ON TRIANGULAR FIBONACCI NUMBERS

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1. Introduction and Results

Vern Hoggatt (see [1]) conjectured that 1, 3, 21, 55 are the only triangular numbers [i.e., positive integers of the form $\frac{1}{2}m(m + 1)$] in the Fibonacci sequence

$$u_{n+2} = u_{n+1} + u_n, \ u_0 = 0, \ u_1 = 1,$$

where n ranges over all integers, positive or negative. In this paper, we solve Hoggatt's problem completely and obtain the following results.

Theorem 1: $8u_n + 1$ is a perfect square if and only if $n = \pm 1, 0, 2, 4, 8, 10$.

Theorem 2: The Fibonacci number u_n is triangular if and only if $n = \pm 1, 2, 4, 8, 10$.

The latter theorem verifies the conjecture of Hoggatt.

The method of the proofs is as follows. Since u_n is a triangular number if and only if $8u_n + 1$ is a perfect square greater than 1, it is sufficient to find all n's such that $8u_n + 1$ is square. To do this, we shall find, for each nonsquare $8u_n + 1$, an integer w_n such that the Jacobi symbol

$$\left(\frac{8u_n + 1}{w_n}\right) = -1.$$

Using elementary congruences we can show that, if $8u_n + 1$ is square, then

$$n \equiv \pm 1 \pmod{2^5 \cdot 5} \text{ if } n \text{ is odd, and}$$

$$n \equiv 0, 2, 4, 8, 10 \pmod{2^5 \cdot 5^2 \cdot 11} \text{ if } n \text{ is even.}$$

We develop a special Jacobi symbol criterion with which we can further show that each congruence class above contains exactly one value of n such that $8u_n + 1$ is a perfect square, i.e., $n = \pm 1, 0, 2, 4, 8, 10$, respectively.

2. Preliminaries

It is well known that the Lucas sequence

$$v_{n+2} = v_{n+1} + v_n, \ v_0 = 2, \ v_1 = 1,$$

where n denotes an integer, is closely related to the Fibonacci sequence, and that the following formulas hold (see [2]):

$$u_{-n} = (-1)^{n+1}u_n, \ v_{-n} = (-1)^n v_n; \tag{1}$$

$$2u_{m+n} = u_m v_n + u_n v_m, \ 2v_{m+n} = 5u_m u_n + v_m v_n; \tag{2}$$

$$u_{2n} = u_n v_n, \ v_{2n} = v_n^2 + 2(-1)^{n+1}; \tag{3}$$

$$v_n^2 - 5u_n^2 = 4(-1)^n; \tag{4}$$

$$u_{2kt+n} \equiv (-1)^t u_n \pmod{v_k}; \tag{5}$$

where n, m, t denote integers and $k \equiv \pm 2 \pmod 6$.

Moreover, since $x = \pm u_n$, $y = \pm v_n$ are the complete set of solutions of the Diophantine equations $5x^2 - y^2 = \pm 4$, the condition $u_n = \frac{1}{2}m(m+1)$ is equivalent to finding all integer solutions of the two Diophantine equations

$$5m^2(m+1)^2 - 4y^2 = \pm 16,$$

i.e., finding all integer points on these two elliptic curves. These problems are also solved in this paper.

3. A Jacobi Symbol Criterion and Its Consequences

In the first place we establish a Jacobi symbol criterion that plays a key role in this paper and then give some of its consequences.

Criterion: If a, n are positive integers such that $n \equiv \pm 2 \pmod 6$, $(a, v_n) = 1$, then

$$\left(\frac{\pm 4au_{2n} + 1}{v_{2n}}\right) = -\left(\frac{8au_n \pm v_n}{64a^2 + 5}\right)$$

whenever the right Jacobi symbol is proper.

Proof: Since $n \equiv \pm 2 \pmod 6$ implies $v_n \equiv 3 \pmod 4$ and $2n \equiv \pm 4 \pmod{12}$ implies $v_{2n} \equiv 7 \pmod 8$, we have

$$\left(\frac{\pm 4au_{2n} + 1}{v_{2n}}\right) = \left(\frac{\pm 8au_{2n} + 2}{v_{2n}}\right) = \left(\frac{\pm 8au_n v_n + v_n^2}{v_{2n}}\right) \quad \text{by (3)}$$

$$= \left(\frac{v_{2n}}{8au_n v_n \pm v_n^2}\right) \quad \text{since } a,\ n > 0 \text{ imply } 8au_n \pm v_n > 0$$

