PELL POLYNOMIAL MATRICES

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1. INTRODUCTION

By defining certain matrices of order 2, we are enabled to derive fresh properties of Pell polynomials $P_n(x)$ and Pell-Lucas polynomials $Q_n(x)$ additional to those obtained by us in [5]. Our work, in summarized form, is an adaptation and extension of some ideas of Walton [6], based on earlier work by Hoggatt and Bicknell-Johnson [2].*

The Pell and Pell-Lucas polynomials which are defined, respectively, by the recurrence relations

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x),\ P_0(x) = 0,\ P_1(x) = 1 \tag{1.1}$$

and

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x),\ Q_0(x) = 2,\ Q_1(x) = 2x \tag{1.2}$$

and some of their basic properties which will be assumed without specific reference, are discussed by us in [3].

To conserve space, we offer our results in a condensed form. This approach has the added virtue of emphasizing techniques.

Convention: For visual ease and simplicity, we abbreviate the functional notation, e.g., $P_n(x) = P_n$, $Q_n(x) = Q_n$.

2. THE ASSOCIATED MATRICES J AND L

Let

$$J = \begin{bmatrix} P_4 & P_2 \\ -P_2 & -P_0 \end{bmatrix}, \tag{2.1}$$

whence, by induction,

$$J^n = P_2^{n-1} \begin{bmatrix} P_{2n+2} & P_{2n} \\ -P_{2n} & -P_{2n-2} \end{bmatrix}. \tag{2.2}$$

Equating corresponding elements in $J^{m+n} = J^m J^n$ gives

$$P_2 P_{2(m+n)} = P_{2(m+1)} P_{2n} - P_{2m} P_{2(n-1)}. \tag{2.3}$$

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*Walton was given a copy of the Hoggatt and Bicknell-Johnson paper while he was writing his thesis. This paper was only published in 1980.

The characteristic equation of J is

$$\lambda^2 - P_4\lambda + P_2^2 = 0, \tag{2.4}$$

so, by the Cayley-Hamilton theorem,

$$J^2 = P_4 J - P_2^2 I. \tag{2.5}$$

Extending (2.5), we have

$$J^{2n+j} = (P_4 J - P_2^2 I)^n J^j, \tag{2.6}$$

whence, by (2.2),

$$P_{4n+2j} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q_2^{n-r} P_{2n-2r+2j}. \tag{2.7}$$

From (2.5),

$$P_4^n J^n = (J^2 + P_2^2 I)^n. \tag{2.8}$$

Equating corresponding matrix elements and simplifying, we get

$$\sum_{r=0}^{n} \binom{n}{r} P_{4r} = Q_2^n P_{2n}. \tag{2.9}$$

Consider, with appeal to (2.5),

$$(J + P_2 I)^2 = (P_4 + 2P_2)J = 8x(x^2 + 1)J. \tag{2.10}$$

Hence,

$$\{8x(x^2 + 1)\}^n J^n = \sum_{r=0}^{2n} \binom{2n}{r} P_2^{2n-r} J^r. \tag{2.11}$$

Now equate corresponding elements. Simplification then yields

$$\sum_{r=0}^{2n} \binom{2n}{r} P_{2r} = 4^n (x^2 + 1)^n P_{2n}. \tag{2.12}$$

Next write

$$L = \begin{bmatrix} P_3 & P_1 \\ -P_1 & -P_{-1} \end{bmatrix} \quad (\text{so } |L| = |J| = -4x^2). \tag{2.13}$$

Then, by (2.2) and (2.13),

$$J^n L = P_2^n \begin{bmatrix} P_{2n+3} & P_{2n+1} \\ -P_{2n+1} & -P_{2n-1} \end{bmatrix}, \tag{2.14}$$

whence

$$J^{2n+j}L = \sum_{r=0}^{n} (-1)^r \binom{n}{r} P_2^{2n} P_4^{n-r} J^{n-r+j} L, \tag{2.15}$$

and so [cf. (2.7)]

$$P_{4n+2j+1} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q_2^{n-r} P_{2n-2r+2j+1}. \tag{2.16}$$

