \in \mathbf{N}$, $a \in {}_kM_n^*$, and $p := n\overline{a} + 1 - n$; if $p \in \mathscr{P}$, then $ap \in {}_{(k+1)}M_n$.

RULE 2: Let $k \in \mathbf{N} \cup \{0\}$, $n \in \mathbf{N}$, $a \in {}_kM_n^*$, and $p := n\overline{a} + A$, $q := n\overline{a} + B$, where $AB = 1 - n + n\overline{a} + n^2\overline{a}^{\,2}$; if $p \in \mathscr{P}$ and $q \in \mathscr{P}$, then $apq \in {}_{(k+2)}M_n$.

RULE 3: Let $k \in \mathbf{N} \cup \{0\}$, $n \in \mathbf{N}$, $a \in {}_kM_n^*$, and $p := n\overline{a} + A$, $q := n\overline{a} + B$, where $AB = 1 + n\overline{a} + n^2\overline{a}^{\,2}$; if $p \in \mathscr{P}$ and $q \in \mathscr{P}$, then $apq \in {}_{(k+2)}M_n^*$.

The proofs of these rules don't require much more than the application of the definitions, and are therefore left to the reader. In fact, the proof of Rule 2 was already given in [7], although the rule itself was formulated there less explicitly.

Rule 1 can be applied for $k \geqslant 1$, but not for $k = 0$, since ${}_0M_n^* = \{1\}$ and $a = 1$ gives $p = 1 \notin \mathscr{P}$. For $k = n = 1$, Rule 1 reads:

$$\text{If } p := 2^{\alpha+1} - 1 \in \mathscr{P}, \text{ then } 2^{\alpha}p \in {}_2M_1,$$

which is Euclid's rule for finding even perfect numbers. For $k = 1$, Rule 1 is equivalent to Rule 0, given in Section 1.

Rules 2 and 3 can both be applied for $k \geqslant 0$. For instance, for $k = 0$, Rule 2 reads:

$$\text{Let } n \in \mathbf{N} \text{ be given; if } p := n + A \in \mathscr{P} \text{ and } q := n + B \in \mathscr{P}, \text{ where } AB = 1 + n^2, \text{ then } pq \in {}_2M_n.$$

For $n = 1$, 2, and 6, this yields the hyperperfect numbers 2×3, 3×7, and 7×43, respectively. Rule 3 reads, for $k = 0$:

$$\text{Let } n \in \mathbf{N} \text{ be given; if } p := n + A \in \mathscr{P} \text{ and } q := n + B \in \mathscr{P}, \text{ where } AB = 1 + n + n^2, \text{ then } pq \in {}_2M_n^*.$$

For $n = 4$ and $n = 10$, we find that $7 \times 11 \in {}_2M_4^*$ and $13 \times 47 \in {}_2M_{10}^*$, respectively.

Rule 3 shows a rather curious "side-effect" for $k \geqslant 1$: if both the numbers p and q in this rule are prime, then not only $apq \in {}_{(k+2)}M_n^*$, but also the number $b := pq$ is an element of ${}_2M_{n\bar{a}}^*$. Indeed, we have

$$\frac{b-1}{\sigma(b)-b} = \frac{pq-1}{p+q+1} = \frac{n^2\bar{a}^2 + n\bar{a}(A+B) + AB - 1}{2n\bar{a} + A + B + 1}$$

$$= \frac{n^2\bar{a}^2 + n\bar{a}(A+B) + n\bar{a} + n^2\bar{a}^2}{2n\bar{a} + A + B + 1} = n\bar{a} \in \mathbf{N}.$$

For example, we know that $7 \times 11 \in {}_2M_4^*$. From Rule 3 with $k = 2$, $n = 4$, and $a = 7 \times 11$, we find that $7 \times 11 \times 547 \times 1291 \in {}_4M_4^*$; the side-effect is that

$$547 \times 1291 \in {}_2M_{(4\times 8\times 12)}^* = {}_2M_{384}^*.$$

In [6] we gave the following additional rule.

RULE 4: Let $t \in \mathbf{N}$ and $p := 6t - 1$, $q := 12t + 1$; if $p \in \mathscr{P}$ and $q \in \mathscr{P}$, then $p^2q \in {}_2M_{(4t-1)}$.

