ON FOLYOMINOES AND FEUDOMINOES

JOHN C. TURNER
University of Waikato, Hamilton, New Zealand

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1. INTRODUCTION

In 1953 Solomon Golomb [1] "invented" polyominoes and gave them to the world in a talk to the Harvard Mathematics Club. Since then polyominoes have given pleasure to tens of thousands, not only through puzzle- and game-type activities carried out with them but also as a source of problems amenable to mathematical study.

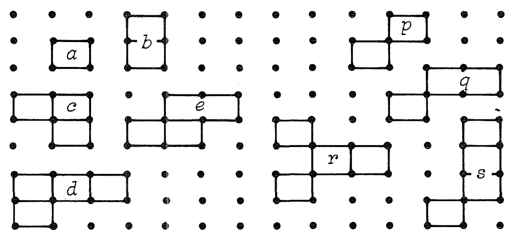
This year we contrived a creative project in combinatorics for a first-year University class. We took the polyominoes and added to them the integers of the Fibonacci sequence in a way to be described below. We christened the resulting objects *folyominoes* and *feudominoes*. In the notes for the project, we wrote: "Thus we have acted as midwife to the birth of twins Folyomino and Feudomino, born of two venerable and well-loved parents, viz. Polyomino and Fibonacci-sequence. We offer the twins to you, to rear, to nourish, and to study; to play with; to build ideas with; to create mathematics with."

In this paper we define the objects of study and describe some of their properties. The linking of the two fields of mathematics will be seen to have given rise to a wealth of new problems, the solution of which can provide the basis for a new field of study. This field might be named *integer sequence geometry*.

2. FOLYOMINOES AND FEUDOMINOES

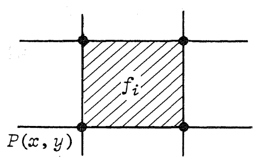
Polyominoes dwell amid the integer points of the Cartesian plane (see [1], [2]). They are formed by connecting unit squares into shapes, by 'glueing' one or more pairs of sides together. Thus, an *n-omino* is a shape consisting of n squares of a large chessboard, connected in such a way that a rook (a chess piece) could be moved from any square of it to any other square of it, in one or more valid rook moves. On the other hand, a *pseudo n-omino* has n unit squares joined together, but this time connection by 'glueing' two vertices is allowed as well as by 'glueing' two sides. In order to traverse all pseudo-polyominoes with a chess piece, we would need to use a king (or a queen) which can make diagonal moves as well as row and column moves.

Examples of both types of n-omino are given below for n = 1, 2, 3, and 4.



a is a 1-omino (monomino)
b is a 2-omino (domino)
c is a 3-omino (tromino)
d and e are 4-ominoes (tetromino)
p is a pseudo-domino
q is a pseudo-tromino
r and s are pseudo-tetrominoes

In order to derive folyominoes from polyominoes, we first place a pair of rectangular axes on the lattice and then assign Fibonacci integers to the unit squares of the positive quadrant by the following *rule*:



The square having the point $P(x, y)$ at its bottom left-hand corner receives the Fibonacci integer f_i, where $i = x + y + 1$ and $f_{i+2} = f_{i+1} + f_i$, with $f_1 = 1 = f_2$.

We may call the result the *Fibonacci lattice*.

Now if we construct a polyomino on this lattice, we may add up the integers in its cells. Let us call the total of the integers in a polyomino p the *value* $v(p)$ of the polyomino.

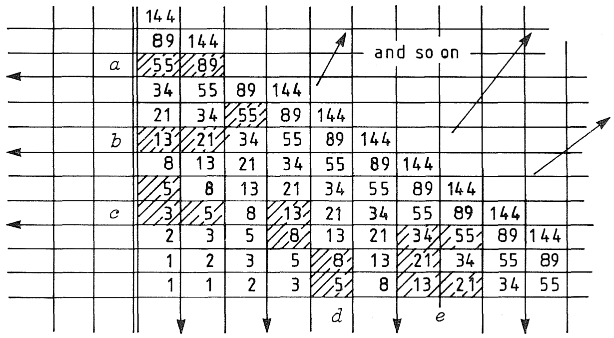
Definitions:

(i) If the value of a polyomino is a Fibonacci integer, the numbered polyomino is a *folyomino*.

