FIBONACCI WORD PATTERNS AND BINARY SEQUENCES

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1. INTRODUCTION

In this paper, we introduce the notion of Fibonacci word patterns, and use these to construct Fibonacci binary sequences. Spaces of the binary sequences are defined, and many properties of the spaces and sequences are obtained. Suggestions are given for using word patterns to generate other types of number sequences.

2. DEFINITIONS

Suppose we are given a *character set* $\mathbf{c} = \{c_1, \ldots, c_k\}$, whose members may be letters or digits. For example, if $k = 2$, and $c_1 = 0$, $c_2 = 1$, the character set is $\mathbf{c} = \{0, 1\}$, which is the binary set usually denoted by $\mathscr{B}$.

Using the characters of $\mathbf{c}$ we can, by juxtaposing characters, form *words*. Then, by juxtaposing words, we can form a pattern of words. A finite pattern of words we shall call a *sentence*.

Definitions:

(i) Given two initial words W_1 and W_2 (called *seed words*), the following recurrence defines an infinite sequence of words:

$$W_{n+2} = W_n W_{n+1}, \; n = 1, 2, \ldots . \tag{1}$$

(ii) The juxtaposition of the first i words generated by recurrence (1) is called a *Fibonacci sentence of length* i.

(iii) The name *Fibonacci word pattern* (or *word sequence*) will be used to denote the infinite juxtaposition $W_1 W_2 W_3 \ldots W_i \ldots$. We shall often use letters A, B for the seed words, and write $F(A, B) \equiv F(W_1, W_2)$ for the Fibonacci pattern. With this notation, the first part of the pattern is $ABABBABABBAB\ldots$, with $W_3 = AB$, $W_4 = BAB$, $W_5 = ABBAB$, and so on. The first four Fibonacci sentences in the pattern are:

A, AB, $ABAB$, and $ABABBAB$.

(iv) If the character set used for the seed words W_1 and W_2 is

$\mathscr{B} = \{0, 1\}$,

the resulting word pattern is a (0, 1)-sequence which we call a *Fibonacci binary pattern* (an FBP).

3. A FIBONACCI BINARY PATTERN

The following example of a Fibonacci binary pattern is the one whose discovery motivated our development of a theory of such patterns.

With $\mathscr{B} = \{0, 1\}$ as the character set, and seed words $A = 0$ and $B = 10$, we obtain the pattern:

$$F(0, 10) = 0100101001001010010\ldots .$$

This particular FBP we have given the symbol ω, after Wythoff. Its interest and importance arise from the following facts.

(i) The positions of the 0's in the sequence are

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, \ldots,$$

which is the sequence $\{a_n\} = \{[n\alpha]\}$, where $n = 1, 2, 3, \ldots$, and where $\alpha = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio.

(ii) The positions of the 1's in the sequence are

$$2, 5, 7, 10, 13, 15, 18, 20, 23, \ldots,$$

which is the sequence $\{b_n\} = \{[n\alpha^2]\}$.

It is well known (see [1], for example) that (a_n, b_n) are the *Wythoff pairs*, much studied in the literature on Fibonacci sequences.

4. SPACES OF FIBONACCI BINARY PATTERNS (FBPs)

Any FBP is determined by choosing two binary words W_1 and W_2 as seeds, and applying the recurrence (1). Let $\mathscr{B}^i$ be the set of all binary words of length i (i.e., words having i characters, each character being either 0 or 1). The number of words in $\mathscr{B}^i$, which we shall denote by $|\mathscr{B}^i|$, is 2^i. Thus, for examples, $\mathscr{B}^1 = \mathscr{B} = \{0, 1\}$ has the two words 0 and 1, and $\mathscr{B}^2 = \{00, 01, 10, 11\}$ has the four words shown.

