Therefore, $f(x) \equiv ax - (a - 1)$ modulo $(N - 1)$. Designate by $f_i(x)$ the position of the card after i repetitions of the permutation. Then, by induction,

$$f^i(x) \equiv a^i x - (a^i - 1) \text{ modulo } (N - 1).$$

It follows that $f^i(x) = x$ if and only if $a^i \equiv 1$ modulo $(N - 1)$.

*****

A GENERAL RECURRENCE RELATION FOR REFLECTIONS IN MULTIPLE GLASS PLATES

Jeffrey A. Brooks
(student)
West Virginia University, Morgantown, WV 26506
(Submitted June 1987)

The number of possible light paths in a stack of two glass plates can be expressed in terms of Fibonacci numbers, as was first pointed out by Moser [1]. If two glass plates are placed together in such a way that each surface can either reflect or transmit light, then the number of distinct paths through the two plates with exactly n internal reflections is F_{n+2}.

Junge and Hoggatt [2] used matrix methods to count reflections in larger numbers of plates. Hoggatt and Bicknell-Johnson [3] used geometric and matrix techniques to count specific sets of reflections. However, these authors did not present a general recurrence relation for the number of distinct light paths with a fixed number of reflections in an arbitrary number of glass plates. Here we shall present such a recurrence relation.

Consider a single ray of light directed into a stack of r glass plates. Let $T_r(n)$ be the number of distinct paths that can be taken by a light ray entering through the top plate, leaving through either the top plate or the bottom plate, and having exactly n internal reflections. Figure 1 illustrates the distinct light paths in two plates with zero, one, two, and three reflections.

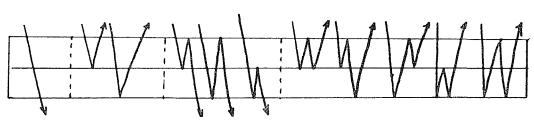

FIGURE 1

As a light ray passes through the stack of plates in a fixed direction, there are a total of r internal surfaces from which it could be reflected. (The surface crossed by the light ray as it enters the stack of plates cannot cause an internal reflection.) Number the reflecting surfaces from 1 to r along the direction of the ray. Figure 2 illustrates this numbering scheme; the path shown consists of reflections from surfaces 2-3-3-2-2.

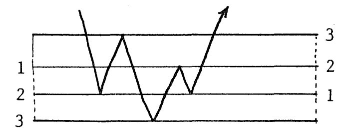

FIGURE 2

Let $G_r(m, n)$ be the number of distinct light paths with exactly n internal reflections such that the n^{th} internal reflection occurs at reflecting surface m. Then, for $n \geq 1$,

$$T_r(n) = \sum_{k=1}^{r} G\ (k,\ n). \tag{1}$$

A light path of length $n + 1$ whose last reflection was from surface m could have undergone its n^{th} reflection at any one of the reflecting surfaces $r - m + 1$ through r. So

$$G_r(m,\ n+1) = \sum_{k=r-m+1}^{r} G_r(k,\ n). \tag{2}$$

Combining (1) and (2), we see that

$$G_r(m,\ n+1) = T_r(n) - \sum_{k=1}^{r-m} G_r(k,\ n). \tag{3}$$

Let σ represent the permutation of $\{1, 2, \ldots, r\}$ that maps $1, 2, 3, 4, \ldots$ onto $r, 1, r-1, 2, \ldots$, and let

$$G_r'(m,\ n) = G_r(\sigma_m,\ n).$$

The functions $\{G_r'(m,\ n) : 1 \leq m \leq r\}$ form a reordering of the $\{G_r'(m,\ n) : 1 \leq m \leq r\}$ which can be expanded recursively in terms of $T_r(n)$.

Let $1 \leq i \leq \lfloor k/2 \rfloor$, where $\lfloor x \rfloor$ is the floor function of Donald Knuth and represents the greatest integer less than or equal to x. Then, applying (2), (3), and the definition of $G_r'(m,\ n)$, we see that:

$$G_r'(1,\ n) = G_r(r,\ n) = T_r(n-1); \tag{4}$$

$$G_r'(2i,\ n) = G_r(i,\ n) = \sum_{k=r-i+1}^{r} G_r(k,\ n-1) \tag{5}$$

$$= \sum_{k=1}^{i} G_r'(2k-1,\ n-1);$$

$$G_r'(2i+1,\ n) = G_r(r-i,\ n) = T_r(n-1) - \sum_{k=1}^{i} G_r(k,\ n-1) \tag{6}$$

$$= T_r(n-1) - \sum_{k=1}^{i} G_r'(2k,\ n-1).$$

By repeatedly applying (4), (5), (6), we can obtain an expansion for $G_r'(m,\ n)$ in terms of $\{T_r(n-k) : 1 \leq k \leq m\}$. Furthermore, the coefficients in the expansion of $G_r'(m,\ n)$ are independent of r. So, for any system of r plates and any $m \leq r$, the coefficients of the expansion of $G_r'(m,\ n)$ in terms of $\{T_r(n-k)\}$, are the same.

Let H_k^j denote the coefficient of $T_r(n-k)$ in the expansion of $G_r'(j,\ n)$. Figure 3 gives the values of H_k^j for $1 \leq k \leq j \leq 8$.

