ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-425 *Proposed by Stanley Rabinowitz, Littleton, MA*

Let $F_n(x)$ be the n^{th} Fibonacci polynomial

$$F_1(x) = 1,\ F_2(x) = x,\ F_{n+2}(x) = xF_{n+1}(x) + F_n(x).$$

Evaluate:

(a) $\displaystyle\int_0^1 F_n(x)\,dx$; (b) $\displaystyle\int_0^1 F_n^2(x)\,dx$.

H-426 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, m, and n be integers. Prove that

$$(F_n F_{m+k-j} - F_m F_{n+k-j})(-1)^m = (F_k F_{j+n-m} - F_j F_{k+n-m})(-1)^j.$$

H-427 *Proposed by Piero Filipponi, Rome, Italy*

Let $C(n, k) = C_1(n, k)$ denote the binomial coefficient $\binom{n}{k}$.

Let $C_2(n, k) = C[C(n, k), k]$ and, in general,

$$C_i(n, k) = C(C\{\ldots[C(n, k), k]\}).$$

For given n and i, is it possible to determine the value k_0 of k for which

$$C_i(n, k_0) > C_i(n, k) \quad (k = 0, 1, \ldots, n;\ k \neq k_0)?$$

SOLUTIONS

Some Triple Sum

H-404 *Proposed by Andreas N. Philippou and Frosso S. Makri, Patras, Greece (Vol. 24, no. 4, November 1986)*

Show that

$$\text{(a)}\ \sum_{r=0}^{n}\sum_{i=0}^{1}\sum_{\substack{n_1,\ _2\ni \\ n_1+2n_2=n-i \\ n_1+n_2=n-r}}\binom{n_1+n_2}{n_1,\ n_2} = F_{n+2},\ n \geq 0;$$

$$\text{(b)} \sum_{r=0}^{n} \sum_{i=0}^{k-1} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n - i \\ n_1 + \cdots + n_k = n - r}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} = F_{n+2}^{(k)}, \; n \geq 0, \; k \geq 2,$$

where $n_1, \ldots, n_k$ are nonnegative integers and $\{F_n^{(k)}\}$ is the sequence of Fibonacci-type polynomials of order k [1].

[1] A. N. Philippou, C. Georghiou, & G. N. Philippou, "Fibonacci-Type Polynomials of Order K with Probability Applications," *The Fibonacci Quarterly* *23*, no. 2 (1985):100-105.

Solution by Tad P. White, Student, UCLA, Los Angeles, CA

(a) Although this is a special case of (b), it can be solved in a slightly simpler manner since the simultaneous equations

$$n_1 + 2n_2 = n - i$$
$$n_1 + n_2 = n - r$$

can be explicitly solved to obtain $n_1 = n + i - 2r$ and $n_2 = r - i$; thus the sum becomes

$$\sum_{r=0}^{n} \sum_{i=0}^{1} \binom{n-r}{r-i} = \sum_{r=0}^{n} \binom{n+1-r}{r},$$

and it is well known that the right-hand side sums to F_{n+2} for $n \geq 0$. However, the details can be omitted since this case is treated in part (b).

(b) Fix $k \geq 2$; we prove this equality in two steps. Let $f(n)$ denote the left-hand side of the equation in question, for our fixed k. First, we show that both sides of the equation are equal for $0 \leq n < k$, and then we show that both sides obey the same kth-order recursion relation, namely

$$f(n) = \sum_{1 \leq l \leq k} f(n-l);$$

we are off to a good start because we know already that $F_n^{(k)}$, and hence $F_{n+2}^{(k)}$, obey this relation.