Suppose that we choose the seed W_1 from $\mathscr{B}^m$, and the second seed W_2 from $\mathscr{B}^n$. There are $2^m \times 2^n = 2^{m+n}$ ways of making this double choice; each choice determines an FBP, which we denote by $F(W_1, W_2)$. We shall use the symbols $\mathscr{F}^{mn}$ to denote the set of all the possible 2^{m+n} FBPs obtained in this way, and call the set the *mn-FBP-space*. Using set notation, the space is defined thus:

$$\mathscr{F}^{mn} = \left\{F(W_1, W_2);\ W_1 \in B^m,\ W_2 \in B^n\right\}, \tag{2}$$

with

$$|\mathscr{F}^{mn}| = 2^{m+n}. \tag{3}$$

5. PROPERTIES OF FBP-SPACES

The FBP-space with the fewest elements is $\mathscr{F}^{11}$. We can list this space completely as follows (we give names to the members in the right-hand column):

Table 1. The First FBP-Space

FBP	First 13 Characters ...		Name(s)
$F(0, 0)$	0000000000000	...	0, z
$F(1, 0)$	1010010100100	...	α
$F(0, 1)$	0101101011011	...	$\overline{\alpha}$ (complement of α)
$F(1, 1)$	1111111111111	...	1, u, $\overline{z}$

Note that the space contains the *zeros sequence* 0 (or z), and its [0, 1]-component, the *units sequence* 1 (or u). It is clear that every mn-FBP-space will contain 0 and 1. It is also clear that whenever an FBP-space contains an element $F(A, B)$, it also contains the complement $F(\overline{A}, \overline{B})$, since, if (A, B) belongs to $\mathscr{B}^m \times \mathscr{B}^n$, so does $(\overline{A}, \overline{B})$. Thus, in $\mathscr{F}^{11}$ we find 0 and α, together with their complements 1 and $\overline{\alpha}$.

We now define equality of two FBPs as follows:

Let $F_1 = \{b_i\}_{i=1}^{\infty}$ and $F_2 = \{c_i\}_{i=1}^{\infty}$.
Then $F_1 = F_2$ if and only if $b_i = c_i \ \forall\, i$.

Proposition 5.1: Let $F_1, F_2 \in \mathscr{F}^{mn}$; then $F_1 = F_2$ iff they have the same seed words.

Proof: Trivial. ■

Thus, there are 2^{m+n} different FBPs in the space F^{mn}; up to complementation, however, there are 2^{m+n-1} different FBPs.

One may note that, if we define addition of two FBPs by

$$F_1 \oplus F_2 \equiv \{b_i + c_i\}_{i=1}^{\infty},$$

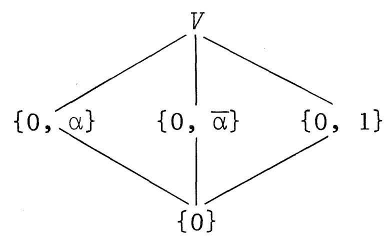
where the binary operation is addition modulus 2 [also known as "exclusive or (XOR)" or "ring sum" addition], the set of elements in any FBP-space form a group under $\oplus$. The details of this group for $\mathscr{F}^{11}$ are shown in the table and graph on the following page.

All the properties noted so far are possessed by pairs of finite binary words of lengths m and n, respectively. To determine something new, which is a property of infinite FBPs and which warrants further study, we ask whether an FBP (other than 0 or 1) occurring in one $\mathscr{F}^{mn}$ space also occurs in another $\mathscr{F}^{mn}$

Group $G(\mathscr{F}^{11})$

$\oplus$	0	α	$\overline{\alpha}$	1
0	0	α	$\overline{\alpha}$	1
α	α	0	1	
$\overline{\alpha}$	$\overline{\alpha}$	1	0	
1	1	$\overline{\alpha}$	α	0

Group Lattice



The Viergruppe (Klein's 4-Group)

space. The answer is "Yes"; every FBP occurs in an infinity of $\mathscr{F}^{mn}$ spaces, as stated in the following theorem.

Theorem 5.1: Let $F(W_1, W_2) \in \mathscr{F}^{mn}$. Then $F(W_1, W_2)$ is also a member of spaces $\mathscr{F}^{rs}$, where

$$(r, s) \in \{(m+n, m+2n), (2m+3n, 3m+5n), \ldots, (p_1m+q_1n, p_2m+q_2n), \ldots\},$$

with the coefficients $\{p_1, q_1, p_2, q_2\}$ being ordered sets of Fibonacci numbers of type $\{f_i, f_{i+1}, f_{i+1}, f_{i+2}\}$.