For example, $t = 1$ and $t = 3$ give $5^2 13 \in {}_2M_3$ and $17^2 37 \in {}_2M_{11}$, respectively. In Section 3 we will prove that with Rules 1, 2, and 4 it is possible to find all HP's of the form $p^\alpha q$, $\alpha \in \mathbf{N}$, below a given bound. We leave it to interested readers to discover why there is no rule (at least for $k \geqslant 1$), analogous to Rule 1, for finding an element of ${}_{(k+1)}M_n^*$ from an element of ${}_kM_n^*$.

From Rules 1-3, it follows that elements of ${}_kM_n$ for some given $k \in \mathbf{N}$ may be found from ${}_{(k-1)}M_n^*$ (with Rule 1) and from ${}_{(k-2)}M_n^*$ (with Rule 2) provided that sufficiently many elements of ${}_{(k-1)}M_n^*$ resp. ${}_{(k-2)}M_n^*$ are available; these can be found with Rule 3 and the "starting" sets ${}_0M_n^*$ and ${}_1M_n^*$ given in (5). We have carried out this "program" for the constructive computation of HP's with three, four, and five different prime factors.

(i) *Construction of elements of* ${}_3M_n$. With Rule 1, we found 34 HP's of the form pqr, from numbers $pq \in {}_2M_n^*$:

the smallest is $61 \times 229 \times 684433 \in {}_3M_{48}$;

the largest one is $9739 \times 13541383 \times 1283583456107389 \in {}_3M_{9732}$.

The elements of ${}_2M_n^*$ were "generated" with Rule 3 from ${}_0M_n^* = \{1\}$. Using Rule 2 we found, from prime powers $p^\alpha \in {}_1M_n^*$, 67 HP's of the form pqr:

five of the smallest are given in [6],

the largest is $8929 \times 79727051 \times 577854714897923 \in {}_3M_{8928}$;

48 HP's of the form p^2qr,

the smallest five are given in [6],

the largest is $7459^2 414994003583 \times 3444400460163740816321 9 \in {}_3M_{7458}$;

9 of the form p^3qr,

the smallest is given in [6],

the largest is $811^3 432596915921 \times 89927962885420066391 \in {}_3M_{810}$;

4 of the form p^4qr,

the smallest is $7^4 30893 \times 36857 \in {}_3M_6$,

the largest is $223^4 553821371657 \times 130059326113901 \in {}_3M_{222}$;

and, furthermore,

$7^6 1340243 \times 2136143 \in {}_3M_6$,

$13^7 815787979 \times 11621986347871 \in {}_3M_{12}$,

and

$19^8 322687706723 \times 11640844402910006759 \in {}_3M_{18}$.

(ii) *Construction of elements of* ${}_4M_n$. In order to construct elements of ${}_4M_n$ with Rule 1, sufficiently many elements of ${}_3M_n^*$ had to be available. This was realized with Rule 3, starting with elements $p^\alpha \in {}_1M_{(p+1)}$, $p \in \mathcal{P}$. The following four HP's with four different prime factors were found:

$3049 \times 9297649 \times 69203101249 \times 5981547458963067824996953 \in {}_4M_{3048}$,

$4201 \times 17692621 \times 7061044981 \times 2204786370880711054109401 \in {}_4M_{4200}$,

$181^2 5991031 \times 579616291 \times 2059102068590772565038 1 \in {}_4M_{180}$,

$181^3 1108889497 \times 33425259193 \times 39781151786825440683346549261 \in {}_4M_{180}$.

By means of Rules 2 and 3, the following three additional elements of ${}_4M_n$ were found:

$1327 \times 6793 \times 10020547039 \times 17769709449589 \in {}_4M_{1110}$ (is in [6]),

$1873 \times 24517 \times 79947392729 \times 80855915754575789 \in {}_4M_{1740}$ (is in [7]),

$5791 \times 10357 \times 222816095543 \times 482764219012881017 \in {}_4M_{3714}$.

(iii) *Construction of an element of* ${}_5M_n$. We have also constructively computed one element of ${}_5M_n$ with Rule 1. The elements of ${}_4M_n^*$ needed for this purpose were computed from ${}_0M_n^*$ by twice applying Rule 3 (first yielding elements of ${}_2M_n^*$, then elements of ${}_4M_n^*$). The HP found is the largest one we know of (apart from the ordinary perfect numbers). It is the 87-digit number:

$$\begin{aligned}&2095497171870781405883328851321934328974054074379064\,14\\&23676492553831733902070878659079 3\\&= 4783 \times 83563 \times 1808560287211 \times 297705496733220305347\\&\times 9737620193207006500935201284805753200507613 01 \in {}_5M_{4524}.\end{aligned}$$

3. CHARACTERIZATION OF ALL HP'S OF THE FORM $p^\alpha q$

The hyperperfect numbers of the form $p^\alpha q$ are characterized by the following theorem.