From (2.5),

$$P_4^n J^n L = \sum_{r=0}^{n} \binom{n}{r} P_2^{2n-2r} J^{2r} L, \tag{2.17}$$

whence, by (2.14),

$$\sum_{r=0}^{n} \binom{n}{r} P_{4r+1} = Q_2^n P_{2n+1}. \tag{2.18}$$

Equation (2.10) leads to

$$(J + P_2 I)^{2n} L = \{8x(x^2 + 1)\}^n J^n L, \tag{2.19}$$

from which

$$\sum_{r=0}^{2n} \binom{2n}{r} P_{2r+1} = 4^n (x^2 + 1)^n P_{2n+1}. \tag{2.20}$$

Again from (2.10),

$$(J + P_2 I)^{2n+1} = \{8x(x^2 + 1)\}^n J^n (J + P_2 I). \tag{2.21}$$

Corresponding entries, when equated, produce

$$\sum_{r=0}^{2n+1} \binom{2n+1}{r} P_{2r} = 4^n (x^2 + 1)^n Q_{2n+1}. \tag{2.22}$$

Multiply both sides of (2.21) by L. In the usual way,

$$\sum_{r=0}^{2n+1} \binom{2n+1}{r} P_{2r+1} = 4^n (x^2 + 1)^n Q_{2n+2}. \tag{2.23}$$

Next, from (2.5), after some algebraic manipulation,

$$\{J - (4x^3 + 2x)I\}^{2n} = (4x^4)^n \cdot 4^n (x^2 + 1)^n I, \tag{2.24}$$

so that

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (2x^2 + 1)^r P_{4n-2r} = 0 \tag{2.25}$$

and

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (2x^2 + 1)^r P_{4n-2r+2} = P_2^{2n+1} (x^2 + 1)^n. \tag{2.26}$$

Now multiply (2.24) by L. Consequently,

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r} (2x^2 + 1)^r P_{4n-2r+1} = x^{2n} \{4(x^2 + 1)\}^n. \tag{2.27}$$

Next, multiply both sides of (2.24) by $J - (4x^3 + 2x)I$. It follows that

$$\sum_{r=0}^{2n+1} (-1)^r \binom{2n+1}{r} (2x^2+1)^r P_{4n-2r+3} = \tfrac{1}{2}(2x)^{2n+2}(x^2+1)^n. \tag{2.28}$$

Other results for P_n, some of them quite complicated, may be found in [4], e.g., formulas obtained by considering J^{ns+j} and $J^{ns}L$. One such formula is

$$P_{2n}^s P_{2s+1} = \sum_{r=0}^{s} \binom{s}{r} P_2^{s+r} P_{2n-2}^r P_{2n(s-r)+1}. \tag{2.29}$$

Observe, in passing, that induction leads to

$$L^n = P_2^{n-1} \begin{bmatrix} P_{n+2} & P_n \\ -P_n & -P_{n-2} \end{bmatrix}. \tag{2.30}$$

3. THE MATRICES K AND M

We are able to derive other identities by defining

$$K = \begin{bmatrix} P_8 & P_4 \\ -P_4 & -P_0 \end{bmatrix}, \quad M = \begin{bmatrix} P_5 & P_1 \\ -P_1 & -P_{-3} \end{bmatrix}, \tag{3.1}$$

and following the techniques used above. The results are listed:

$$K^n = P_4^{n-1} \begin{bmatrix} P_{4n+4} & P_{4n} \\ -P_{4n} & -P_{4n-4} \end{bmatrix} \tag{3.2}$$

$$P_4 P_{4(m+n)} = P_{4(m+1)} P_{4n} - P_{4m} P_{4(n-1)} \tag{3.3}$$

$$K^{2n} = (P_8 K - P_4^2 I)^n \tag{3.4}$$

$$P_4^n P_{8n} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} P_8^{n-r} P_4^r P_{4(n-r)} \tag{3.5}$$

$$P_4^n P_{8n+4} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} P_8^{n-r} P_4^r P_{4(n+1-r)} \tag{3.6}$$