$$= \left(\frac{v_{2n}}{v_n}\right)\left(\frac{v_{2n}}{8au_n \pm v_n}\right) = \left(\frac{-2}{v_n}\right)\left(\frac{\frac{1}{2}(5u_n^2 + v_n^2)}{8au_n \pm v_n}\right) \quad \text{by (2)}$$

$$= -\left(\frac{2}{v_n}\right)\left(\frac{a}{8au_n \pm v_n}\right)\left(\frac{40au_n^2 + 8av_n^2}{8au_n \pm v_n}\right) = -\left(\frac{2}{v_n}\right)\left(\frac{a}{8au_n \pm v_n}\right)\left(\frac{\lambda(64a^2 + 5)u_n v_n}{8au_n \pm v_n}\right)$$

$$= \pm\left(\frac{2}{v_n}\right)\left(\frac{a}{8au_n \pm v_n}\right)\left(\frac{8au_n \pm v_n}{64a^2 + 5}\right)\left(\frac{u_n v_n}{8au_n \pm v_n}\right).$$

If $u_n \equiv 1 \pmod 4$, then

$$\left(\frac{u_n}{8au_n \pm v_n}\right) = \left(\frac{8au_n \pm v_n}{u_n}\right) = \left(\frac{v_n}{u_n}\right) = \left(\frac{u_n}{v_n}\right);$$

If $u_n \equiv 3 \pmod 4$, then

$$\left(\frac{u_n}{8au_n \pm v_n}\right) = \mp\left(\frac{8au_n \pm v_n}{u_n}\right) = -\left(\frac{v_n}{u_n}\right) = \left(\frac{u_n}{v_n}\right).$$

Hence, we always have $\left(\dfrac{u_n}{8au_n \pm v_n}\right) = \left(\dfrac{u_n}{v_n}\right)$.

Since $\left(\dfrac{v_n}{8au_n \pm v_n}\right) = \mp\left(\dfrac{8au_n \pm v_n}{v_n}\right) = \lambda\left(\dfrac{2a}{v_n}\right)\left(\dfrac{u_n}{v_n}\right)$, we get

$$\left(\frac{\pm 4au_{2n}+1}{v_{2n}}\right) = -\left(\frac{a}{v_n}\right)\left(\frac{a}{8au_n \pm v_n}\right)\left(\frac{8au_n \pm v_n}{64a^2+5}\right) = -\left(\frac{a}{8au_{2n} \pm v_n^2}\right)\left(\frac{8au_n \pm v_n}{64a^2+5}\right).$$

Moreover, put $a = 2^s b$, $s \geq 0$, $2 \nmid b$. If $b \equiv 1 \pmod 4$, then

$$\left(\frac{a}{8au_{2n} \pm v_n^2}\right) = \left(\frac{b}{8au_{2n} \pm v_n^2}\right) = \left(\frac{8au_{2n} \pm v_n^2}{b}\right) = 1;$$

If $b \equiv 3 \pmod 4$, then

$$\left(\frac{a}{8au_{2n} \pm v_n^2}\right) = \left(\frac{b}{8au_{2n} \pm v_n^2}\right) = \pm\left(\frac{8au_{2n} \pm v_n^2}{b}\right) = 1,$$

the same as above, so we finally obtain

$$\left(\frac{\pm 4au_{2n}+1}{v_{2n}}\right) = -\left(\frac{8au_n \pm v_n}{64a^2+5}\right).$$

The proof is complete. □

Now we derive some consequences of this criterion.

Lemma 1: If $n \equiv \pm 1 \pmod{2^5 \cdot 5}$, then $8u_n + 1$ is a square only for $n = \pm 1$.

Proof: We first consider the case $n \equiv 1 \pmod{2^5 \cdot 5}$. If $n \neq 1$, put

$$n = \delta(n-1) \cdot 3^r \cdot 2 \cdot 5m + 1,$$

where $\delta(n-1)$ denotes the sign of $n-1$, and $r \geq 0$, $3 \nmid m$, then $m > 0$ and $m \equiv \pm 16 \pmod{48}$. We shall carry out the proof in two cases depending on the congruence class of $\delta(n-1) \cdot 3^r \pmod 4$.