(ii) If the value of a pseudo-polyomino is a Fibonacci integer, the numbered pseudo-polyomino is a *feudomino*.

In the following diagram, we show the positive quadrant of the Fibonacci lattice, with three example folyominoes marked on it.

The numbering of the lattice could be extended into the other three quadrants. Here, however, all our problems and discoveries will be confined to the positive quadrant.



a is a 2-folyomino (total 55 + 89 = 144 = f_{12})
b is a 3-feudomino (total 13 + 21 + 55 = 89 = f_{11})
c is a 3-folyomino
d is a 4-feudomino
e is a 5-folyomino

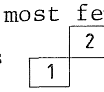
Note that a folyomino is also a feudomino (since a king as well as a rook can traverse a folyomino); but a feudomino with at least one vertex connection cannot also be called a folyomino, since it cannot be traversed by a rook.

3. FIRST CLASSIFICATION

Tables 1 and 2 show all the folyominoes having n = 1, 2, 3, 4, or 5 cells, and the feudominoes with n = 1, 2, 3, or 4 cells.

It should be noted that each folyomino is a representative of an infinite class. with any class, the members all have the same shape but differ in their values. For example, the 2-folyominoes $\boxed{ \mid }$ form the class

$$\{\boxed{1 \mid 1}, \boxed{1 \mid 2}, \boxed{2 \mid 3}, \ldots, \boxed{f_i \mid f_{i+1}}, \ldots\};$$

their values form the set $\{f_{i+2} : i = 1, 2, \ldots\}$. The same is true of most feudominoes; however, there are some unique feudominoes. One example is ; we give other examples in Table 2.

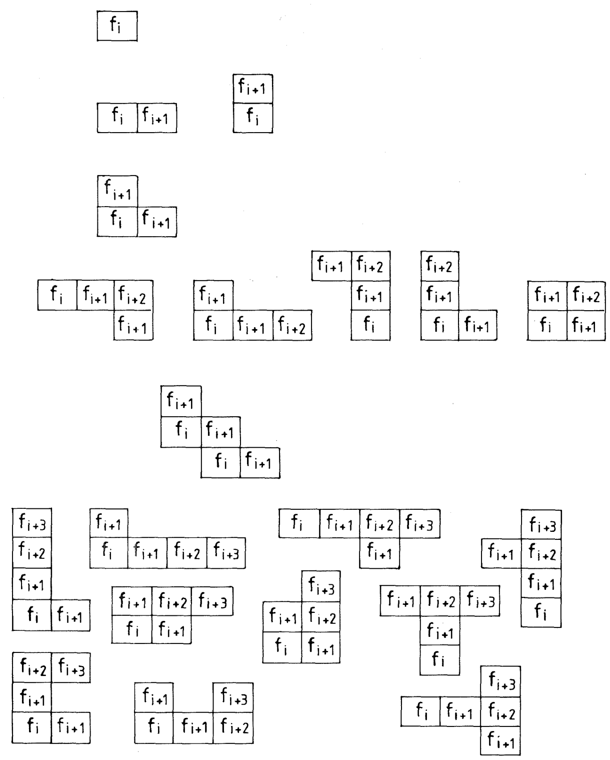
Notes:

(i) A polyomino has *size* (i.e., the number of cells in it) and *orientation* in the plane. One can translate it from one part of the plane to another; one can rotate it through 90°, 180°, or 270°; one can flip it over; and it still remains the same polyomino unless one expressly forbids one or the other of these transformations.