Proof: We shall write A, B for W_1, W_2, to avoid subscripts, and begin by proving a lemma.

Lemma: $F(A, B) = ABF(AB, BAB)$. (4)

This follows immediately from (1), since the recurrence generation of words produces $W_3 = AB$, and then $W_4 = BAB$; thus, $F(W_3, W_4)$ is the continuation of $F(A, B)$ after words A and B are juxtaposed.

We shall now prove that

$$F(A, B) = F(AB, ABB). \tag{5}$$

Using (4) on the left-hand side, we obtain

$$F(A, B) = ABF(AB, BAB) = ABx_1, x_2, x_3, \ldots, \text{ say;}$$

and the right-hand side of (5) is

$$F(AB, ABB) = y_1, y_2, y_3, \ldots, \text{ say; where each } x_i, y_j \in \{A, B\}.$$

We have to show that $y_1 = A$, $y_2 = B$, $y_3 = x_1, \ldots, y_i = x_{i-2}, \ldots$. To show that this is so, we shall replace the two ABs in the x seed words by C and C^*, respectively, and those in the y seed words by D and D^*, respectively. Then the expanded sequences are

$$x\colon\ F(AB, BAB) = F(C, BC^*) = C, BC^*, CBC^*, BC^*CBC^*, \ldots;$$

$$y:\quad F(AB,\ ABB) = F(D,\ D^*B) = D,\ D^*B,\ DD^*B,\ D^*BDD^*B,\ \ldots\ .$$

Comparison of the elements of these two expansions completes the proof of (5).

Note now that, if $A \in \mathscr{B}^m$, and $B \in \mathscr{B}^n$, $F(AB,\ ABB) \in \mathscr{F}^{(m+n)(m+2n)}$; and if we replace AB by A' and ABB by B', we can use the same proof to show that

$$F(AB,\ ABB) = F(A'B',\ A'B'B') = F(ABABB,\ ABABBABB) \in \mathscr{F}^{rs},$$

with $r = 2m + 3n$ and $s = 3m + 5n$.

Inductive argument establishes that this process can be continued indefinitely, with r and s being Fibonacci integers as claimed. ■

Corollaries:

(i) From (5) we see that we can write $F(A,\ B) = F(S_i,\ T_i)$, where $(S_i,\ T_i)$ are obtainable from the following double recurrence system:

$$S_{i+1} = S_i T_i, \text{ with } S_1 = A,\ T_1 = B, \text{ and } T_{i+1} = S_{i+1}T_i. \tag{6}$$

Let us denote the *length of a word* W (i.e., the number of characters it contains) by $\ell(W)$. Then, if $\ell(A) = m$ and $\ell(B) = n$, by Theorem 5.1 we have

$$\ell(S_i) = f_i m + f_{i+1}n \quad\text{and}\quad \ell(T_i) = f_{i+1}m + f_{i+2}n. \tag{7}$$

Thus, since S_i is repeated infinitely often, the first $(f_i m + f_{i+1}n)$ characters of the FBP, for $i = 1, 2, \ldots$, occur together infinitely often later in the sequence.

Indeed, if we take *any* subsequence $\{b_j,\ b_{j+1},\ \ldots,\ b_k\}$ of an FBP, and if we choose i large enough, the subsequence will be included in S_i, and hence will be repeated infinitely often. We call this property of FBPs the *strong recurrence property*.

(ii) Let us define scalar multiplication of a sequence of words thus: If α is a scalar, then $\alpha(W_1,\ W_2,\ \ldots) = W_1W_1 \ldots W_1W_2W_2 \ldots W_2\ \ldots$, each word being taken α times before continuing the sequence with the next word.

With this notation, repeated application of the lemma in Theorem 5.1 shows that

$$F(W_1,\ W_2) = 2(W_3,\ W_6,\ \ldots,\ W_{3i},\ \ldots), \tag{8}$$

where $W_3 = W_1W_2$, etc.

We may say that any FBP has a *scalar factor of* 2, with a meaning which is clear from (8).