Case 1: $\delta(n-1) \cdot 3^r \equiv 1 \pmod 4$. Let $k = 5m$ if $m \equiv 16 \pmod{48}$ or $k = m$ if $m \equiv 32 \pmod{48}$, then we always have $k \equiv 32 \pmod{48}$. Using (5) and (2), we obtain

$$8u_n + 1 \equiv 8u_{2k+1} + 1 \equiv 4(u_{2k} + v_{2k}) + 1 \equiv 4u_{2k} + 1 \pmod{v_{2k}}.$$

Using the Criterion, we get (evidently the conditions are satisfied)

$$\left(\frac{8u_n+1}{v_{2k}}\right) = \left(\frac{4u_{2k}+1}{v_{2k}}\right) = -\left(\frac{8u_k+v_k}{69}\right)$$

Take modulo 69 to $\{8u_n + v_n\}$, the sequence of the residues has period 48, and $k \equiv 32 \pmod{48}$ implies $8u_k + v_k \equiv 38 \pmod{69}$, then we get

$$\left(\frac{8u_n+1}{v_k}\right) = -\left(\frac{38}{69}\right) = -1$$

so that $8u_n + 1$ is not a square in this case.

Case 2: $\delta(n-1) \cdot 3^r \equiv 3 \pmod 4$. In this case, let $k = m$ if $m \equiv 16 \pmod{48}$ or $k = 5m$ if $m \equiv 32 \pmod{48}$ so that $k \equiv 16 \pmod{48}$ always. Similarly, by (5), (2), and the Criterion, we have

$$\left(\frac{8u_n+1}{v_{2k}}\right) = \left(\frac{-4u_{2k}+1}{v_{2k}}\right) = -\left(\frac{8u_k - v_k}{69}\right).$$

Since the sequence of residues of $\{8u_n - v_n\}$ (mod 69) has period 48 and $k \equiv 16 \pmod{48}$ implies $8u_k - v_k \equiv 31 \pmod{69}$, we get

$$\left(\frac{8u_n + 1}{v_{2k}}\right) = -\left(\frac{31}{69}\right) = -1.$$

Hence $8u_n + 1$ is also not a square in this case.

Secondly, if $n \equiv -1 \pmod{2^5 \cdot 5}$ and $n \neq -1$, by (1) we can write

$$8u_n + 1 = 8u_{-n} + 1.$$

Since $-n \equiv 1 \pmod{2^5 \cdot 5}$ and $-n \neq 1$, it cannot possibly be a square according to the argument above.

Finally, when $n = \pm 1$, both give $8u_n + 1 = 3^2$, which completes the proof. □

In the remainder of this section we suppose that n is even. Note that if n is negative and even, then $8u_n + 1$ is negative, so it cannot be a square; hence, we may assume that $n \geq 0$.

Lemma 2: If $n \equiv 0 \pmod{2^2 \cdot 5^2}$, then $8u_n + 1$ is a square only for $n = 0$.

Proof: If $n > 0$, put $n = 2 \cdot 5^2 \cdot 2^s \cdot \ell$, $2 \nmid \ell$, $s \geq 1$, and let

$$k = \begin{cases} 2^s & \text{if } s \equiv 0 \pmod{3}, \\ 5^2 \cdot 2^s & \text{if } s \equiv 1 \pmod{3}, \\ 5 \cdot 2^s & \text{if } s \equiv 2 \pmod{3}, \end{cases}$$

then $k \equiv \pm 6 \pmod{14}$. Since $(2, v_k) = 1$, $k \equiv \pm 2 \pmod 6$, by (5) and the Criterion we get

$$\left(\frac{8u_n + 1}{v_{2k}}\right) = \left(\frac{\pm 8u_{2k} + 1}{v_{2k}}\right) = -\left(\frac{16u_k \pm v_k}{9 \cdot 29}\right) = -\left(\frac{16u_k \pm v_k}{29}\right).$$

[It is easy to check that $(16u_n \pm v_n, 3) = 1$ for any even n.]

Simple calculations show that both of the residue sequences $\{16u_n \pm v_n\}$ modulo 29 have period 14. If $k \equiv 6 \pmod{14}$, then

$$16u_k + v_k \equiv 1 \pmod{29},\ 16u_k - v_k \equiv -6 \pmod{29};$$

if $k \equiv -6 \pmod{14}$, then

$$16u_k + v_k \equiv 6 \pmod{29},\ 16u_k - v_k \equiv -1 \pmod{29}.$$

Since $(\pm 1/29) = (\pm 6/29) = 1$, we obtain

$$\left(\frac{8u_n + 1}{v_{2k}}\right) = -1,$$

so that $8u_n + 1$ is not a square.

The case $n = 0$ gives $8u_n + 1 = 1^2$, which completes the proof. □

Lemma 3: If $n \equiv 2 \pmod{2^5 \cdot 5^2}$, then $8u + 1$ is a square only for $n = 2$.