A folyomino, on the other hand, also has a *value* (i.e., the total of its cell values); so, under any of the above transformations its value may change. Let us agree that, if the value remains the same after *some* rotations and/or flippings, then the differently oriented folyominoes are *equivalent*. Otherwise, they are inequivalent. (Recall that when we speak of a folyomino, we refer to a representative of an infinite class of folyominoes having the same shape and orientation. We define all members of such a class to be equivalent, too.)

TABLE 1. FOLYOMINOES UP TO $n = 5$

n — Folyominoes — Value

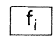
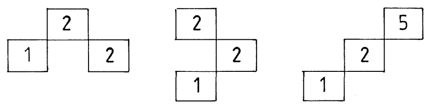
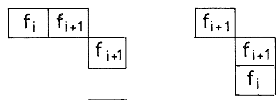
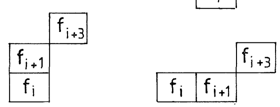
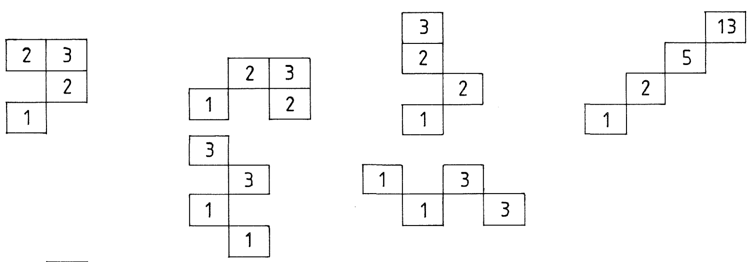
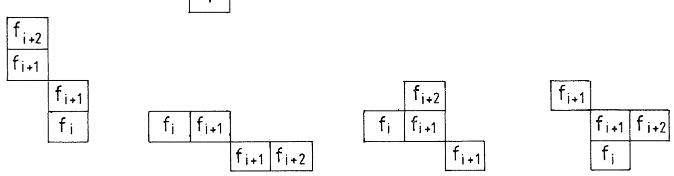
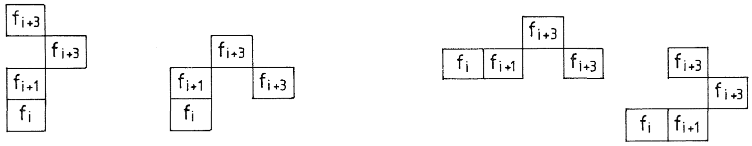
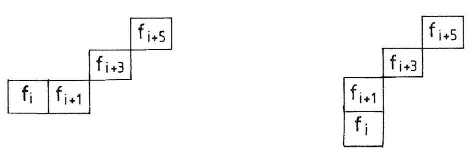


n	Value
1	f_i
2	f_{i+2}
3	f_{i+3}
4	f_{i+4}
5	f_{i+4}, f_{i+5}

(ii) Referring to Table 1, we see that the numbers of inequivalent folyominoes, for $n = 1, \ldots, 5$, is as given in the table below. We give also the number of different n-polyominoes, for comparison.

n	1	2	3	4	5
# folyominoes	1	1	1	2	6
# polyominoes	1	1	2	5	12

TABLE 2. FEUDOMINOES UP TO n=4

n	Feudominoes (N.B. The Folyominoes in Table 1 are also Feudominoes)	Values
1		f_i
2		2,3 (unique)
3		5,5,8 (unique)
		f_{i+3}
		f_{i+4}
4		8,8,8,8,8,21 (unique
		f_{i+4}
		f_{i+5}
		f_{i+6}

(iii) Combining the information of Tables 1 and 2 for $n = 1, \ldots, 4$, and leaving aside the unique feudominoes, we get the following table:

n	1	2	3	4
# feudominoes	1	1	3	7
# pseudo-ominoes	1	2	5	22

(iv) Note that usually if a folyomino is not a square, it has one equivalent folyomino (it is always the case with the feudominoes in Table 2); there are 5 exceptions in Table 1, for $n = 3$ and for $n = 5$.