Theorem: Let $m := p^\alpha q$ ($\alpha \in \mathbf{N}, p \in \mathscr{P}, q \in \mathscr{P}$) be a hyperperfect number, then

(i) $\alpha = 1 \Rightarrow (\exists n \in \mathbf{N}$ with $m \in {}_2M_n$ such that $p = n + A$, $q = n + B$, with $AB = 1 + n^2)$;

(ii) $\alpha = 2 \Rightarrow (\exists t \in \mathbf{N}$ with $m \in {}_2M_{(4t-1)}$ and $p = 6t - 1$ and $q = 12t + 1)$ $\vee$ ($m \in {}_2M_{(p-1)}$ with $q = p^3 - p + 1$);

(iii) $\alpha > 2 \Rightarrow (m \in {}_2M_{(p-1)}$ with $q = p^{\alpha+1} - p + 1)$.

Proof: (i) This case follows immediately from Rule 2 (with $k = 0$).
(ii) If p^2q is hyperperfect, then the number $(p^2q - 1)/((p + 1)(p + q))$ must be a positive integer. Consider the function

$$f(x, y) := \frac{x^2y - 1}{(x + 1)(x + y)}, \quad x, y \in \mathbf{N}.$$

To characterize all pairs x, y for which $f(x, y) \in \mathbf{N}$, we can safely take $x \geqslant 2$ and $y \geqslant 2$. Let $x \geqslant 2$ be fixed, then we have for all $y \geqslant 2$,

$$f(x, y) < \frac{x^2y}{(x + 1)(x + y)} < \frac{x^2}{x + 1} = x - 1 + \frac{1}{x + 1}.$$

Hence, the largest integral value which could possibly be assumed by f is $x - 1$, and one easily checks that this value is actually assumed for $y = x^3 - x + 1$. So we have found

$$f(x, x^3 - x + 1) = x - 1, \; x \in \mathbf{N}, \; x \geqslant 2. \tag{6}$$

One also easily checks that f is monotonically increasing in y (x fixed), so that

$$2 \leqslant y \leqslant x^3 - x + 1. \tag{7}$$

Now, in order to have $f \in \mathbf{N}$, it is necessary that $x + 1$ divides $x^2y - 1$, or, equivalently, that $x + 1$ divides $y - 1$, since

$$\frac{x^2y - 1}{x + 1} = y(x - 1) + \frac{y - 1}{x + 1}.$$

Therefore, we have $y = k(x + 1) + 1$, with $k \in \mathbf{N}$ and $1 \leqslant k \leqslant x(x - 1)$ by (7). Substitution of this into f yields

$$f(x, y) = \frac{kx^2 + x - 1}{(k + 1)(x + 1)} = x - 1 - \frac{x^2 - x - k}{(k + 1)(x + 1)} =: x - 1 - g(x, k).$$

It follows that $x + 1$ must divide $x^2 - x - k$, or, equivalently, that $x + 1$ must divide $k - 2$. Hence, $k = j(x + 1) + 2$, with $j \in \mathbf{N} \cup \{0\}$ and $0 \leqslant j \leqslant x - 2$. Substitution of this into g yields

$$g(x, j(x + 1) + 2) = \frac{x - 2 - j}{j(x + 1) + 3}.$$

This function is decreasing in j, and for $j = 0, 1, \ldots, x - 2$ it assumes the values:

$$\begin{aligned} g(x, 2) &= (x - 2)/3, \\ g(x, x + 3) &= \frac{x - 3}{x - 4} < 1, \\ &\vdots \\ g(x, x(x - 1)) &= 0. \end{aligned}$$

It follows that there is precisely one more possibility [in addition to (6)] for f to be a positive integer, viz., when $j = 0$, $k = 2$, $y = 2x + 3$, and $x \pmod 3 = 2$. So we have found

$$f(3t - 1, 6t + 1) = 2t - 1, \ t \in \mathbf{N}. \tag{8}$$

The statement in the Theorem now easily follows from (6) and (8).

