ELEMENTARY PROBLEMS AND SOLUTIONS

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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-586 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Show that $5\sum_{k=0}^{n} F_{k+1}F_{n+1-k} = (n + 1)F_{n+3} + (n + 3)F_{n+1}$.

B-587 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Let $y = \sum_{n=0}^{\infty} F_n x^n/n!$ and $z = \sum_{n=0}^{\infty} L_n x^n/n!$.

Show that $y'' = y' + y$ and $z'' = z' + z$.

B-588 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Find the y and z of Problem B-587 in closed form.

B-589 *Proposed by Herta T. Freitag, Roanoke, VA*

The number $N = 0434782608695652173913$ has the property that the digits of KN are a permutation of the digits of N for $K = 1, 2, \ldots, m$. Determine the largest such m.

B-590 *Proposed by Herta T. Frietag, Roanoke, VA*

Generalize on Problem B-589 and describe a method for predicting the leftmost digit of KN.

B-591 *Proposed by Mihaly Bencze, Jud. Brasa, Romania*

Let $F(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ with each a_n in $\{0, 1\}$.

Prove that $f(x) \neq 0$ for all x in $-1/a < x < 1/a$, where $a = (1 + \sqrt{5})/2$.

SOLUTIONS

Constant Modulo 5

B-562 *Proposed by Herta T. Freitag, Roanoke, VA*

Let c_n be the integer in $\{0, 1, 2, 3, 4\}$ such that

$$c_n \equiv L_{2n} + [n/2] - [(n - 1)/2] \pmod{5},$$

where $[x]$ is the greatest integer in x. Determine c_n as a function of n.

Solution by J. Suck, Essen, Germany

$c_n = 3$ for all $n \in Z$. From the very definition, we see that $L_n \equiv 2, 1, 3, 4 \pmod{5}$ for $n \equiv 0, 1, 2, 3$, respectively, (mod 4). Hence

$$L_{2n} \equiv \begin{cases} 2 & \text{for } n \text{ even} \\ 3 & \text{for } n \text{ odd.} \end{cases}$$

But for n even,

$$\left[\frac{n}{2}\right] - \left[\frac{n}{2} - \frac{1}{2}\right] = \frac{n}{2} - \left(\frac{n}{2} - 1\right) = 1,$$

and for n odd,

$$\left[\frac{n-1}{2} + \frac{1}{2}\right] - \left[\frac{n-1}{2}\right] = \frac{n-1}{2} - \frac{n-1}{2} = 0.$$

So,

$$L_{2n} + \left[\frac{n}{2}\right] - \left[\frac{n-1}{2}\right] \equiv \begin{cases} 2 + 1, & n \text{ even} \\ 3 + 0, & n \text{ odd} \end{cases} = 3 \pmod{5}.$$

Also solved by Paul S. Bruckman, Lászlô Cseh, L. A. G. Dresel, Piero Filipponi, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Imre Merényi, Bob Prielipp, Heinz-Jürgen Seiffert, and the proposer.

2 of 3 Are Multiples of 4

B-563 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_n = \sum_{i=1}^{n} L_{2i+1}L_{2i-2}$. For which values of n is S_n exactly divisible by 4?

Solution by J. Suck, Essen, Germany

From the definition of the Lucas numbers we see that if $k \equiv 0, 1, 2, 3, 4, 5 \pmod{6}$, then $L_k \equiv 2, 1, 3, 0, 3, 3 \pmod{4}$, respectively. Hence, if $i \equiv 1$, 2, 0 (mod 3), then $L_{2i+1}L_{2i-2} \equiv 0 \cdot 2 \equiv 0,\ 3 \cdot 3 \equiv 1,\ 1 \cdot 3 \equiv 3 \pmod 4$, respectively. This, of course, implies that $S_n \equiv 0 \pmod 4$ if and only if $n \equiv 1$ or 0 (mod 3) and $S_n \equiv 1$ otherwise.

Also solved by Paul S. Bruckman, Lászlő Cseh, L. A. G. Dresel, Piero Filipponi, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Bob Prielipp, Heinz-Jürgen Seiffert, and the proposer.

Summing $[aF_k]$

B-564 *Proposed by Lászlő Cseh, Cluj, Romania*

Let $a = (1 + \sqrt{5})/2$ and $[x]$ be the greatest integer in x. Prove that

$$[aF_1] + [aF_2] + \cdots + [aF_n] = F_{n+3} - [(n + 4)/2].$$

Solution by Paul S. Bruckman, Fair Oaks, CA

First we note that $aF_k = 5^{-1/2}(a^{k+1} - b^{k+1} + b^k(b - a)) = F_{k+1} - b^k$. Since $-1 < b < 0$, thus $[aF_{2k}] = F_{2k+1} - 1$, $[aF_{2k+1}] = F_{2k+2}$, or $[aF_k] = F_{k+1} - e_k$, where e_k is the characteristic function of the even integers.

Let $S_n \equiv \sum_{k=1}^{n} [aF_k]$. Then

$$S_n = \sum_{k=1}^{n} (F_{k+1} - e_k) = \sum_{k=1}^{n} (F_{k+3} - F_{k+2}) - \left[\frac{n}{2}\right] = F_{n+3} - F_3 - \left[\frac{n}{2}\right]$$
$$= F_{n+3} - \left[\frac{n+4}{2}\right]. \quad \text{Q.E.D.}$$

Also solved by Piero Filipponi, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Imre Merényi, Bob Prielipp, Heinz-Jürgen Seiffert, J. Suck, and the proposer.