(v) All four 3-cell feudominoes have the same shape; but two different values occur. It never happens, among the folyominoes of Table 1, that two folyominoes have the same shape and have different values. We ask whether it is possible to construct such a pair of folyominoes.

4. TILING PROBLEMS

Many of the attractive problems concerning polyominoes involve finding how to use certain sets of them in order to fill a given shape exactly. For example, there are just 12 different pentominoes, and one problem is to use a set of these to fill (i.e., to *tile*) a 6×10 rectangle. It has been shown that there are 2339 different ways of doing this (although it is surprisingly difficult to find even one of these, if one cuts the pentominoes out of cardboard and attempts a jig-saw approach to the problem!).

With folyominoes, the number and types of possible tiling problem multiply, because not only can one aim to tile a given shape with them, but also one can aim to achieve certain kinds of total value for the shape (e.g., a Fibonacci number of a particular kind.) Further, one can aim to produce a sequence of shapes that have a given sequence of integer values; we discuss below, in Sections 5 and 6, two problems of this kind.

First we discuss problems of tiling (i) squares, (ii) rectangles, and (iii) the quarter-plane.

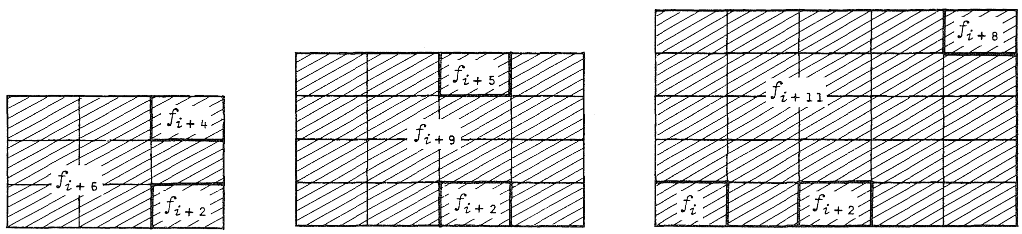
(i) Tiling an $n \times n$ square: Every 1×1 square is, of course, a folyomino. So, too, is every 2×2 square, since each has the following arrangement:

f_{i+1}	f_{i+2}
f_i	f_{i+1}

which has value f_{i+4}. It is worth noting here that if we were to create an r-bonacci lattice, assigning integers from an r-bonacci sequence to the cells, then every $r \times r$ square would have a value that was an integer of the sequence. We give a tribonacci example of this in Section 7.

Therefore, when n is 1 or 2, the $n \times n$ square can be tiled with a single folyomino. A natural question to ask is: What is the minimal number, say ϕ, of feudominoes required to tile a given square? We have not yet found a general answer to this question; however, the answers for small n may be found by inspection. A table, and example minimal tilings for n = 3, 4, 5 follow:

n	1	2	3	4	5
ϕ	1	2	3	3	4



Using the fact that $f_r + \cdots + f_s = f_{s+2} - f_{r+1}$, we easily find the following formula for the value of an $n \times n$ square which has f_i in its least-value cell:

$$V_{n \times n} = f_{i+2n+2} - 2f_{i+n+2} + f_{i+2}.$$

To find a solution for ϕ for a given value of n, we have to find a minimal partition of $V_{n \times n}$ using Fibonacci integers as addends.

(ii) Tiling an $m \times n$ rectangle: Rather than ask for the minimum number of feudominoes required to tile a given shape, as in (i), we ask what is the *total* number that can be found in the shape, differing in any way.

Let Φ_{mn} be the total required for an $m \times n$ rectangle. It is easy to show that $\Phi_{m1} = 2m - 1$; but we have not yet found a formula for Φ_{m2}, even when adding the restriction that only folyominoes be counted.