Now that we know any given FBP occurs in an infinite number of $\mathscr{F}^{mn}$ spaces, we may ask how many new FBPs can be found in a given space $\mathscr{F}^{mn}$, new in the sense that they have not already occurred in an *earlier* space. To give meaning to "earlier," we define an ordering of the FBP spaces by the following ordering of (m, n) pairs:

$m \backslash n$	1	2	3	4	...
1	(1, 1) →	(1, 2)	(1, 3) →	(1, 4)	...
2	(2, 1)	(2, 2)	(2, 3)	⋮	...
3	(3, 1)	⋮	⋮		...
⋮	⋮				...

Using the symbol < for the order relation, we can now write

$$\mathscr{F}^{11} < \mathscr{F}^{12} < \mathscr{F}^{21} < \mathscr{F}^{31} < \mathscr{F}^{22} < \cdots . \tag{9}$$

At this point, we will add to the difficulty of determining how many new FBPs occur in a given $\mathscr{F}^{mn}$ by defining "new" in a broader sense than "not equal to an earlier one." To do this, however, we need to introduce the concept of *eventual equality*.

Consider the two sequences

$$F_1 = c_1 c_2 c_3 c_4 \ldots, \quad \text{and} \quad F_2 = xyzc_1 c_2 c_3 c_4 \ldots,$$

where after xyz the sequence for F_2 continues exactly as for F_1. We shall say that F_1 and F_2 are "eventually equal." In general, we define eventually-equal sequences thus:

Let F_1, F_2 be any two FBPs; if $F_1 = B_1F$ and $F_2 = B_2F$, where F is a FBP and B_1, B_2 are binary words (possibly empty), then F_1 and F_2 are *eventually equal*. We shall write this as

$$F_1 \overset{\text{ev}}{=} F_2.$$

We now define an *equivalence relation* for FBPs thus:

Let F_1, F_2 be any two FBPs; then $F_1 \equiv F_2$ if either $F_1 = F_2$ or $F_1 \overset{\text{ev}}{=} F_2$. Otherwise, $F_1 \not\equiv F_2$.

With this notion of equivalence and inequivalence of FBPs, we can sort members of FBP spaces into equivalence classes and attempt to count the classes.

Examples:

(1)
$$\begin{aligned} F(1, 0) &= 1, 0, 10, 010, 10010, \ldots \\ &= 1F(0, 10) \\ &= 10F(10, 010) \text{ etc. (by lemma, Theorem 5.1),} \\ &= F(10, 100) \\ &= F(10100, 10100100) \text{ etc, (by Theorem 5.1);} \end{aligned}$$

therefore,

$$\begin{aligned} F(1, 0) &\equiv F(0, 10) \equiv F(10, 010) \equiv \cdots \\ &\equiv F(10, 100) \equiv F(10100, 10100100) \equiv \cdots . \end{aligned}$$

(ii) The following table lists the FBPs in the first five spaces, showing only the new ones that appear in each space. The [0, 1]-complements of the sequences are listed in bar-notation at the end of each space. Thus, $\alpha = F(1, 0)$ is described in full, but $\overline{\alpha} = F(0, 1)$ is merely listed with all other complements at the end of the $\mathscr{F}^{11}$ section.