Assuming first that $0 \leq n \leq k-1$, the upper limit of the summation over i can be replaced with n, since if $i > n$, the condition $n_1 + \cdots + kn_k = n - i$ is not satisfied by any k-tuple $(n_1, \ldots, n_k)$. Also, the condition that $n_1 + \cdots + n_k = n - r$ for some r with $0 \leq r \leq n$ is vacuously satisfied by every k-tuple $(n_1, \ldots, n_k)$ satisfying $n_1 + \cdots + kn_k = n - i$ for some $i \leq n$, so we may remove both this condition and the summation over r. Therefore,

$$\begin{aligned} f(n) &= \sum_{i=0}^{n} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n - i}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} \\ &= \sum_{i=0}^{n} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = i}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} \\ &= \sum_{i=0}^{n} F_{i+1}^{(k)}. \end{aligned}$$

Since $F_{n+1}^{(k)} = \sum_{i=1}^{n} F_i^{(k)}$ for $n \leq k$, we conclude that $f(n) = F_{n+2}^{(k)}$ for $0 \leq n \leq k-1$.

We now derive a recursion relation for $f(n)$. We make use of the following property of multinomial coefficients:

$$\binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} = \sum_{1 \leq l \leq k} \binom{n_1 + \cdots + n_k - 1}{n_1, \ldots, n_{l-1}, n_l - 1, n_{l+1}, \ldots, n_k}.$$

We will follow the convention that a multinomial coefficient vanishes when any entry is negative, so that this identity remains valid whenever each n_k is nonnegative. Substituting this in the formula defining $f(n)$, we find

$$f(n) = \sum_{1\le l\le k} \sum_{r=0}^{n} \sum_{i=0}^{k-1} \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\cdots+kn_k=n-i \\ n_1+\cdots+n_k=n-r}} \binom{n_1+\cdots+n_k-1}{n_1,\ldots,n_{l-1},n_l-1,n_{l+1},\ldots,n_k}$$

Letting m_i denote n_i for $i \neq l$ and $m_l = n_l - 1$, this becomes

$$= \sum_{1\le l\le k} \sum_{r=0}^{n} \sum_{i=0}^{k-1} \sum_{\substack{m_1,\ldots,m_k \ni \\ m_1+2m_2+\cdots+km_k=n-l-i \\ m_1+\cdots+m_k=n-1-r}} \binom{m_1+\cdots+m_k}{m_1,\ldots,m_k}$$

Letting s now denote $r+1-l$,

$$= \sum_{1\le l\le k} \sum_{s=1-l}^{n+1-l} \sum_{i=0}^{k-1} \sum_{\substack{m_1,\ldots,m_k \ni \\ m_1+2m_2+\cdots+km_k=n-l-i \\ m_1+\cdots+m_k=n-l-s}} \binom{m_1+\cdots+m_k}{m_1,\ldots,m_k}$$

The terms with $s < 0$ and $s = n+1-l$ contribute zero to the sum, so we may eliminate them to obtain

$$= \sum_{1\le l\le k} \left[\sum_{s=0}^{n-l} \sum_{i=0}^{k-1} \sum_{\substack{m_1,\ldots,m_k \ni \\ m_1+2m_2+\cdots+km_k=n-l-i \\ m_1+\cdots+m_k=n-l-s}} \binom{m_1+\cdots+m_k}{m_1,\ldots,m_k} \right]$$

$$= \sum_{1\le l\le k} f(n-l).$$

Thus $f(n)$ and $F_{n+2}^{(k)}$ obey the same kth order recursion relation, and agree for $0 \le n \le k-1$. Thus $f(n) = F_{n+2}^{(k)}$ for all $k \ge 2$ and $n \ge 0$.

Also solved by P. Bruckman, C. Georghiou, and the proposers.

General Ize

H-405 *Proposed by Piero Filipponi, Rome, Italy*
(Vol. 24, no. 4, November 1986)

(i) Generalize Problem B-564 by finding a closed form expression for

$$\sum_{n=1}^{N} [\alpha^k F_n], \quad (N = 1, 2, \ldots; \ k = 1, 2, \ldots)$$

where $\alpha = (1+\sqrt{5})/2$, F_n is the n^{th} Fibonacci number, and $[x]$ denotes the greatest integer not exceeding x.

(ii) Generalize the above sum to negative values of k.

(iii) Can this sum be further generalized to any rational value of the exponent of α?