(iii) Tiling the quarter-plane: Referring to the 'positive', or 'north-east', portion of the plane only, simple tiling problems are: Tile the quarter-plane using only

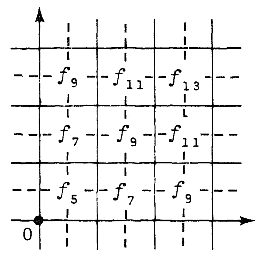
(a) even-valued folyominoes;

(b) odd-valued folyominoes;

(c) folyominoes with even-subscripted F-values;

(d) folyominoes with odd-subscripted F-values.

We easily found solutions for (a), (b), and (d); but for (c) our only solution so far uses a 5-feudomino of value f_6. The simplest solution for (d) uses the 2×2 squares, thus:



Incidentally, this solution with odd-subscripted folyominoes suggests the following generalization. Defining a *lolyomino* to be a polyomino whose value is an integer of the Lucas sequence, $\{L_i\} = 1, 3, 4, 7, \ldots$, the diagram above immediately gives a tiling in terms of even-subscripted lolyominoes. This follows from the fact that $L_i = f_{i-1} + f_{i+1}$; so placing two 2×2 squares side by side gives a lolyomino. Thus, $\boxed{f_5}\boxed{f_7} = \boxed{\quad L_6 \quad}$; and the required type of quarter-plane tiling, using 2×4 lolyominoes of even-subscript values, is immediately evident.

We turn now to a new kind of tiling problem: Given any integer, does a shape (i.e., a combination of cells) exist whose total value equals the integer, and which can be tiled by distinct folyominoes? We shall call this the *integer tiling problem*; and, in view of Zeckendorf's theorem on Fibonacci partitioning of the integers, it is easy to arrive at a solution.

5. ZECKENDORF INTEGER TILINGS

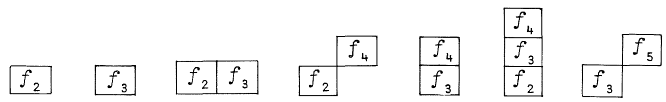
Zeckendorf's theorem (see [4] for details) tells us that any integer can be partitioned into distinct Fibonacci integers in such a way that there is no gap larger than one in the sequence of f_i-values used in the partition, with all sequences beginning with $f_2 = 1$ or $f_3 = 2$.

We construct the required partitions recursively as follows: Let the partition of 1, namely f_2, be written as a set $P_1 = \{f_2\}$; and the partitions of 2 and 3 be written as $P_2 = \{f_3\}$, $P_3 = \{f_1, f_2\}$, respectively. Then the partitions of the next three ($= f_4$) integers are given by:

$$P_4 = P_1 \cup \{f_4\}; \quad P_5 = P_2 \cup \{f_4\}; \quad P_6 = P_3 \cup \{f_4\}.$$

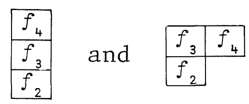
The partitions of the next five ($= f_5$) integers are given by taking the union of $\{f_5\}$ with each of P_2, P_3, P_4, P_5, and P_6, in turn. For the next eight ($= f_6$), we take the union $\{f_6\}$ of $P_4, P_5, \ldots, P_{11}$, in turn. And so on.

Using the same recurrence procedure, and with each union taking the corresponding cells from the Fibonacci lattice, we can construct shapes which constitute Zeckendorf tilings for each integer. The tilings for $n = 1, \ldots, 7$ are shown below:



1 2 3 4 5 6 7

Note that, for $n = 6$, two types of tile arise, viz:



Therefore, the answer to the integer tiling problem is: for each integer, a Zeckendorf tiling can be constructed. Some integers have more than one type of Zeckendorf tiling (Z-tiling).

Now that we have shown how to construct Zeckendorf integer tilings, we can classify the integers according to defined properties of their respective tilings. Four interesting properties are:

ϕ = minimal number of folyominoes in a Z-tiling;
δ = number of diagonal connections in a Z-tiling;
τ = number of types of Z-tiling (different up to rotations and flippings of the shape only) of a given integer;
σ = size (i.e., number of cells used) of a Z-tiling.