Proof: If $n > 2$, put $n = 3^r \cdot 2 \cdot 5^2 \cdot \ell + 2$, $3 \nmid \ell$, $\ell > 0$, then $\ell \equiv \pm 16 \pmod{48}$. Let $k = \ell$ or 5ℓ or $5^2\ell$, which will be determined later. Since $4 \mid k$ implies $(3, v_k) = 1$, and clearly $k \equiv \pm 2 \pmod 6$, we obtain, using (5), (2), and the Criterion

$$\left(\frac{8u_n + 1}{v_{2k}}\right) = \left(\frac{\pm 8u_{2k+2} + 1}{v_{2k}}\right) = \left(\frac{\pm 12u_{2k} + 1}{v_{2k}}\right) = -\left(\frac{24u_k \pm v_k}{581}\right).$$

Taking $\{24u_n \pm v_n\}$ modulo 581, we obtain two residue sequences with the same period 336 and having the following table:

n	(mod 336)	80	112	128	208	224	256
$24u_n + v_n$	(mod 581)	65	401	436	359	261	170
$24u_n - v_n$	(mod 581)	411	320	222	145	180	516

It is easy to check that

$$\left(\frac{24u_n \pm v_n}{581}\right) = 1$$

for all six of these residue classes n (mod 336).

Since $336 = 48 \cdot 7$, we see that $\ell \equiv \pm 16 \pmod{48}$ are equivalent to $\ell \equiv 16$, 32, 64, 80, 112, 128, 160, 176, 208, 224, 256, 272, 304, 320 (mod 336). We choose k as follows:

$$k = \begin{cases} \ell & \text{if } \ell \equiv 80, 112, 128, 208, 224, 256 \pmod{336} \\ 5\ell & \text{if } \ell \equiv 16, 160, 176, 320 \pmod{336} \\ 5^2\ell & \text{if } \ell \equiv 32, 64, 272, 304 \pmod{336}. \end{cases}$$

With this choice k must be congruent to one of 80, 112, 128, 208, 224, and 256 modulo 336. Thus, we get

$$\left(\frac{8u_n + 1}{v_{2k}}\right) = -\left(\frac{24u_k \pm v_k}{581}\right) = -1,$$

so that $8u_n + 1$ is not a square.

Finally, the case $n = 2$ gives $8u_n + 1 = 3^2$. The proof is complete. □

Lemma 4: If $n \equiv 4 \pmod{2^5}$, then $8u_n + 1$ is a square only for $n = 4$.

Proof: If $n > 4$, we put $n = 2 \cdot 3^r \cdot k + 4$, $3 \nmid k$, then $k \equiv \pm 16 \pmod{48}$. According to (5), we have

$$8u_n + 1 \equiv -8u_4 + 1 \equiv -23 \pmod{v_k}.$$

Simple calculations show that the sequence of residues $\{v_k\}$ modulo 23 has period 48 and that $k \equiv \pm 16 \pmod{48}$ implies that $v_k \equiv -1 \pmod{23}$. Hence,

$$\left(\frac{8u_n + 1}{v_k}\right) = \left(\frac{-23}{v_k}\right) = \left(\frac{v_k}{23}\right) = \left(\frac{-1}{23}\right) = -1,$$

so that $8u_n + 1$ is not a square in this case.

When $n = 4$, $8u_n + 1 = 5^2$. The proof is complete. □

Lemma 5: If $n \equiv 8 \pmod{2^5 \cdot 5}$, then $8u_n + 1$ is a square only for $n = 8$.

Proof: If $n > 8$, we put $n = 2 \cdot 3^r \cdot 5\ell + 8$, $3 \nmid \ell$, then $\ell \equiv \pm 16 \pmod{48}$. Let $k = \ell$ or 5ℓ, which will be determined later. For both cases, we have, by (5),

$$8u_n + 1 \equiv -8u_8 + 1 \equiv -167 \pmod{v_k}.$$

The sequence $\{v_n\}$ modulo 167 is periodic with period 336, and the following table holds.

n (mod 336)	±32	±64	±80	±112	±160
v_n (mod 167)	125	91	17	166	120

It is easy to verify that all values in the second row are quadratic nonresidues modulo 167. Let A denote the set consisting of the residue classes in the first row. We now choose k such that its residue modulo 336 is in A.

The condition $\ell \equiv \pm 16 \pmod{48}$ is equivalent to $\ell \equiv 16$, 32, 64, 80, 112, 128, 160, 176, 208, 224, 256, 272, 304, 320 (mod 336), and all of these residue classes, except four classes, are in A. For these classes, we let $k = \ell$. The four exceptions are $\ell \equiv 16, 128, 208, 320 \pmod{336}$, for which we choose $k = 5\ell$ so that $k \equiv 80, -32, 32, -80 \pmod{336}$, respectively, which are also in A. Thus, for every choice of k, v_k is a quadratic nonresidue modulo 167. Hence,

$$\left(\frac{8u_n + 1}{v_k}\right) = \left(\frac{-167}{v_k}\right) = \left(\frac{v_k}{167}\right) = -1,$$

and $8u_n + 1$ is not a square.

