ON THE REDUCTION OF A LINEAR RECURRENCE OF ORDER r

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1. INTRODUCTION

J. R. Bastida shows in his paper [1] that, if $u \in R$, $u > 1$, and $(x_n)_{n \geqslant 0}$ is a sequence given by

$$x_{n+1} = ux_n + \sqrt{(u^2 - 1)(x_n^2 - x_0^2) + (x_1 - ux_0)^2}, \quad n \geqslant 0, \tag{1}$$

then $x_{n+2} = 2ux_{n+1} - x_n$, $n \geqslant 0$. So, if the numbers u, x_0, and x_1 are integers, it results that x_n is an integer for any $n \geqslant 0$.

Bastida and DeLeon [2] establish sufficient conditions for the numbers u, t, x_0, and x_1 such that the linear recurrence

$$x_{n+2} = 2ux_{n+1} - tx_n \tag{2}$$

can be reduced to a relation of form (1), between x_n and x_{n+1}. Consequently, the relation's two consecutive terms of Fibonacci, Lucas, and Pell sequences are given in [2].

S. Roy [6] finds this relation for the Fibonacci sequence using hyperbolic functions.

In this paper we shall prove that if a sequence $(x_n)_{n \geqslant 1}$ satisfies a linear recurrence of order $r \geqslant 2$, then there exists a polynomial relation between any r consecutive terms. This shows that the linear recurrence of order r was reduced to a nonlinear recurrence of order $r - 1$.

From a practical point of view, for $r \geqslant 3$, expressing x_n in the function of $x_{n-1}, \ldots, x_{n-r+1}$ is difficult, because we must solve an algebraic equation of degree $\geqslant 3$ and choose the "good solution."

If $r = 2$, we can do this in many important cases. An application of this case is a generalization of the result given in [3].

2. THE MAIN RESULT

Let $(x_n)_{n \geqslant 1}$ be a sequence given by the linear recurrence of order r,

$$x_n = \sum_{k=1}^{r} a_k x_{n-r+k-1}, \quad n \geqslant r + 1, \; x_i = \alpha_i, \quad 1 \leqslant i \leqslant r, \tag{3}$$

where $\alpha_1, \ldots, \alpha_r$ and $a_1, \ldots, a_r$ are given real numbers (they can also be complex numbers or elements of an arbitrary commutative field). Suppose $a_1 \neq 0$.

For $n \geqslant r$, we consider the determinant

$$D_n = \begin{vmatrix} x_{n-r+1} & x_{n-r+2} & \cdots & x_{n-1} & x_n \\ x_{n-r+2} & x_{n-r+3} & \cdots & x_n & x_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1} & x_n & \cdots & x_{n+r-3} & x_{n+r-2} \\ x_n & x_{n+1} & \cdots & x_{n+r-2} & x_{n+r-1} \end{vmatrix} \tag{4}$$

and then prove the following theorem.

Theorem 1. Let $(x_n)_{n\geqslant 1}$ be a sequence given by (3) and let D_n be given by (4). Then, for any $n \geqslant r$, we have the r relation

$$D_n = (-1)^{(r-1)(n-r)} a_1^{n-r} D_r \tag{5}$$

Proof: Following the method of [4], [5], and [7] (for $r = 2$), we introduce the matrix

$$A_n = \begin{bmatrix} x_{n-r+1} & x_{n-r+2} & \cdots & x_{n-1} & x_n \\ x_{n-r+2} & x_{n-r+3} & \cdots & x_n & x_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1} & x_n & \cdots & x_{n+r-3} & x_{n+r-2} \\ x_n & x_{n+1} & \cdots & x_{n+r-2} & x_{n+r-1} \end{bmatrix}. \tag{6}$$

It is easy to see that

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_{r-2} & a_{r-1} & a_r \end{bmatrix} A_n = A_{n+1}, \tag{7}$$

so that

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_{r-2} & a_{r-1} & a_r \end{bmatrix} A_r = A_n. \tag{8}$$

Passing to determinants in (8), we obtain

$$((-1)^{r-1}a_1)^{n-r}D_r = D_n \quad \text{for } n \geq r;$$

that is, the relation (5) is true.