Remark: ϕ, δ, and σ are invariant over tiling types, and $\delta = 0$ for $n = f_i - 2$ and $f_i - 3$, $i \geqslant 5$.

We will conclude this section by tabulating the four properties for the Z-tilings of $n = 1, \ldots, 19$. A recurrence formula can be written down to generate the sequence of σ values.

Table 3. Properties of Z-Tilings

n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
ϕ:	1	1	1	2	1	2	2	1	2	2	2	3	2	2	2	2	3	2	3
δ:	0	0	0	1	0	0	1	1	1	0	0	2	1	1	2	1	1	0	0
τ:	1	1	1	1	1	2	1	1	1	2	3	1	1	2	1	1	2	3	6
σ:	1	1	2	2	2	3	2	3	3	3	4	3	3	4	3	4	4	4	5

6. LATTICE PATHS AND F-CHAINS

There is a large literature on the combinatoric theory of paths defined on rectangular lattices, and [3] gives a good review of this. It is natural that we should now combine the notion of folyomino with that of paths on a Fibonacci lattice.

Definitions:

(i) A *simple path* on a Fibonacci lattice is a sequence of distinct cells on the lattice, each cell arrived at being adjacent (horizontally, vertically, or diagonally) to the previous cell. In chess terms, then, a simple path is a king's tour with no repetitions of cells. Let us use the symbols c_1, c_2, ..., c_n to describe a simple path starting at c_1 and ending at cell c_n; the sequence of cell values will be described by v_1, v_2, ..., v_n. The *length* of a simple path is the number of cells in it. The *value* of the path is

$$V_n = \sum_{i=1}^{n} v_i.$$

The r^{th} *partial path* is c_1, c_2, ..., c_r, with $1 \leqslant r \leqslant n$, having value

$$V_r = \sum_{i=1}^{r} v_i.$$

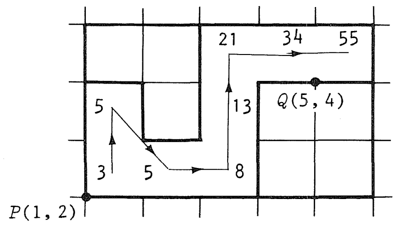
(ii) An *F-chain* is a simple path on a Fibonacci lattice such that all its partial paths are feudominoes; that is, all the partial path values $V_1, V_2, \ldots, V_n$ are Fibonacci integers.

Counting the F-chains

We will address the basic problem only, namely that of counting the number of F-chains that start at $P(i, j)$, in the cell c_1 having value f_{i+j+1}, and end at $Q(r, s)$, in the cell c_n having value f_{r+s+1}. We assume that $0 \leqslant i \leqslant r$ and $0 \leqslant j \leqslant s$. There are many cases to consider, if one looks at the different possible steps from cell to cell; if one does or does not allow unique steps [e.g., $P(1, 1)$ to $0(0, 0)$, involving the value sum $f_3 + f_1$]; if one imposes boundaries that a path cannot cross. To keep this introduction short, we give solutions for just two cases.

Case 1: Only steps in one of four directions ↑, →, ↖, ↘ (i.e., N, E, NW, SE) are allowed; and all the paths are to lie entirely within the boundary of the rectangle determined by the diagonal PQ.

Solution: We refer to the example in which $i = 1$, $j = 2$, $r = 5$, and $s = 4$; the inference to the general solution given at the end is elementary.



The F-chain shown in the diagram has partial path values:

3, 8, 13, 21, 34, 55, 89, 144

Note that the first two steps of all F-chains from P are forced to be either N, SE (giving partial value $3+5+5 = 13$) or E, NW (again giving partial value 13). From there on, all paths can proceed by only N or E steps. To get from lower 5-cell to the 55-cell, starting with value 13, required two N-steps and three E-steps. The number of different ways of doing this is equal to the number of different arrangements of the symbols N N E E E, which is $\binom{5}{2}$. Similarly, to get from the upper 5-cell to the 55-cell requires one N-step and four E-steps; the number of ways of doing this is $\binom{5}{1}$. Hence, the total number of F-chains from P to Q is $\binom{5}{1} + \binom{5}{2} = 15$.