Fibonacci-Pell Products Summed

B-565 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Let $P_0, P_1, \ldots$ be the sequence of Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \in \{2, 3, \ldots\}$. Show that

$$9\sum_{k=0}^{n} P_k F_k = P_{n+2}F_n + P_{n+1}F_{n+2} + P_n F_{n-1} - P_{n-1}F_{n+1}.$$

Solution by Paul S. Bruckman, Fair Oaks, CA

Let R_n denote the right member in the statement of the problem. Then

$$R_n = (2P_{n+1} + P_n)F_n + P_{n+1}(F_{n+1} + F_n) + P_n(F_{n+1} - F_n) - (P_{n+1} - 2P_n)F_{n+1};$$

after simplification, this reduces to

$$R_n = 3(P_{n+1}F_n + P_n F_{n+1}). \tag{1}$$

Therefore,

$$\Delta R_n \equiv R_{n+1} - R_n = 3(P_{n+2}F_{n+1} - P_{n+1}F_n + P_{n+1}F_{n+2} - P_nF_{n+1})$$
$$= 3\{(2P_{n+1} + P_n)F_{n+1} - P_{n+1}F_n + P_{n+1}(F_{n+1} + F_n) - P_nF_{n+1}\},$$

which reduces to

$$\Delta R_n = 9P_{n+1}F_{n+1}. \tag{2}$$

On the other hand, let S_n denote the left member in the statement of the problem. Clearly,

$$\Delta S_n = 9P_{n+1}F_{n+1}. \tag{3}$$

Since $\Delta R_n = \Delta S_n$, this implies that

$$R_n = S_n + c,\ n = 0,\ 1,\ 2,\ \ldots, \tag{4}$$

for some constant c (independent of n). Since $P_0 = F_0 = 0$, thus

$$R_0 = 0 \quad \text{and} \quad S_0 = 9P_0F_0 = 0.$$

Setting $n = 0$ in (4), we find that $0 = R_0 = S_0 + c = c$, i.e., $c = 0$. Therefore,

$$R_n = S_n \text{ for all } n. \quad \text{Q.E.D.} \tag{5}$$

Also solved by L. A. G. Dresel, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Heinz-Jürgen Seiffert, and the proposer.

Lucas-Pell Products Summed

B-566 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*

Let P_n be as in B-565. Show that

$$9\sum_{k=0}^{n} P_kL_k = P_{n+2}L_n + P_{n+1}L_{n+2} + P_nL_{n-1} - P_{n-1}L_{n+1} - 6.$$

Solution by Paul S. Bruckman, Fair Oaks, CA

The proof is similar to that of B-565. Using the same notation, we find, as before, that

$$\Delta R_n = 9P_{n+1}L_{n+1} = \Delta S_n, \tag{1}$$

and

$$R_n = S_n + c,\ n = 0,\ 1,\ 2,\ \ldots, \tag{2}$$

for some constant c (independent of n).

Also, however, we have the following relation, which differs from (1) in the solution of B-565:

$$R_n = 3(P_{n+1}L_n + P_nL_{n+1}) - 6. \tag{3}$$

As before, $S_0 = 9P_0L_0 = 0$; also, using (3), $R_0 = 3(1 \cdot 2 + 0 \cdot 1) - 6 = 0$. Setting $n = 0$ in (2), as before, we find that $c = 0$. Thus,

$$R_n = S_n \text{ for all } n. \quad \text{Q.E.D.} \tag{4}$$

Also solved by L. A. G. Dresel, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, J. Suck, and the proposer.

Relatives of Hermite Polynomials

B-567 *Proposed by P. Rubio, Dragados Y Construcciones, Madrid, Spain*

Let $a_0 = a_1 = 1$ and $a_{n+1} = a_n + na_{n-1}$ for n in $Z^+ = \{1, 2, \ldots\}$. Find a simple formula for

$$G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k.$$

Solution by L. A. G. Dresel, Reading. England

Putting $A_k = a_k/k!$, we have

$$G(x) = \sum_{k=0}^{\infty} A_k x^k,$$

where $A_0 = A_1 = 1$ and $(n + 1)A_{n+1} = A_n + A_{n-1}$ for $n = 1, 2, \ldots$. It follows that the series for $G(x)$ is convergent and differentiable, and

$$\frac{dG}{dx} = \sum_{k=0}^{\infty} (k + 1)A_{k+1}x^k = A_1 + \sum_{k=1}^{\infty} (A_k + A_{k-1})x^k = \sum_{k=0}^{\infty} (A_k x^k + A_k x^{k+1})$$
$$= (1 + x)G.$$

Since $G(0) = 1$, we can integrate the differential equation for G to obtain

$$G(x) = e^{x + \frac{1}{2}x^2}.$$

Also solved by Duane Broline, Paul S. Bruckman, Odoardo Brugia & Piero Filipponi, Dario Castellanos, László Cseh, Alberto Facchini, J. Foster, C. Georghiou, L. Kuipers, J. Z. Lee & J. S. Lee, Imre Merényi, Heinz-Jürgen Seiffert, J. Suck, David Zeitlin, and the proposer.

Editorial Note: Castellanos and Zeitlin pointed out that $a_n = 2^{-n/2}i^n H_n(-i/\sqrt{2})$, where the H_n are the Hermite polynomials. Bruckman, Seiffert, and Zeitlin gave the explicit formula:

$$a_n = n! \sum_{k=0}^{[n/2]} (1/2^k(n - 2k)!k!).$$

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