$$P_8^n P_{4n} = P_4^n \sum_{r=0}^{n} \binom{n}{r} P_{8r} \tag{3.7}$$

$$\sum_{r=0}^{2n} \binom{2n}{r} P_{4r} = Q_2^{2n} P_{4n} \tag{3.8}$$

$$\sum_{r=0}^{2n+1} \binom{2n+1}{r} P_{4r} = Q_2^{2n+1} P_{4n+2} \tag{3.9}$$

$$K^n M = P_4^n \begin{bmatrix} P_{4n+5} & P_{4n+1} \\ -P_{4n+1} & -P_{4n-3} \end{bmatrix} \tag{3.10}$$

$$\sum_{r=0}^{2n}\binom{2n}{r}P_{4r+1} = Q_2^{2n}P_{4n+1} \tag{3.11}$$

$$\sum_{r=0}^{2n+1}\binom{2n+1}{r}P_{4r+1} = Q_2^{2n+1}P_{4n+3} \tag{3.12}$$

$$M^n = P_4^{n-1}\begin{bmatrix} P_{n+4} & P_n \\ -P_n & -P_{n-4} \end{bmatrix} \tag{3.13}$$

Additional information on the matrix K is given in Mahon [4].

4. THE MATRICES N AND U

In like manner, by defining the matrices

$$N = \begin{bmatrix} P_6 & P_2 \\ -P_2 & -P_{-2} \end{bmatrix}, \quad U = \begin{bmatrix} P_7 & P_3 \\ -P_3 & -P_{-1} \end{bmatrix}, \tag{4.1}$$

and again using techniques similar to those above, we prove further identities which are listed:

$$K^nN = P_4^n\begin{bmatrix} P_{4n+6} & P_{4n+2} \\ -P_{4n+2} & -P_{4n-2} \end{bmatrix} \tag{4.2}$$

$$\sum_{r=0}^{2n}\binom{2n}{r}P_{4r+2} = Q_2^{2n}P_{4n+2} \tag{4.3}$$

$$\sum_{r=0}^{2n+1}\binom{2n+1}{r}P_{4r+2} = Q_2^{2n+1}P_{4n+4} \tag{4.4}$$

$$K^nU = P_4^n\begin{bmatrix} P_{4n+7} & P_{4n+3} \\ -P_{4n+3} & -P_{4n-1} \end{bmatrix} \tag{4.5}$$

$$\sum_{r=0}^{2n}\binom{2n}{r}P_{4r+3} = Q_2^{2n}P_{4n+3} \tag{4.6}$$

$$\sum_{r=0}^{2n+1}\binom{2n+1}{r}P_{4r+3} = Q_2^{2n+1}P_{4n+5} \tag{4.7}$$

See [4] for further, more complicated results.

From what has been said in the above sections, it appears that there is a chain of matrices of the type given which would produce formulas of (perhaps) minor interest.

5. THE MATRIX W

We now introduce a matrix having the property of generating Pell and Pell-Lucas polynomials simultaneously. It was suggested by a problem proposed by Ferns [1].

$$W = \begin{bmatrix} 2x & 1 \\ 4(x^2+1) & 2x \end{bmatrix} \qquad (|W| = -4). \tag{5.1}$$

Induction leads to

$$W^n = 2^{n-1}\begin{bmatrix} Q_n & P_n \\ 4(x^2+1)P_n & Q_n \end{bmatrix}. \tag{5.2}$$

Then

$$W^n\begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2^n\begin{bmatrix} P_n \\ Q_n \end{bmatrix}. \tag{5.3}$$

Now

$$W^{m+n} = 2^{m+n-1}\begin{bmatrix} Q_{m+n} & P_{m+n} \\ 4(x^2+1)P_{m+n} & Q_{m+n} \end{bmatrix} \text{ by (5.2)} \tag{5.4}$$

$$= 2^{m+n-2}\begin{bmatrix} Q_m & P_m \\ 4(x^2+1)P_m & Q_m \end{bmatrix}\begin{bmatrix} Q_n & P_n \\ 4(x^2+1)P_n & Q_n \end{bmatrix} \text{ by (5.2) also.}$$

Corresponding entries give formulas (3.18) and (3.19) for P_{m+n} and Q_{m+n}, respectively, appearing in [3].