Generalizing, the number of F-chains from $P(i, j)$ to $Q(r, s)$ is given by:

$$\binom{r+s-i-j-1}{s-j-1} + \binom{r+s-i-j-1}{s-j} = \binom{m}{n},$$

where $m = (r+s) - (i+j)$ and $n = s-j$, $m, n > 0$.

The value of each F-chain is the same, namely f_{r+s+3}. This is remarkable, in that the value is independent of i and j. Thus, we can state the following proposition regarding F-chains.

Proposition: Given $Q(r, s)$, and any other point $P(i, j)$ with $0 \leqslant i < r$ and $0 \leqslant j < s$. All F-chains from P to Q, with the conditions of Case 1, have the same value f_{r+s+3}.

Case 2: Only steps in one of the five directions $\uparrow$, $\rightarrow$, $\nwarrow$, $\searrow$, $\nearrow$ (i.e., N, E, NW, SE, NE) are allowed; $i \geqslant 1$ and $j \geqslant 1$; and no boundary conditions imposed.

Solution: Allowing for the NE steps (which were not allowed in Case 1) and removing boundary conditions leads to many more possibilities for constructing F-chains from $P(i, j)$ to $Q(r, s)$. We give the solution in terms of two coupled partial recurrence equations. To explain them, we must first define the following three counting functions.

(i) $A(i, j)$ is the number of F-chains from $P(i, j)$ to $Q(r, s)$, with all cells having their usual assigned F-values.

(ii) $B(i, j)$ is the number of F-chains from P to Q, with the first cell in each chain having value f_{i+j+2} and the others having their usual values.

(iii) $C(i, j)$ is the number of F-chains from P to Q, with the first cell having value f_{i+j+3} and the others having their usual values.

Now let us consider the first steps of F-chains from $P(i, j)$, beginning with the first cell-value f_{i+j+1}. There are two possibilities; namely, either a step N leading to cell $(i, j + 1)$ and partial value $V_2 = f_{i+j+2}$, or else a step E leading to cell $(i + 1, j)$, again with partial value $V_2 = f_{i+j+2}$. We can, therefore, write down the equation:

$$A(i, j) = B(i, j + 1) + B(i + 1, j) \tag{1}$$

Considering F-chains starting from $P(i, j)$ with the first cell having value f_{i+j+2}, we see that three different first steps are possible, the first two being to cells $(i - 1, j + 1)$ or $(i + 1, j - 1)$ in which cases the partial values $V_2 = f_{i+j+3}$ are achieved; the third is to cell $(i + 1, j + 1)$, achieving $V_2 = f_{i+j+4}$. From this information we can write down the equation:

$$B(i, j) = C(i - 1, j + 1) + C(i + 1, j - 1) + B(i + 1, j + 1) \tag{2}$$

Finally, we need an equation for $C(i, j)$. In fact, as explained above in Case 1, we can obtain a formula for it, thus,

$$C(i, j) = \binom{m}{n}, \tag{3}$$

where $m = (r + s) - (i + j)$ and $n = s - j$. (N.B. It is no accident that this number is precisely the same as the total for Case 1, as a moment's reflection on the two cases will show.)

Putting formula (3) into equation (2) gives:

$$B(i, j) = B(i + 1, j + 1) + \binom{m}{n - 1} + \binom{m}{n + 1} \tag{4}$$

For any given pair of values of (r, s), we can use equation (4) to compute a table of values $B(i, j)$; then, finally, using equation (1) with a particular pair (i, j) will give us the total $A(i, j)$, which is the object of the study.

As mentioned earlier, there are many other problems we could pose about F-chains, the solutions of which we could seek by means of lattice-path counting methods; but we must leave them here.