Table 2. Inequivalent FBPs in the First Five Spaces

$$N = |\mathscr{F}^{mn}| = 2^{m+n}$$

Space $\mathscr{F}^{mn}$ (m, n)	Sequence $F(A, B)$ (first thirteen characters)	Descriptive Names
(1, 1) $N = 4$	$F(0, 0)$ = 0000000000000 $F(1, 0)$ = 1010010100100 0, z = 1111111111111 $\overline{\alpha}$	0, z, zero α, alpha 1, u, unity
(1, 2) $N = 8$	$F(1, 00)$ = 1001000010010 $\overline{\beta}$ The other six in this space are 0, $F(0, 10) \overset{ev}{=} \alpha$, $F(0, 01) \overset{ev}{=} \alpha$, and their complements.	β, beta
(2, 1) $N = 8$	$F(01, 0)$ = 0100100010010 $F(11, 0)$ = 1101100110110 $\overline{\gamma}$, $\overline{\varepsilon}$ The other four in this space are 0, $F(10, 0) = \gamma$, 1, $\overline{\gamma}$.	γ, gamma ε, epsilon
(3, 1) $N = 16$	$F(100, 0)$ = 1000010000100 $F(011, 0)$ = 0110011000110 $F(101, 0)$ = 1010101001010 $F(111, 0)$ = 1110111001110 $\overline{\zeta}$, $\overline{\eta}$, $\overline{\mu}$, $\overline{\nu}$ The other eight in this space are 0, $F(010, 0) \overset{ev}{=} F(001, 0) \overset{ev}{=} \zeta$, $F(110, 0) = \eta$, and their complements.	ζ, zeta η, eta μ, mu ν, nu
(2, 2) $N = 16$	$F(00, 01)$ = 0001000101000 $F(00, 11)$ = 0011001111001 $F(01, 10)$ = 0110011010011 $F(01, 11)$ = 0111011111011 $F(01, 01)$ = 0101010101010 $\overline{\pi}$, $\overline{\rho}$, $\overline{\sigma}$, $\overline{\tau}$, $\overline{c}_1$, 0, 1 $F(00, 10) \overset{ev}{=} \pi$, $F(10, 11) \overset{ev}{=} \tau$, and their complements.	π, phi ρ, rho σ, sigma τ, tau c_1, first cyclic

Notes:

(i) The first cyclic FBPs are 0, 1, c_1, $\overline{c}_1$. We show later that cyclic sequences can only occur when $m = n$.

(ii) The list count of "esentially new" (i.e., up to complementation) FBPs that are noncyclic grows by the following increments: 1, 1, 2, 4, 4, ... as we proceed through the ordered FBP spaces.

We have not yet found a general formula for these increments. However, we have a useful sequence parameter for determining whether or not two FBPs may be equivalent, namely the limit density of the words of the binary sequences. We describe this parameter next.

6. THE DENSITY OF AN FBP

Consider the FBP given by $F(A, B)$, where A, B are binary seed words having weights (numbers of 1's) $\omega(A) = a$ and $\omega(B) = b$, respectively. Let the lengths (numbers of characters) of A, B be $\ell(A) = m$ and $\ell(B) = n$, respectively. Let $F(A, B) = W_1W_2W_3, \ldots, W_i, \ldots$, the W_i being the words generated by the Fibonacci recurrence.

Definitions:

(i) The *density* of word W_i is $\delta_i \equiv \dfrac{\omega(W_i)}{\ell(W_i)}$.

(ii) The *density* of $F(A, B)$ is $\delta \equiv \lim\limits_{i\to\infty} \delta_i$, assuming such a limit exists.

Theorem 6.1: The density of $F(A, B)$ is

$$\delta = \frac{a + b}{m + n} = c + d\alpha,$$

where

$$\alpha = \frac{1}{2}(1 + \sqrt{5}), \quad c = \frac{1}{\Delta}\begin{vmatrix} a & n \\ b & m + n \end{vmatrix}, \quad d = \frac{1}{\Delta}\begin{vmatrix} m & a \\ n & b \end{vmatrix},$$

and

$$\Delta = \begin{vmatrix} m & n \\ n & m + n \end{vmatrix} = m^2 - n^2 + mn.$$

Proof: The i^{th} word W_i of the Fibonacci word pattern $F(A, B)$ contains f_{i-2} A's and f_{i-1} B's; this follows by induction from the recurrence construction of the pattern. Therefore,

$$\delta_i = \frac{\omega(W_i)}{\ell(W_i)} = \frac{f_{i-2}a + f_{i-1}b}{f_{i-2}m + f_{i-1}n}.$$

Dividing numerator and denominator by f_{i-2} and taking limits gives

$$\delta = \lim_{i\to\infty} \delta_i = \frac{a + b\alpha}{m + n\alpha},$$

with α the golden ratio $\frac{1}{2}(1 + \sqrt{5})$.