Finally, for $n = 8$, $8u_n + 1 = 13^2$, which completes the proof. □

Lemma 6: If $n \equiv 10 \pmod{2^2 \cdot 5 \cdot 11}$, then $8u_n + 1$ is a square only for $n = 10$.

Proof: In the first place, by taking $\{v_n\}$ modulo 439 we get a sequence of residues with period 438 and having the following table:

n (mod 438)	2	8	16	44	56	64	94	178	230	256	296	302	332	356	376
v_n (mod 439)	3	47	12	306	54	407	395	24	79	101	394	202	184	135	74

Let B denote the set consisting of all fifteen residue classes modulo 438 in the first row. Simple calculations show that, for each n in B, v_n is a quadratic nonresidue modulo 439.

Now suppose that $8u_n + 1$ is a square. If $n > 10$, put $n = 2 \cdot \ell \cdot 5 \cdot 11 \cdot 2^t + 10$, $2 \nmid \ell$, $t \geq 1$. The sequence $\{2^t\}$ modulo 438 is periodic with period 18 with respect to t and we obtain the following table:

t (mod 18)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2^t (mod 438)	$\underline{2}$	4	$\underline{8}$	$\underline{16}$	32	$\underline{64}$	128	$\underline{256}$	74	148	$\underline{296}$	154	308	$\underline{178}$	$\underline{356}$	274	110	220
$5 \cdot 2^t$ (mod 438)										$\underline{302}$		$\underline{332}$				$\underline{56}$		
$11 \cdot 2^t$ (mod 438)		$\underline{44}$					$\underline{94}$		$\underline{376}$									$\underline{230}$
$5 \cdot 11 \cdot 2^t$ (mod 438)					$\underline{8}$								$\underline{296}$				$\underline{356}$	

where the underlined residue classes modulo 438 are in B. If we take k as follows:

$$k = \begin{cases} 2^t & \text{if } t \equiv 1, 3, 4, 6, 8, 11, 14, 15 \pmod{18} \\ 5 \cdot 2^t & \text{if } t \equiv 10, 12, 16 \pmod{18} \\ 11 \cdot 2^t & \text{if } t \equiv 0, 2, 7, 9 \pmod{18} \\ 5 \cdot 11 \cdot 2^t & \text{if } t \equiv 5, 13, 17 \pmod{18}, \end{cases}$$

then the residue of k modulo 438 is in B, that is, v_k is a quadratic nonresidue modulo 439. Thus, by (5), we get

$$8u_n + 1 \equiv -8u_{10} + 1 \equiv -439 \pmod{v_k},$$

and

$$\left(\frac{8u_n + 1}{v_k}\right) = \left(\frac{-439}{v_k}\right) = \left(\frac{v_k}{439}\right) = -1,$$

so $8u_n + 1$ is not a square. In the remaining case $n = 10$, we have $8u_n + 1 = 21^2$. The proof is complete. □

Lemmas 2 to 6 immediately imply the following result:

Corollary 1: Assume that $n \equiv 0, 2, 4, 8, 10 \pmod{2^5 \cdot 5^2 \cdot 11}$, then $8u_n + 1$ is a square only for $n = 0, 2, 4, 8, 10$. □

4. Some Lemmas Obtained by Congruent Calculations

The lemmas in this section provide a system of necessary conditions for $8u_n + 1$ to be a square. We prove them mainly by the following process of calculation: First we study $\{8u_n + 1\}$ modulo a_1. We get a sequence with period k_1 (with respect to n), in which we eliminate every residue class modulo k_1 of n for which $8u_n + 1$ is a quadratic nonresidue modulo a_1. Next we study $\{8u_n + 1\}$ modulo a_2, and get a sequence with period k_2. For our purpose, a_2 will be chosen in such a way so that $k_1 | k_2$. Then we eliminate every residue class modulo k_2 of n from those left in the preceding step, for which $8u_n + 1$ is a quadratic nonresidue modulo a_2. We repeat this procedure until we reach the desired results.

Remark: Most of the a_i will be chosen to be prime and the calculations may then be carried out directly from the recurrence relation

$$8u_{n+2} + 1 = (8u_{n+1} + 1) + (8u_n + 1) - 1.$$

Lemma 7: If $8u_n + 1$ is a square, then $n \equiv \pm 1, 0, 2, 4, 8, 10 \pmod{2^5 \cdot 5}$.