(iii) As in the proof of (ii), we now have to find out for which values of x, $y \in \mathbf{N}$, $x \geqslant 2$, and $y \geqslant 2$, the function $f(x, y) \in \mathbf{N}$, where

$$f(x, y) := \frac{x^\alpha y - 1}{(x^{\alpha-1} + \cdots + 1)(x + y)}, \ \alpha > 2.$$

For fixed $x \geqslant 2$, we have

$$f(x, y) < \frac{x^\alpha}{x^{\alpha-1} + \cdots + 1} = x - 1 + \frac{1}{x^{\alpha-1} + \cdots + 1}.$$

As in the proof of (ii) we find that $f(x, y) = x - 1$ for $y = x^{\alpha+1} - x + 1$ and that $2 \leqslant y \leqslant x^{\alpha+1} - x + 1$. Furthermore, $x^{\alpha-1} + \cdots + 1$ must divide $x^{\alpha}y - 1$, so that $y = k(x^{\alpha-1} + \cdots + 1) + 1$, with $1 \leqslant k \leqslant x(x - 1)$. Substitution of this into f yields a certain function g, in the same way as in the proof of (ii), but in this case g can only assume integral values for $k = x(x - 1)$. This implies the statement in the Theorem, case (iii). Q.E.D.

It is easy to see that the characterizations given in this Theorem are equivalent to Rule 2 ($k = 0$) when $\alpha = 1$, to Rule 4 or Rule 1 ($k = 1$) when $\alpha = 2$, and to Rule 1 ($k = 1$) when $\alpha > 2$.

This Theorem enables us to find very cheaply all HP's of the form $p^{\alpha}q$, $\alpha \in \mathbf{N}$, below a given bound. For example, to find all HP's in M_n of the form pq below 10^8, we only have to check whether

$$p := n + A \in \mathscr{P} \quad \text{and} \quad q := n + B \in \mathscr{P}$$

for all possible factorizations of $AB = 1 + n^2$, for $1 \leqslant n \leqslant 4999$. This range of n follows from the fact that if $pq \in M_n$ then $pq > 4n^2$. The following additional restrictions can be imposed on n:

(i) n should be 1 or even since, if n is odd and $n \geqslant 3$, then $n^2 + 1 \equiv 2 \pmod 4$, so that one of A or B is odd and one of p or q is even and $\geqslant 4$.

(ii) If $n \geqslant 3$, then $n \equiv 0 \pmod 3$, since if $n \equiv 1$ or $2 \pmod 3$, then $n^2 + 1 \equiv 2 \pmod 3$, so that one of A or B is $\equiv 1 \pmod 3$ and the other is $\equiv 2 \pmod 3$; consequently, one of p or q is $\equiv 0 \pmod 3$ and > 3.

Hence, the only values of n to be checked are $n = 1$, $n = 2$, and $n = 6t$, $1 \leqslant t \leqslant 833$. It took about 6 seconds CPU-time on a CDC CYBER 175 computer to check these values of n, and to generate in this way all HP's of the form pq below 10^8.

4. EXHAUSTIVE COMPUTER SEARCHES

From the rules given in Section 2, it follows that it is of importance to know elements of M^* when one wants to find elements of M. Therefore, we have carried out an exhaustive computer search for all elements of M^* below the bound 10^8. Because of (5) the search was restricted to elements with at least two different prime factors. A check was done to determine whether $(m - 1)/(\sigma(m) - m) \in \mathbf{N}$, for all $m \leqslant 10^8$ with $\omega(m) \geqslant 2$. Since the most time-consuming part is the computation of $\sigma(m)$, a second check was done to determine whether $(m - 1)/(\sigma(m) - m - 1) \in \mathbf{N}$ [in the case where $(m - 1)/(\sigma(m) - m) \notin \mathbf{N}$]. If so, m was an HP; thus, our program also produced, almost for free, all HP's below 10^8. (The search took about 100 hours of "idle" computer time on a CDC CYBER 175.) The results are as follows.