Setting $\dfrac{a + b\alpha}{m + n\alpha} = c + d\alpha$, algebra gives

$$a + b\alpha = (cm + dn) + \alpha[cn + d(m + n)],$$

using the fact that $\alpha^2 = \alpha + 1$.

Equating coefficients of α^0 and α^1 gives

$$a = cm + dn \quad \text{and} \quad b = cn + d(m + n).$$

Solving for c, d by the method of determinants gives the formulas required. ■

Before presenting a table of densities for the first fifteen FBPs, we make three remarks and state a proposition on the density of a complement sequence.

Remarks:

(i) It is clear that if $F_1 \overset{\text{ev}}{=} F_2$, the densities of F_1 and F_2 are equal, because the limit is applied to W_i, and beyond certain points in both sequences all characters correspond.

(ii) It might seem a better procedure to define density by

$$\delta(F) = \lim_{i \to \infty} \frac{\omega(S_i)}{\ell(S_i)},$$

where S_i is the Fibonacci sentence $W_1 W_2 \ldots W_i$. In fact, perhaps surprisingly, this limit is the same as the one derived above, which can be proved using the identity

$$\sum_{r=1}^{i} f_r = f_{i+2} - 1.$$

(iii) From the definition of δ it is evident that $0 \leqslant \delta(F) \leqslant 1$ for all F.

Proposition: Let $F \equiv F(A, B)$ have density $\delta(F) = c + d\alpha$ as in Theorem 6.1. Then the [0, 1]-complement sequence $\overline{F} \equiv F(\overline{A}, \overline{B})$ has density

$$\delta(\overline{F}) = 1 - \delta(F) = \frac{(m - a) + (n - b)\alpha}{m + n\alpha} = (1 - c) - d\alpha. \tag{10}$$

Proof: The proof follows immediately from consideration of the composition of W_i. ■

We could say that $\delta(F)$ is a measure of the density of 1's in the sequence F, and $\delta(\overline{F})$ is a measure of the density of 0's in F. (See Table 3.)

We have used the density parameter in two ways. First, when we checked for equivalence of two FBPs to produce Table 2. From Remark (i) above we know that two FBPs are *inequivalent* if they have different densities. However, the converse is not true, as can be seen by scanning Table 3; η and μ have equal densities, as have σ and c_1. To distinguish between equal density pairs, one must compare their patterns of 0's and 1's. Thus:

$$\eta = 0110011000110\ldots \quad \text{and} \quad \mu = 1010101001010\ldots$$

are clearly distinct, since the former contains pairs of 1's while the latter does not.

Similarly for $\sigma = 01100110\ldots$ and $c_1 = 01010101\ldots$.

Table 3. Densities of the First Fifteen FBPs

Sequence	Parameter Values m, n	a, b	c, d	δ (to 3 d.p.)
$0 = F(0, 0)$	1, 1	0, 0	0, 0	0
$\alpha = F(1, 0)$	1, 1	1, 0	2, −1	0.382
$1 = F(1, 1)$	1, 1	1, 1	1, 0	1
$\beta = F(1, 00)$	1, 2	1, 0	−3, 2	0.236
$\gamma = F(01, 0)$	2, 1	1, 0	$\frac{1}{5}(3, -1)$	0.276
$\varepsilon = F(11, 0)$	2, 1	2, 0	$\frac{2}{5}(3, -1)$	0.553
$\zeta = F(100, 0)$	3, 1	1, 0	$\frac{1}{11}(4, -1)$	0.217
$\eta = F(011, 0)$	3, 1	2, 0	$\frac{2}{11}(4, -1)$	0.433
$\mu = F(101, 0)$	3, 1	2, 0	$\frac{2}{11}(4, -1)$	0.433
$\nu = F(111, 0)$	3, 1	3, 0	$\frac{3}{11}(4, -1)$	0.650
$\pi = F(00, 01)$	2, 2	0, 1	$\frac{-1}{2}, \frac{-1}{2}$	0.309
$\rho = F(00, 11)$	2, 2	0, 2	−1, −1	0.618
$\sigma = F(01, 10)$	2, 2	1, 1	$\frac{1}{2}, 0$	0.5
$\tau = F(01, 11)$	2, 2	1, 2	$0, \frac{1}{2}$	0.809
$c_1 = F(01, 01)$	2, 2	1, 1	$\frac{1}{2}, 0$	0.5

Our second use of δ was to study the question: "Given an FBP, how many equivalent forms has it for a fixed m, and for a fixed n (we have already seen that it has an infinite number of equivalent forms when m and n are allowed to vary)?". Again, we have no general answer to this question, but examining the density of an FBP provides a useful start. We give one example.