Proof:

(i) Modulo 11. The sequence of residues of $\{8u_n + 1\}$ has period 10. We can eliminate $n \equiv 3, 5, 6, 7 \pmod{10}$ since they imply, respectively,

$$8u_n + 1 \equiv 6, 8, 10, 6 \pmod{11},$$

all of which are quadratic nonresidues modulo 11, so there remain $n \equiv \pm 1, 0, 2, 4, 8 \pmod{10}$.

For brevity, we shall omit the sentences about periods in what follows since they can be inferred from the other information given, e.g., mod 10 in the above step.

(ii) Modulo 5. Eliminate $n \equiv 9, 11, 12, 14, 18 \pmod{20}$, which imply

$$8u_n + 1 \equiv \pm 2 \pmod 5,$$

which are quadratic nonresidues modulo 5, so there remain $n \equiv \pm 1, 0, 2, 4, 8, 10 \pmod{20}$.

(iii) Modulo 3. Eliminate $n \equiv 3, 5, 6 \pmod 8$, which imply

$$8u_n + 1 \equiv 2 \pmod 3,$$

which is a quadratic nonresidue modulo 3, so eliminate $n \equiv 19, 21, 22, 30 \pmod{40}$ and there remain $n \equiv \pm 1, 0, 2, 4, 8, 10, 20, 24, 28 \pmod{40}$.

(iv) Modulo 2161. Eliminate $n \equiv 28, 39, 41, 42, 44, 60, 68 \pmod{80}$ since they imply, respectively,

$$8u_n + 1 \equiv 1153, 2154, 2154, 2154, 2138, 2067, 1010 \pmod{2161},$$

which are quadratic nonresidues modulo 2161, so there remain $n \equiv \pm 1, 0, 2, 4, 8, 10, 20, 24, 40, 48, 50, 64 \pmod{80}$.

(v) Modulo 3041. Eliminate $n \equiv 24, 40, 50, 64, 79, 81, 82, 84, 88, 90, 100, 104, 120, 128 \pmod{160}$ since they imply, respectively,

$$8u_n + 1 \equiv -57, 2590, 2613, 1815, -7, -7, -7, -23, 2874, 2602, 619, 59, 447, 1500 \pmod{3041},$$

which are quadratic nonresidues modulo 3041.

Modulo 1601. Eliminate $n \equiv 130, 144 \pmod{160}$ since they imply, respectively,

$$8u_n + 1 \equiv 639, 110 \pmod{1601},$$

which are quadratic nonresidues modulo 1601.

Hence, there remain $n \equiv \pm 1, 0, 2, 4, 8, 10, 20, 48, 80 \pmod{160}$.

(vi) Modulo 2207. Eliminate $n \equiv 48, 80, 208, 240 \pmod{320}$ since they imply

$$8u_n + 1 \equiv 933 \text{ or } 1276 \pmod{2207},$$

both of which are quadratic nonresidues modulo 2207, so eliminate $n \equiv 48$ and $80 \pmod{160}$ and there remain $n \equiv \pm 1, 0, 2, 4, 8, 10, 20 \pmod{160}$.

(vii) Now we eliminate $n \equiv 20 \pmod{160}$ by the following calculation. Put $n = 160m + 20$, since $80 \equiv 2 \pmod 6$; by (5), $u_{160m+20} \equiv \pm u_{20} \pmod{v_{80}}$, where the sign + or - depends on whether m is even or odd. Using (3) and (4), we get

$$\begin{aligned}\left(\frac{8u_{20}+1}{v_{80}}\right) &= \left(\frac{v_{80}}{8u_{20}+1}\right) = \left(\frac{(v_{20}^2-2)^2-2}{8u_{20}+1}\right) = \left(\frac{(5u_{20}^2+2)^2-2}{8u_{20}+1}\right)\\ &= \left(\frac{(5\cdot(8u_{20})^2+2\cdot 8^2)^2-2\cdot 8^4}{8u_{20}+1}\right)\\ &= \left(\frac{(5+2\cdot 8^2)^2-2\cdot 8^4}{8u_{20}+1}\right) = \left(\frac{9497}{8u_{20}+1}\right) = \left(\frac{9497}{54121}\right) = -1.\end{aligned}$$

Similarly,

$$\left(\frac{-8u_{20}+1}{v_{80}}\right) = \left(\frac{v_{80}}{8u_{20}-1}\right) = \left(\frac{9497}{8u_{20}-1}\right) = \left(\frac{9497}{54119}\right) = -1.$$

Hence $8u_n + 1$ must not be a square when $n \equiv 20 \pmod{160}$, and, finally, there remain $n \equiv \pm 1, 0, 2, 4, 8, 10 \pmod{160}$. This completes the proof. □

In the following two lemmas, we suppose that n is even.