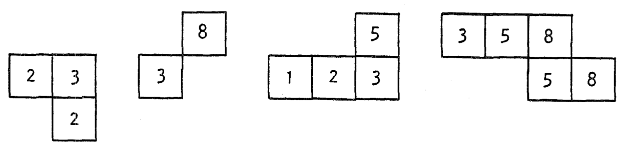
7. SUMMARY AND EXTENSIONS

We have shown how an integer sequence can be assigned to a lattice, and be used to give values to polyominoes constructed on the lattice. We chose to use the Fibonacci sequence, and studied tiling and path problems related to the folyominoes which resulted.

Many interesting possibilities suggest themselves for varying and extending our studies. We end by briefly indicating some of these.

Lucas polyominoes (lolyominoes)

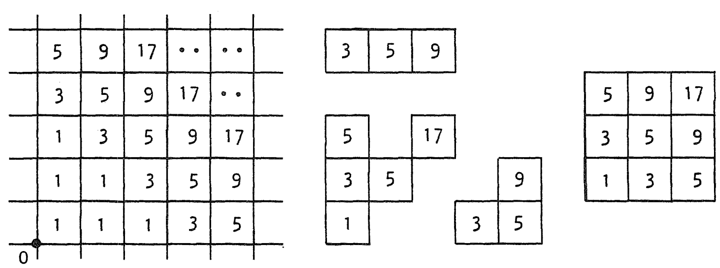
We have defined a *lolyomino* to be a polyomino whose value is a member of the Lucas sequence 1, 3, 4, 7, 11, 18, ... Examples of lolyominoes found on the Fibonacci lattice are:



We can study lolyominoes on the Fibonacci lattice. Likewise, we can use the Lucas sequence to produce a Lucas lattice: then we can study folyominoes on the Lucas lattice. It is clear that interesting comparisons and dual relations between the two systems will abound.

Integer sequence geometry

In Section 4(i), we noted a result concerning polyominoes defined on r-bonacci lattices. To give one example of such a lattice, with $r = 3$ and the sequence 1, 1, 1, 3, 5, 9, 17, 31, ..., we show a portion of the lattice, and a few small-size *trolyominoes*.



Note that the 3 × 3 square is a trolyomino, as claimed in 4(i). Note also that only odd-sized trolyominoes are possible: this is easily proved true, for all single cells have an odd value, and any combination of an even number of them would have an even total value. Since all members of this tribonacci sequence are odd, an even-valued combination of cells cannot be a trolyomino.

Finally, we do not have to stay with r-bonacci sequences. Generally, we can use the sequence s_1, s_2, s_3, ... to define an S-lattice thus: Our definition of what constitutes an S-polyomino (see the figure on the following page) will depend on whatever property or properties of the sequence $\{S_i\}$ we wish to highlight. Then our discoveries concerning the S-polyominoes (or *solyominoes*) will constitute results in *integer sequence geometry*.

	.	.	.	
	S_3	S_4	S_5	..
	S_2	S_3	S_4	..
	S_1	S_2	S_3	..
0				

Generalizations will take place when we compare results on solyominoes drawn from a class of S-lattices, defined using a class of related integer sequences. An obvious candidate for such studies is a class of F-lattices, using the sequences $F(a, b)$ defined by

$$F_1 = a,\ F_2 = b,\ F_{i+2} = F_{i+1} + F_i, \quad (a, b) \in Z \times Z.$$

REFERENCES

1. S. W. Golomb. *Polyominoes*. New York: Scribner, 1965.
2. D. A. Klarner, ed. *The Mathematical Gardner*. New York: Prindle, Weber & Schmidt, 1981.
3. S. G. Mohanty. *Lattice Path Counting and Applications*. New York: Academic Press, 1979.
4. J. L. Brown, Jr. "Zeckendorf's Theorem and Some Applications." *The Fibonacci Quarterly* 2, no. 3 (1964):163-168.

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