Example: Find all the equivalents of $\alpha = F(1, 0)$ in spaces $\mathscr{F}^{mn}$, for $1 \leqslant m \leqslant 2$, $1 \leqslant n \leqslant 4$.

Solution: The parameters of α and $\delta = 0.382$ with $(m, n) = (1, 1)$, and $(a, b) = (1, 0)$; therefore, any other FBP is a candidate for equivalence if

$$0.382 = \frac{1}{1 + \alpha} = \frac{a + b\alpha}{m + n\alpha}.$$

Equating coefficients of α^0 and α^1 gives conditions for a and b as follows:

$$\left.\begin{aligned} a &= 2m - n \\ b &= m - n \end{aligned}\right\} \text{ with } 0 < m, \quad 0 < n. \tag{11}$$

Thus, feasible solutions for (m, n) are the lattice points on and between lines $m = n$ and $m = \frac{1}{2}n$. For fixed m, the values for n are $m, m + 1, \ldots, 2m$. To solve our problem, we need only look at the following (m, n)-points:

$(1, 1)$, $(1, 2)$, $(2, 2)$, $(2, 3)$, and $(2, 4)$.

From (11) we compute the corresponding (a, b)-values; then we can write out all possible FBPs having the same density as α. Finally, we can check these for equivalences. Table 4 shows the FBPs with $\delta = \delta(\alpha)$.

Table 4. The FBPs with Density Equal to $\delta(\alpha) = 0.382$

(m, n)	Parameter Values $a = 2m - n$	$b = n - m$	Fibonacci Binary Patterns (FBPs)
1, 1	1	0	$F(1, 0) = \alpha$
1, 2	0	1	$F(0, 10)$, $F(0, 01)$
			(both are $\overset{\text{ev}}{=} \alpha$)
2, 2	2	0	$F(11, 00) = 2\alpha$
2, 3	1	1	$F(10, 100)$, $F(01, 100)$
			$F(10, 010)$, $F(01, 010)$
			$F(10, 001)$, $F(01, 001)$
2, 4	0	2	$F(00, 1100)$, $F(00, 0110)$
			$F(00, 0011)$, $F(00, 1001)$
			$F(00, 1010)$, $F(00, 0101)$

Combinatoric Formula: The total number of FBPs with density $\delta(\alpha)$ is given by the formula

$$\sum_{m=1}^{m^*} \sum_{n=m}^{2m} \binom{m}{2m - n}\binom{n}{n - m} = \sum_{m=1}^{m^*} \sum_{n=m}^{2m} \frac{[n]_m}{(2m - n)!\,(n - m)!}, \tag{12}$$

where m^* is a given upper limit for m, and $[n]_m$ is the falling factorial

$n(n - 1) \cdots (n - m + 1)$.

Proof: We obtained the limits for m and n above. The binomial coefficients count the numbers of ways in which the a 1's and b 1's can be placed in the seed words A and B, respectively. ■

To complete the solution to our problem, we have to examine all the FBPs found, to check for equivalences. By inspection, we find the following set of inequivalent sequences which have $\delta = \delta(\alpha) = 0.382$, for $m = 1, 2$, $n = 1, 2, 3$ 4.

$$\{F(1, 0) = \alpha,\ F(11, 00) = 2\alpha,\ F(01, 100),\ F(10, 001),$$
$$F(01, 001),\ F(00, 1100),\ F(00, 1001),\ F(00, 1010)\}.$$

The cardinal number of this set is 8, which is half the total number of equal-density FBPs found.

