RECURRENCE RELATIONS FOR A POWER SERIES

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For integers $n \geq 2$, $r \geq 0$, let

$$S_n(r) = \sum_{k=1}^{\infty} k^r n^{-k}.$$

It was proved in [1] that, for all $r \geq 1$,

$$S_n(r) = \frac{n}{n-1}\left[\binom{r}{1}S_n(r-1) - \binom{r}{2}S_n(r-2) + \cdots + (-1)^{r+1}\binom{r}{r}S_n(0)\right].$$

The purpose of this paper is to extend this result for arithmetic progressions and also to obtain a related formula with no alternate signs.

Let a and q be real numbers, and let $(a_k)_{k\geq 0}$ be the arithmetic progression

$$(a + kq)_{k\geq 0}.$$

If $|x| < 1$ and $r \in \{0, 1, \ldots\}$, we define

$$S_r(x) = \sum_{k=1}^{\infty} a_k^r x^k. \tag{1}$$

In this note we establish two recurrence relations for the series (1). Namely:

$$S_r(x) = \frac{x}{1-x}\left[(a+q)^r + \binom{r}{1}qS_{r-1}(x) + \binom{r}{2}q^2S_{r-2}(x) + \cdots + q^rS_0(x)\right] \tag{2}$$

and

$$S_r(x) = \frac{1}{1-x}\left[a^rx + \binom{r}{1}qS_{r-1}(x) - \binom{r}{2}q^2S_{r-2}(x) + \cdots + (-1)^{r+1}q^rS_0(x)\right]. \tag{3}$$

Let us denote by $S_r(x, m)$ the m-adic partial sum, i.e.,

$$S_r(x, m) = \sum_{k=1}^{m} a_k^r x^k.$$

Proof of (2): We first deduce a functional equation for $S_r(x, m)$.

$$\begin{aligned}
S_r(x, m+p) &= \sum_{k=1}^{m+p} a_k^r x^k = \sum_{k=1}^{m} a_k^r x^k + \sum_{k=m+1}^{m+p} a_k^r x^k \\
&= S_r(x, m) + x^m \sum_{i=1}^{p} a_{m+i}^r x^i \\
&= S_r(x, m) + x^m \sum_{i=1}^{p} (mq + a_i)^r x^i \\
&= S_r(x, m) + x^m \sum_{i=1}^{p} \sum_{j=0}^{r} \binom{r}{j}(mq)^j a_i^{r-j} x^i \\
&= S_r(x, m) + x^m \sum_{j=0}^{r} \binom{r}{j}(mq)^j \sum_{i=1}^{p} a_i^{r-j} x^i
\end{aligned}$$

$$= S_r(x, m) + x^m \sum_{j=0}^{r} \binom{r}{j} (mq)^j S_{r-j}(x, p).$$

For $m = 1$, we obtain

$$S_r(x, p + 1) = (a + q)^r x + x \sum_{j=0}^{r} \binom{r}{j} q^j S_{r-j}(x, p)$$

$$= (a + q)^r x + x S_r(x, p) + x \sum_{j=1}^{r} \binom{r}{j} q^j S_{r-j}(x, p).$$

Now, if $p \to \infty$, we have

$$S_r(x) = (a + q)^r x + x S_r(x) + x \sum_{j=1}^{r} \binom{r}{j} q^j S_{r-j}(x),$$

or

$$S_r(x) = \frac{x}{1 - x}\left[(a + q)^r + \binom{r}{1} q S_{r-1}(x) + \binom{r}{2} q^2 S_{r-2}(x) + \cdots + q^r S_0(x)\right],$$

which was to be proved.