Remark: As to (iii), it can be proved that

$$[\alpha^{1/k} F_n] = F_n, \text{ if } 1 \leqslant n \leqslant [(\ln\sqrt{5} - \ln(\alpha^{1/k} - 1))/\ln\alpha].$$

References

1. V. E. Hoggatt, Jr., & M. Bicknell-Johnson, "Representation of Integers in Terms of Greatest Integer Functions and the Golden Section Ratio," *The Fibonacci Quarterly* 17, no. 4 (1979):306-318.
2. V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers* (Boston: Houghton Mifflin Company, 1969).

Partial solution by the proposer

First, recall that [1, Lemma 2]

$$[\alpha^k F_n] = \begin{cases} F_{n+k} & (n \text{ odd}) \\ F_{n+k} - 1 & (n \text{ even}). \end{cases} \quad (k \geqq 2;\ n \geqq k) \tag{1}$$

It can be noted that, since the relationship [1, Lemma 1]

$$[\alpha F_n] = \begin{cases} F_{n+1} & (n \text{ odd}) \\ F_{n+1} - 1 & (n \text{ even}) \end{cases} \quad (n \geqq 2) \tag{2}$$

clearly holds also for $n = 1$, (a) holds for $k = 1$ as well.

Then, we find an expression for $[\alpha^k F_n]$ in the case of $1 \leqq n \leqq k - 1$. Using the Binet form, the equality

$$\alpha^k F_n = F_{k+n} - \beta^n F_k \tag{3}$$

can be proved [1, Lemma 3]. Again, using the Binet form, we obtain

$$\beta^n F_k = \frac{\beta^n(\alpha^k - \beta^k)}{\sqrt{5}} = \frac{(-1)^n \alpha^{k-n} - \beta^{k+n}}{\sqrt{5}} + \frac{(-1)^n(\beta^{k-n} - \beta^{k-n})}{\sqrt{5}}$$

$$= (-1)^n F_{k-n} + \frac{(-1)^n \beta^{k-n} - \beta^{k+n}}{\sqrt{5}} = (-1)^n F_{k-n} + \alpha.$$

Since it is readily seen that

$$\begin{cases} 0 < x < 1 & (k \text{ even}) \\ -1 < x < 0 & (k \text{ odd}), \end{cases} \quad (1 \leqq n \leqq k - 1) \tag{4}$$

from (3) and (4), we can write

$$[\alpha^k F_n] = \begin{cases} F_{k+n} - F_{k-n} - 1 & (n \text{ even}, k \text{ even}) \\ F_{k+n} - F_{k-n} & (n \text{ even}, k \text{ odd}) \\ F_{k+n} + F_{k-n} - 1 & (n \text{ odd}, k \text{ even}) \\ F_{k+n} + F_{k-n} & (n \text{ odd}, k \text{ odd}) \end{cases} \quad (1 \leqq n \leqq k - 1)$$

from which, by Hoggatt's I_{24} and I_{22} [2], we get

$$[\alpha^k F_n] = \begin{cases} L_k F_n & (k \text{ odd}) \\ L_k F_n - 1 & (k \text{ even}). \end{cases} \quad (1 \leqq n \leqq k - 1) \tag{5}$$

Now, let us distinguish the following two cases.

Case 1: $k > N$

From (5) and Hoggatt's I_1 [2], we have

$$\sum_{n=1}^{N}[\alpha^k F_n] = \begin{cases} L_k \sum_{n=1}^{N} F_n = L_k(F_{N+2} - 1) & (k \text{ odd}) \\ L_k \sum_{n=1}^{N} F_n - N = L_k(F_{N+2} - 1) - N & (k \text{ even}) \end{cases}$$

which can be rewritten in the following more compact form:

$$\sum_{n=1}^{N}[\alpha^k F_n] = L_k(F_{N+2} - 1) - N\frac{(-1)^k + 1}{2} \quad (\text{if } k > N). \tag{6}$$