The characteristic equation for W is

$$\lambda^2 - 4x\lambda - 4 = 0, \tag{5.5}$$

whence, by the Cayley-Hamilton theorem,

$$W^2 - 4xW - 4I = 0, \tag{5.6}$$

so

$$W^{2n} = 4^n(xW + I)^n. \tag{5.7}$$

Algebraic manipulation, after multiplication by W^j, produces the formulas for P_{2n+j} and Q_{2n+j}, (3.28) and (3.29), in [3].

Induction, with the aid of (5.6), yields

$$W^n = 2^{n-1}(P_nW + 2P_{n-1}I). \tag{5.8}$$

Considering W^{ns+j} and tidying up, we have

$$W^{ns+j} = 2^{(n-1)s}\sum_{r=0}^{s}\binom{s}{r}P_n^rP_{n-1}^{s-r}2^{s-r}W^{r+j}, \tag{5.9}$$

giving

$$P_{ns+j} = \sum_{r=0}^{s} \binom{s}{r} P_n^r P_{n-1}^{s-r} P_{r+j}, \tag{5.10}$$

and

$$Q_{ns+j} = \sum_{r=0}^{s} \binom{s}{r} P_n^r P_{n-1}^{s-r} Q_{r+j}. \tag{5.11}$$

Further,

$$\begin{aligned}\sum_{r=0}^{2n} \binom{2n}{r} (xW)^{r+j} 2^{2n-r} &= (xW + 2I)^{2n} W^j \\ &= (x^2W^2 + 4xW + 4I)^n W^j \\ &= (x^2 + 1)^n W^{2n+j}, \quad \text{by (5.6).}\end{aligned} \tag{5.12}$$

Accordingly,

$$\sum_{r=0}^{2n} \binom{2n}{r} x^r P_{r+j} = (x^2 + 1)^n P_{2n+j} \tag{5.13}$$

and

$$\sum_{r=0}^{2n} \binom{2n}{r} x^r Q_{r+j} = (x^2 + 1)^n Q_{2n+j}. \tag{5.14}$$

From (5.12),

$$\sum_{r=0}^{2n+1} \binom{2n+1}{r} (xW)^r 2^{2n+1-r} = (x^2 + 1)^n W^{2n} (xW + 2I) \tag{5.15}$$

and we deduce

$$\sum_{r=0}^{2n+1} \binom{2n+1}{r} x^r P_r = \tfrac{1}{2}(x^2 + 1)^n Q_{2n+1} \tag{5.16}$$

and

$$\sum_{r=0}^{2n+1} \binom{2n+1}{r} x^r Q_r = 2(x^2 + 1)^{n+1} P_{2n+1}. \tag{5.17}$$

Also, from (5.6),

$$(4xW)^n = (W^2 - 4I)^n, \tag{5.18}$$

whence

$$(2x)^n P_n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} P_{2n-2r} \tag{5.19}$$

and

$$(2x)^n Q_n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q_{2n-2r}. \tag{5.20}$$

Let us revert momentarily to (5.8).

Rearrange (5.8) and raise to the s^{th} power to obtain

$$2^{(n-1)s} P_n^s W^s = \sum_{r=0}^{s} (-1)^r \binom{s}{r} 2^{nr} P_{n-1}^r W^{n(s-r)}. \tag{5.21}$$

Identities such as

$$P_n^s Q_s = \sum_{r=0}^{s} (-1)^r \binom{s}{r} P_{n-1}^r Q_{n(s-r)} \tag{5.22}$$

and

$$P^{s}P_{s+j} = \sum_{r=0}^{s}(-1)^{r}\binom{s}{r}P_{n-1}^{r}P_{n(s-r)+j} \tag{5.23}$$

flow from (5.21).

The above information, together with complementary material in [5], offers some details of the finite summation of Pell and Pell-Lucas polynomials by means of matrices. Clearly, the topics treated are far from complete. For instance, (5.1) extends naturally to

$$W_m = \begin{bmatrix} Q_m & 1 \\ Q_m^2 + 4(-1)^{m-1} & Q_m \end{bmatrix} \quad [|W_m| = 4(-1)^m], \tag{5.24}$$

from which new properties of our polynomials may be derived. Enough has been said, however, to indicate techniques for further development.

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