Proof of (3): We proceed as follows:

$$S_r(x, m) = \sum_{k=1}^{m} a_k^r x^k = \sum_{k=1}^{m} x^k \sum_{i=1}^{k} (a_i^r - a_{i-1}^r) + \sum_{k=1}^{m} a^r x^k$$

$$= \sum_{k=1}^{m} x^k \sum_{i=1}^{k} [a_i^r - (a_i - q)^r] + \sum_{k=1}^{m} a^r x^k$$

$$= \sum_{k=1}^{m} x^k \sum_{i=1}^{k} \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} a_i^{r-j} q^j + a^r \sum_{k=1}^{m} x^k$$

$$= \sum_{k=1}^{m} \sum_{i=1}^{k} \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} a_i^{r-j} q^j x^k + a^r \sum_{k=1}^{m} x^k$$

$$= \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} q^j \sum_{i=1}^{m} \sum_{k=1}^{m} a_i^{r-j} x^k + a^r \sum_{k=1}^{m} x^k$$

$$= \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} q^j \sum_{i=1}^{m} a_i^{r-j} \sum_{k=1}^{m} x^k + a^r \sum_{k=1}^{m} x^k$$

$$= \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} q^j \sum_{i=1}^{m} \frac{1}{1 - x} (a_i^{r-j} x^i - a_i^{r-j} x^{m+1}) + a^r \sum_{k=1}^{m} x^k$$

$$= \frac{1}{1 - x} \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} q^j [S_{r-j}(x, m) - x^{m+1} s_{r-j}(m)] + a^r \sum_{k=1}^{m} x^k,$$

where

$$s_{r-j}(m) = \sum_{i=1}^{m} a_i^{r-j}.$$

So $S_r(x, m)$ can be written as

$$S_r(x, m) = \frac{1}{1 - x} \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} q^j S_{r-j}(x, m) - T(x, m) + a^r \sum_{k=1}^{m} x^k, \tag{4}$$

with

with

$$T(x,\ m) = \frac{x^{m+1}}{1 - x}\sum_{j=1}^{r}(-1)^{j+1}\binom{r}{j}q^j s_{r-j}(m).$$

We will show that

$$T(x,\ m) \to 0 \text{ as } m \to \infty. \tag{5}$$

We have

$$a^r = (a_1 - q)^r = a_1^r - \binom{r}{1}a_1^{r-1}q + \cdots + (-1)^r q^r$$

$$a_1^r = (a_2 - q)^r = a_2^r - \binom{r}{1}a_2^{r-1}q + \cdots + (-1)^r q^r$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m-1}^r = (a_m - q)^r = a_m^r - \binom{r}{1}a_m^{r-1}q + \cdots + (-1)^r q^r.$$

So that

$$a^r = a_m^r - \binom{r}{1}qs_{r-1}(m) + \cdots + (-1)^r q^r s_0(m)$$

or

$$\sum_{j=1}^{r}(-1)^{j+1}\binom{r}{j}q^j s_{r-j}(m) = a_m^r - a^r$$

and (5) is now clear.

Let $m \to \infty$ in (4). It then follows that

$$S_r(x) = \frac{1}{1 - x}\left[a^r x + \binom{r}{1}qS_{r-1}(x) - \binom{r}{2}q^2 S_{r-2}(x) + \cdots + (-1)^{r+1}q^r S_0(x)\right],$$

is exactly (3).

Remark: Of course, one can consider

$$\overline{S}_r(x) = \sum_{k=0}^{\infty} a_k^r x^k \quad (|x| < 1,\ r \geq 0),$$

and obtain

$$\overline{S}_r(x) = \frac{x}{1 - x}\left[a^r + \binom{r}{1}q\overline{S}_{r-1}(x) + \binom{r}{2}q^2\overline{S}_{r-2}(x) + \cdots + q^r\overline{S}_0(x)\right] + a^r, \tag{2'}$$

and

$$\overline{S}_r(x) = \frac{1}{1 - x}\left[(a - q)^r + \binom{r}{1}q\overline{S}_{r-1}(x) - \binom{r}{2}q^2\overline{S}_{r-2}(x) + \cdots + (-1)^{r+1}q^r\overline{S}_0(x)\right], \tag{3'}$$

respectively.

Reference

1. L. Cseh & I. Merényi. Problem E 3153. *Amer. Math. Monthly* 93 (1986).

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