Theorem 2. Let $(x_n)_{n\geq 1}$ be the sequence given by the linear recurrence (3). There exists a polynomial function of degree r, $F_r: R^r \to R$, such that the relation

$$F_r(x_n, x_{n-1}, \ldots, x_{n-r+1}) = (-1)^{(r-1)(n-r)}a_1^{n-r}F_r(\alpha_r, \alpha_{r-1}, \ldots, \alpha_1) \qquad (9)$$

is true for every $n \geq r$.

Proof: Observe that, from the recurrence (3), we can compute the value of D_r knowing $\alpha_1, \alpha_2, \ldots, \alpha_r$. Also, from the recurrence (3), we can express successively all elements of D_n as a function of the terms $x_n, x_{n-1}, \ldots, x_{n-r+1}$ of the sequence $(x_n)_{n\geq 1}$. Thus there exists a polynomial function of degree r, $F_r: R^r \to R$ such that the relation (9) is true.

If we suppose that the equation

$$F_r(x_n, x_{n-1}, \ldots, x_{n-r+1}) = (-1)^{(r-1)(n-r)}a_1^{n-r}F_r(\alpha_r, \ldots, \alpha_1)$$

can be resolved with respect to x_n, we find that x_n depends only on the terms $x_{n-1}, x_{n-2}, \ldots, x_{n-r+1}$.

If this is possible, the expression of x_n is, in general, very complicated.

When $r = 2$, we obtain

$$F_2(x, y) = x^2 - a_2xy - a_1y^2, \qquad (10)$$

and it results that, for the sequence $(x_n)_{n\geq 1}$ given by

$$x_n = a_1x_{n-2} + a_2x_{n-1}, \quad n \geq 3, \quad x_1 = \alpha_1, \quad x_2 = \alpha_2, \qquad (11)$$

the relation $F_2(x_n, x_{n-1}) = (-1)^n a_1^{n-2}F_2(\alpha_2, \alpha_1)$ holds. The last relation is the first result of [2], where it was proved by mathematical induction. If we write this relation explicitly, we obtain

$$(2x_n - a_2x_{n-1})^2 = (a_2^2 + 4a_1)x_{n-1}^2 + 4(-1)^{n-1}a_1^{n-2}(a_1\alpha_1^2 + a_2\alpha_1\alpha_2 - \alpha_2^2). \qquad (12)$$

From the relation (12), under some supplementary conditions concerning the sequence $(x_n)_{n\geq 1}$, we can express x_n in terms of x_{n-1}.

Again, from (12), it follows that if the sequence satisfies (11), where $a_1, a_2, \alpha_1, \alpha_2 \in N$, then for any $n \geq 3$,

$$(a_2^2 + 4a_1)x_{n-1}^2 + 4(-1)^{n-1}a_1^{n-2}(a_1\alpha_1^2 + a_2\alpha_1\alpha_2 - \alpha_2^2)$$

is a square. This result is an extension of [3].

In the particular case $r = 3$, after elementary calculation, we obtain

$$\begin{aligned} F_3(x, y, z) = -x^3 &- (a_1 + a_2a_3)y^3 - a_1^2z^3 + 2a_3x^2y + a_2x^2z \\ &- (a_2^2 + a_1a_3)y^2z - (a_3^2 - a_2)xy^2 \\ &- a_1a_3xz^2 - 2a_1a_2yz^2 + (3a_1 - a_2a_3)xyz. \end{aligned}$$

So from relation (9), we get that, for the linear recurrence

$$x_n = a_1x_{n-3} + a_2x_{n-2} + a_3x_{n-1}, \quad n \geq 4, \quad x_1 = \alpha_1, \quad x_2 = \alpha_2, \quad x_3 = \alpha_3, \qquad (13)$$

the relation $F_3(x_n, x_{n-1}, x_{n-2}) = a_1^{n-3}F_3(\alpha_3, \alpha_2, \alpha_1)$ is true.

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