Lemma 8: If n is even and $8u_n + 1$ is a square, then we have $n \equiv 0, 2, 4, 8, 10 \pmod{2^2 \cdot 5^2}$.

Proof: We begin from the second step of the proof of Lemma 7. Note that since n is even, there remain $n \equiv 0, 2, 4, 8, 10 \pmod{20}$.

(i) Modulo 101. Eliminate $n \equiv 12, 18, 20, 24, 32, 38, 40, 42, 44, 48 \pmod{50}$ since they imply, respectively,

$$8u_n + 1 \equiv 42, 69, 86, 73, 34, 61, 66, 35, 38, 94 \pmod{101},$$

which are quadratic nonresidues modulo 101.

Modulo 151. Eliminate $n \equiv 22, 28, 34 \pmod{50}$ since they imply, respectively,

$$8u_n + 1 \equiv 51, 102, 108 \pmod{151},$$

which are quadratic nonresidues modulo 151.

Hence, there remain $n \equiv 0, 2, 4, 8, 10, 30, 50, 60, 64, 80 \pmod{100}$.

(ii) Modulo 3001. Eliminate $n \equiv 60$ and $80 \pmod{100}$ since they imply, respectively,

$$8u_n + 1 \equiv 2562 \text{ and } 2900 \pmod{3001},$$

both of which are quadratic nonresidues modulo 3001.

Modulo 25. Eliminate $n \equiv 64 \pmod{100}$ since it implies

$$8u_n + 1 \equiv 10 \pmod{25},$$

which is a quadratic nonresidue modulo 25.
Hence, there remain $n \equiv 0, 2, 4, 8, 10, 30, 50 \pmod{100}$.

(iii) Modulo 401. Eliminate $n \equiv 30, 50, 130, 150 \pmod{200}$ since they imply, respectively,

$$8u_n + 1 \equiv 122, 165, 281, 238 \pmod{401},$$

which are quadratic nonresidues modulo 401. Hence, at last, there remain $n \equiv 0, 2, 4, 8, 10 \pmod{100}$, which completes the proof. □

Lemma 9: If n is even and $8u_n + 1$ is a square, then we have $n \equiv 0, 2, 4, 8, 10 \pmod{2^2 \cdot 5 \cdot 11}$.

Proof:

(i) Modulo 199. Eliminate $n \equiv 16, 18, 20 \pmod{22}$ since they imply, respectively,

$$8u_n + 1 \equiv 136, 176, 192 \pmod{199},$$

which are quadratic nonresidues modulo 199. There remain $n \equiv 0, 2, 4, 6, 8, 10, 12, 14 \pmod{22}$.

(ii) Modulo 89. Eliminate $n \equiv 6, 24, 26, 28, 32, 34 \pmod{44}$ since they imply, respectively,

$$8u_n + 1 \equiv 65, 82, 66, 26, 6, 6 \pmod{89},$$

which are quadratic nonresidues modulo 89, so there remain $n \equiv 0, 2, 4, 8, 10, 12, 14, 22, 30, 36 \pmod{44}$.

(iii) In the first two steps of the proof of Lemma 7 we have shown that it is necessary for $n \equiv 0, 2, 4, 8, 10 \pmod{20}$, so that there further remain $n \equiv 0, 2, 4, 8, 10, 22, 30, 44, 48, 80, 88, 90, 100, 102, 110, 124, 140, 142, 144, 168, 180, 184, 188, 190 \pmod{220}$.

(iv) Modulo 661. Eliminate $n \equiv 44, 48, 124, 144, 180, 184 \pmod{220}$ since they imply, respectively,

$$8u_n + 1 \equiv 544, 214, 290, 447, 379, 546 \pmod{661},$$

which are quadratic nonresidues modulo 661.
Modulo 331. Eliminate $n \equiv 30, 58, 88, 102 \pmod{110}$ since they imply, respectively,

$$8u_n + 1 \equiv 242, 231, 312, 164 \pmod{331},$$

which are quadratic nonresidues modulo 331. Thus, we can eliminate $n \equiv 30, 88, 102, 140, 168 \pmod{220}$.
Modulo 474541. Eliminate $n \equiv 80, 90, 142, 188 \pmod{220}$ since they imply, respectively,

$$8u_n + 1 \equiv 12747, 54121, 131546, 131546 \pmod{474541},$$

which are quadratic nonresidues modulo 474541.
Hence there remain $n \equiv 0, 2, 4, 8, 10, 22, 100, 110, 190 \pmod{220}$.