Case 2: $1 \leqq k \leqq N$

From (6), we can write

$$\begin{aligned} \sum_{n=1}^{N}[\alpha^k F_n] &= \sum_{n=1}^{k-1}[\alpha^k F_n] + \sum_{n=k}^{N}[\alpha^k F_n] \\ &= L_k(F_{k+1} - 1) - (k - 1)\frac{(-1)^k + 1}{2} + \sum_{n=k}^{N}[\alpha^k F_n]. \end{aligned} \tag{7}$$

From (1) we have

$$\sum_{n=k}^{N}[\alpha^k F_n] = \sum_{n=1}^{N-k+1} F_{2k+n-1} - \begin{cases} \left[\dfrac{N - k + 1}{2}\right] & (k \text{ odd}) \\ \left[\dfrac{N - k + 2}{2}\right] & (k \text{ even}), \end{cases}$$

which, by Hoggatt's I_1 [2] can be rewritten as (cf. Prob. B-564, for $k = 1$)

$$\sum_{n=k}^{N}[\alpha^k F_n] = F_{N+k+2} - F_{2k+1} - \left[\frac{2N - 2k + 3 + (-1)^k}{4}\right] \tag{8}$$

Combining (7) and (8), we obtain

$$\begin{aligned} \sum_{n=1}^{N}[\alpha^k F_n] = {} & L_k(F_{k+1} - 1) + F_{N+k+2} - F_{2k+1} \\ & - (k - 1)\frac{(-1)^k + 1}{2} - \left[\frac{2N - 2k + 3 + (-1)^k}{4}\right], \end{aligned}$$

that is,

$$\begin{aligned} \sum_{n=1}^{N}[\alpha^k F_n] = {} & L_k(F_{k+1} - 1) + F_{N+k+2} - F_{2k+1} \\ (\text{if } 1 \leqq k \leqq N) \quad & - \frac{2N + (2k - 1)(-1)^k + (-1)^N}{4}. \end{aligned} \tag{9}$$

The problem can be further generalized to negative values of the exponent k. The proof can be obtained by reasoning similar to the preceding and is omitted for the sake of brevity. Se we offer the following

Conjecture: For N and k positive integers,

$$\sum_{n=1}^{N}[\alpha^{-k}F_n] = \begin{cases} F_{N-k+2} - \left[\dfrac{N-k+3}{2}\right], & \text{if } N > k+1 \\ 0, & \text{if } N \leqq k+1. \end{cases}$$

Also partially solved by P. Bruckman.

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BOOK REVIEW

by A.F. Horadam, University of New England, Armidale, Australia 2351

Leonardo Pisano (Fibonacci)—The Book of Squares

(an annotated translation into modern English)—L.E. Sigler, Academic Press 1987.

This is the first complete translation into English of Fibonacci's masterpiece, *Liber quadratorum* ("The Book of Squares"), which was written in 1225. Until the nineteenth century when he acquired the nickname Fibonacci, the author, who was born in Pisa and christened Leonardo, was universally known as Leonardo Pisano. He is better-known for his *Liber abbaci* in which the Fibonacci numbers first appear.

The volume under review consists of three main parts, namely; a short biographical sketch of Fibonacci, an English translation of *Liber quadratorum,* and a commentary on this translation ("The Book of Squares"). The Latin text followed by Sigler is that used by Boncompagni who found the MS in the Ambrosian Library in Milan when preparing the first printed edition of Fibonacci's writings in 1857-62.

Sigler's commentary is particularly useful as it provides in detail an explanation of Fibonacci's text in modern mathematical notation and terminology. Fibonacci had no algebraic symbolism to help him. Following Euclid, he represented numbers geometrically as line-segments. It is truly remarkable how far he could progress with this limited mathematical equipment. His achievements in this book justly confirm him as the greatest exponent of number theory, particularly in indeterminate analysis, in the Middle Ages.

A representative, and famous, problem posed and solved in the text is: Find a square number from which, when 5 is added or subtracted, there always arises a square number.

According to the translator, "a knowledge of secondary school mathematics, algebra and geometry ought to be adequate preparation for the reading and understanding of this book."

We are indebted to Sigler for making this English translation available. For many, it could open up a new world of delight.