7. GENERALIZATIONS, FURTHER PROPERTIES OF $F(A, B)$; APPLICATIONS

In this final section, we give density formulas, without proofs, for two new kinds of binary pattern; then we list propositions concerning run-lengths of A and B in the pattern $F(A, B)$. Details of these results may be found in [2] and [3]. We also indicate briefly how word patterns can be used to generate number sequences. Two ways of doing this are given; we are investigating others. We believe that studies of number sequences derived from word patterns will be very fruitful, in that they will provide classes of sequences with interesting properties related to those of word patterns. Developing links between theories of word patterns and theories of number sequences will prove beneficial to both topics.

(1) The density of an FBP with $W_1 = rA$ and $W_2 = sB$

Let $W_1 = AA \ldots A$ (with A taken r times) and $W_2 = BB \ldots B$ (with B taken s times), with A, B being binary words. Then

$$\delta(F(rA, sB)) = \frac{ra + sb\alpha}{rm + sn\alpha}. \tag{13}$$

(2) Tribonacci binary patterns

$T(W_1, W_2, W_3)$ is the tribonacci word pattern

$$W_1W_2W_3 \ldots W_n \ldots, \text{ where } W_n \equiv W_{n-3}W_{n-2}W_{n-1}.$$

If the seed words W_1, W_2, and W_3 have the binary character set, we have a tribonacci binary pattern (a TBP) whose density is given by

$$\delta(T) = \frac{\tau\omega_1 + (\tau + 1)\omega_2 + \tau^2\omega_3}{\tau\ell_1 + (\tau + 1)\ell_2 + \tau^2\ell_3}, \tag{14}$$

where $\tau = 1.839$ is the positive root of $x^3 - x^2 - x - 1 = 0$. It is clear that we can extend these definitions and formulas to give n-bonacci patterns and their densities.

(3) Further properties of $F(A, B) = ABABBABBBBAB\ldots$

The following propositions concerning runs within the pattern are easily proved:

All A-runs have length 1; all B-runs have length 1 or 2. (15)

The number of A's in the i^{th} word of the pattern is f_{i-2}; the number of B's is f_{i-1}, with $f_{-1} = 1$, $f_0 = 0$. (16)

The number of B-runs of length 2 in W_{2i+1} is $f_{2(i-1)}$, and in W_{2i+2} is $f_{2i-1} - 1$, $i = 1, 2, \ldots$. (17)

The number of B-runs of length 1 in W_i can be determined using (16) and (17).

Consider the i^{th} Fibonacci sentence $S_i = W_1 W_2 \ldots W_i$. The number of B-runs of length 1 in S_i is $f_{i-2} + 1$, of length 2 is $f_{i-1} - 1$, and of either length is f_i, for $i > 1$. (18)

Define the *chaos* χ_i of W_i to be the number of transpositions of adjacent letters required to set the word into the form $AA\ldots ABB\ldots B$. Then χ_i satisfies the recurrence $\chi_i - \chi_{i-2} - \chi_{i-1} = f_{i-3}^2$, $i \geqslant 4$, with $\chi_1 = \chi_2 = \chi_3 = 0$.

(4) Two applications in number theory

(i) Generation of r-tuple integer sequences

In [2] we show generally how FBPs may be used to generate sequences of r-tuples of integers, whose properties we have only begun to study. One simple example must suffice here, with $r = 2$.

Suppose we use seed words $W_1 = a$, $W_2 = ba$, then consider the *positions* of a and b, respectively, in the resulting Fibonacci word pattern. Thus, the word pattern is

$$F(a, ba) = abaababaaba\ldots,$$

and the a-positions are 1, 3, 4, 6, 8, 9, 11, ... with the b-positions being 2, 5, 7, 10, etc. Taking these in pairs, we get the 2-tuple sequence

$$(1, 2), (3, 5), (4, 7), (6, 10), \text{etc.}$$

We see that $F(a, ba)$ in this manner generates the Wythoff-pairs sequence.

It is clear how we can generate 3-tuple sequences if we use character set $\{a, b, c\}$; and so on.

(ii) The Fibonacci reals

If we take any Fibonacci binary pattern and place a decimal point in front of it, we obtain a binary representation of a real number in the interval (0, 1). We believe the class of all such numbers, namely the *Fibonacci reals* to be worthy of study.

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