(v) Modulo 307. Eliminate $n \equiv 14, 22, 58, 66 \pmod{88}$ since they imply, respectively,

$$8u_n + 1 \equiv 254, 162, 55, 147 \pmod{307},$$

which are quadratic nonresidues modulo 307. These are equivalent to $n \equiv 14, 22 \pmod{44}$, so that we can eliminate $n \equiv 22, 110, 190 \pmod{220}$ from those left in the foregoing step and then there remain $n \equiv 0, 2, 4, 8, 10, 100 \pmod{220}$.

(vi) Modulo 881. Eliminate $n \equiv 12, 56, 100, 144 \pmod{176}$ since they imply, respectively,

$$8u_n + 1 \equiv 272, 293, 611, 590 \pmod{881},$$

which are quadratic nonresidues modulo 881. These are equivalent to $n \equiv 12 \pmod{44}$, so that we can eliminate $n \equiv 100 \pmod{220}$.

Finally, there remain $n \equiv 0, 2, 4, 8, 10 \pmod{220}$. This completes the proof. □

From Lemmas 7 to 9, we can derive the following corollary.

Corollary 2: If n is even, and if $8u_n + 1$ is a square, then $n \equiv 0, 2, 4, 8, 10 \pmod{2^5 \cdot 5^2 \cdot 11}$.

Proof: Suppose that $8u_n + 1$ is a square, n is even. According to Lemmas 7 to 9, n must satisfy the following congruences simultaneously:

$$\begin{cases} n \equiv c_1 \pmod{2^5 \cdot 5} \\ n \equiv c_2 \pmod{2^2 \cdot 5^2} \\ n \equiv c_3 \pmod{2^2 \cdot 5 \cdot 11} \end{cases} \quad c_1, c_2, c_3 \in \{0, 2, 4, 8, 10\}.$$

Because the greatest common divisor of the three modulos is 20 and the absolute value of the difference of any two numbers in $\{0, 2, 4, 8, 10\}$ cannot exceed 10, we conclude that $c_1 = c_2 = c_3$. Moreover, since the least common multiple of the three modulos is $2^5 \cdot 5^2 \cdot 11$, we finally obtain $n \equiv 0, 2, 4, 8, 10 \pmod{2^5 \cdot 5^2 \cdot 11}$. The proof is complete. □

5. Proofs of Theorems

Now we give the proofs of the theorems in Section 1.

Proof of Theorem 1: Suppose $8u_n + 1$ is a square, the conclusion follows from Lemma 7 and Lemma 1 when n is odd, and from Corollary 2 and Corollary 1 when n is even. □

Proof of Theorem 2: The proof follows immediately from Theorem 1, by excluding $u_0 = 0$, since a triangular number is positive.

In fact,

$$u_{\pm 1} = u_2 = 1 \cdot 2/2, \ u_4 = 2 \cdot 3/2, \ u_8 = 6 \cdot 7/2, \ u_{10} = 10 \cdot 11/2. \ \square$$

Finally, we give two corollaries as the Diophantine equation interpretations of Theorem 2.

Corollary 3: The Diophantine equation

$$5x^2(x + 1)^2 - 4y^2 = 16 \tag{6}$$

has only the integer solutions $(x, y) = (-2, \pm 1), (1, \pm 1)$.

Proof: According to (4) and the explanation at the end of Section 2, equation (6) implies $\frac{1}{2}x(x + 1) = u_n$ and n is odd, thus it follows from Theorem 2 that $\frac{1}{2}x(x + 1) = 1$, which gives $x = -2$ or 1. □

Corollary 4: The Diophantine equation

$$5x^2(x + 1)^2 - 4y^2 = -16 \tag{7}$$

has only the integer solutions $(x, y) = (-1, \pm 2)$, $(0, \pm 2)$, $(-2, \pm 3)$, $(1, \pm 3)$, $(-3, \pm 7)$, $(2, \pm 7)$, $(-7, \pm 47)$, $(6, \pm 47)$, $(-11, \pm 123)$, and $(10, \pm 123)$.

Proof: With the same reason as in Corollary 3, equation (7) implies $\frac{1}{2}x(x + 1) = u_n$ and n is even, so $\frac{1}{2}x(x + 1) = 0, 1, 3, 21$, or 55 by Theorem 2 (adding $u_0 = 0$). Thus, we get $x = -1, 0, -2, 1, -3, 2, -7, 6, -11, 10$, which give all integer solutions of equation (7). □

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