Apart from the ordinary perfect numbers, there are 146 HP's below 10^8. Only two of them have the form $p^\alpha qr$:

$$13 \times 269 \times 449 \in {}_3M_{12} \quad \text{and} \quad 7^2 383 \times 3203 \in {}_3M_6;$$

these were also found in the searches described in Section 2. All others have the form characterized in Section 3, and could have been found with a search based on that characterization (using the fact that if $p^\alpha q \in {}_2M_n$, then $p > n$ and $q > n$). A question that naturally arises is the following: Are there any HP's that can*not* be constructed with one of Rules 1, 2, or 4?[2]

There are 312 numbers $m \leqslant 10^8$ which belong to M^* and which have $\omega(m) \geqslant 2$. Of these, 306 have the form pq and could have been (and, as a check, actually were) found very cheaply with Rule 3 of Section 2. The others are:

$$7 \times 61 \times 229 \in {}_3M_6^*,\ 113 \times 127 \times 2269 \in {}_3M_{58}^*,$$

$$149 \times 463 \times 659 \in {}_3M_{96}^*,\ 19 \times 373 \times 10357 \in {}_3M_{18}^*,$$

$$151 \times 373 \times 1487 \in {}_3M_{100}^*,\ 7 \times 11 \times 547 \times 1291 \in {}_4M_4^*;$$

the second, third, and fifth numbers could not have been found using Rule 3.

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[2] *The referee has answered this question in the affirmative by giving the example* $12161963773 = 191 \times 373 \times 170711 \in M_{126}$.

5. HYPERCYCLES

A possible generalization of hyperperfect numbers can be obtained as follows. Let $n \in \mathbf{N}$ be given, and define the function $f_n : \mathbf{N}\backslash\{1\} \Rightarrow \mathbf{N}$ as

$$f_n(m) := 1 + n[\sigma(m) - m - 1], \quad m \in \mathbf{N}\backslash\{1\}. \tag{9}$$

Starting with some $m_0 \in \mathbf{N}\backslash\{1\}$, one might investigate the sequence

$$m_0,\ f_n(m_0),\ f_n(f_n(m_0)),\ \ldots\ . \tag{10}$$

For $n = 1$, this is the well-known aliquot sequence of m_0, which can have cycles of length 1 (perfect numbers), length 2 (amicable pairs), and others. In order to get some impression of the cyclic behavior for $n > 1$, we have computed, for $2 \leqslant n \leqslant 20$, five terms of all sequences (10) with starting term $m_0 \leqslant 10^6$, and we have registered the cycles with length $\geqslant 2$ and $\leqslant 5$ in the following table.

TABLE 1

HYPERCYCLES*

n	k	$m_0, m_1, \ldots, m_{k-1}$
5	2	$19461 = 3 \times 13 \times 499$, $42691 = 11 \times 3881$
7	3	$925 = 5^2 37$, $1765 = 5 \times 353$, $2507 = 23 \times 109$
8	2	$28145 = 5 \times 13 \times 433$, $66481 = 19 \times 3499$
	3	$238705 = 5 \times 47741$, $381969 = 3^3 7 \times 43 \times 47$, $2350961 = 79 \times 29759$
	4	$94225 = 5^2 3769$, $181153 = 7^2 3697$, $237057 = 3 \times 31 \times 2549$, $714737 = 61 \times 11717$
	2	$3452337 = 3^2 7 \times 54799$, $17974897 = 53 \times 229 \times 1481$
9	2	$469 = 7 \times 67$, $667 = 23 \times 29$
	2	$1315 = 5 \times 263$, $2413 = 19 \times 127$
	2	$1477 = 7 \times 211$, $1963 = 13 \times 151$
	2	$2737 = 7 \times 17 \times 23$, $6463 = 23 \times 281$
10	3	$1981 = 7 \times 283$, $2901 = 3 \times 967$, $9701 = 89 \times 109$
12	2	$697 = 17 \times 41$, $2041 = 13 \times 157$
	2	$3913 = 7 \times 13 \times 43$, $12169 = 43 \times 283$
	2	$54265 = 5 \times 10853$, $130297 = 29 \times 4493$
14	2	$1261 = 13 \times 97$, $1541 = 23 \times 67$
	3	$508453 = 11 \times 17 \times 2719$, $1106925 = 3 \times 5^2 14759$, $10126397 = 281 \times 36037$

*Different numbers $m_0, m_1, \ldots, m_{k-1}$ such that $m_k = m_0$, where $m_{i+1} := f_n(m_i)$, f_n defined in (9).

TABLE 1 (continued)

n	k	$m_0, m_1, \ldots, m_{k-1}$
19	2	$9197 = 17 \times 541$, $10603 = 23 \times 461$
	4	$184491 = 3^3 6833$, $1688493 = 3 \times 562831$, $10693847 = 709 \times 15083$, $300049 = 31 \times 9679$
	2	$5151775 = 5^2 251 \times 821$, $24124073 = 89 \times 271057$

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