H_k^j	$k=1$	2	3	4	5	6	7	8	9	10
$j=1$	1									
2	0	1								
3	1	0	−1							
4	0	2	0	−1						
5	1	0	−3	0	1					
6	0	3	0	−4	0	1				
7	1	0	−6	0	5	0	−1			
8	0	4	0	−10	0	6	0	−1		
9	1	0	−10	0	15	0	−7	0	1	
10	0	5	0	−20	0	21	0	−8	0	1

FIGURE 3

Before proceeding, we must introduce a notation for iterated sums of integers. For m, $n \ge 1$, define the n^{th}-iterated sum from 1 to n, denoted $S(m, n)$, by

$$S(m, n) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} \cdots \sum_{i_n=1}^{i_{n-1}} 1.$$

By convention, we let $S(0, n) = 1$ for all n. Note that $S(m, n)$ obeys the following identity:

$$\sum_{i=1}^{m} S(n, i) = S(n + 1, m).$$

Theorem 1: If $j \equiv k \pmod 2$, then

$$H_k^j = (-1)^{\lfloor (k-1)/2 \rfloor} S(k - 1, \lfloor (j - k)/2 \rfloor + 1).$$

Otherwise, $H_k^j = 0$.

Proof: By induction on k.

Suppose $k = 1$. Then H_k^j is the coefficient of $T_r(n - 1)$ in the expansion of $G_r'(j, n)$. If j is odd, then $H_k^j = 1$, since none of the terms in the summation in (6) can depend on $T_r(n - 1)$. If j is even, then $H_k^j = 0$, since none of the terms in the summation in (5) can depend on $T\ (n - 1)$. In either case, the statement of the theorem is satisfied.

Suppose $k > 1$. Assume the statement of the theorem is true for $k' \le k$. Four cases must be considered:

1. Suppose j and k are both even. Let $j = 2i$ where $1 \le i \le \lfloor j/2 \rfloor$. Then

$$\begin{aligned}
H_k^j = \sum_{m=1}^{i} H_{k-1}^{2m-1} &= \sum_{m=1}^{i} (-1)^{\left\lfloor \frac{(k-1)-1}{2} \right\rfloor} S\left(k - 2, \left\lfloor \frac{(2m - 1) - (k - 1)}{2} \right\rfloor + 1\right) \\
&= (-1)^{\left\lfloor \frac{k-2}{2} \right\rfloor} \sum_{m=1}^{i} S\left(k - 2, \left\lfloor \frac{2m - k}{2} \right\rfloor + 1\right) \\
&= (-1)^{\left\lfloor \frac{k-2}{2} \right\rfloor} S\left(k - 1, \left\lfloor \frac{2i - k}{2} \right\rfloor + 1\right) \\
&= (-1)^{\left\lfloor \frac{k-1}{2} \right\rfloor} S\left(k - 1, \left\lfloor \frac{j - k}{2} \right\rfloor + 1\right)
\end{aligned}$$

where in the last step we used the fact that k even implies $\left\lfloor \frac{k-2}{2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor$.

2. Suppose j and k are both odd. Let $j = 2i + 1$ where $1 \le i \le \lfloor j/2 \rfloor$. Then

$$\begin{aligned}
H_k^j &= -\sum_{m=1}^{i} H_{k-1}^{2m} = -\sum_{m=1}^{i} (-1)^{\left\lfloor \frac{(k-1)-1}{2} \right\rfloor} S\left(k-2, \left\lfloor \frac{2m-(k-1)}{2} \right\rfloor + 1\right) \\
&= (-1)^{\left\lfloor \frac{k}{2} \right\rfloor} \sum_{m=1}^{i} S\left(k-2, \left\lfloor \frac{(2m+1)-k}{2} \right\rfloor + 1\right) \\
&= (-1)^{\left\lfloor \frac{k}{2} \right\rfloor} S\left(k-1, \left\lfloor \frac{(2i+1)-k}{2} \right\rfloor + 1\right) \\
&= (-1)^{\left\lfloor \frac{k-1}{2} \right\rfloor} S\left(k-1, \left\lfloor \frac{j-k}{2} \right\rfloor + 1\right)
\end{aligned}$$

where in the last step we used the fact that k odd implies $\left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor$.

3. Suppose j is even and k is odd. Let $j = 2i$ where $1 \le i \le \lfloor j/2 \rfloor$. Then

$$H_k^j = \sum_{m=1}^{i} H_{k-1}^{2m-1} = 0.$$

4. Suppose j is odd and k is even. Let $j = 2i + 1$ where $1 \le i \le \lfloor j/2 \rfloor$. Then

$$H_k^j = \sum_{m=1}^{i} H_{k-1}^{2m} = 0.$$

This completes the proof of Theorem 1.

Theorem 2: $T_r(n) = \sum_{k=1}^{r} (-1)^{\lfloor (k-1)/2 \rfloor} S(k, \lfloor (r-k)/2 \rfloor + 1) T_r(n-k)$.

Proof:
$$\begin{aligned}
T_r(n) &= \sum_{m=1}^{r} G_r(m, n) = \sum_{m=1}^{r} G_r'(m, n) = \sum_{m=1}^{r} \left(\sum_{k=1}^{m} H_k^m T_r(n-k) \right) \\
&= \sum_{k=1}^{r} \left(\sum_{m=k}^{r} H_k^m \right) T_r(n-k) \\
&= \sum_{k=1}^{r} \left(\sum_{m=k}^{r} (-1)^{\lfloor (k-1)/2 \rfloor} S(k-1, \lfloor (m-k)/2 \rfloor + 1)\, T_r(n-k) \right) \\
&= \sum_{k=1}^{r} (-1)^{\lfloor (k-1)/2 \rfloor} S(k, \lfloor (r-k)/2 \rfloor + 1) T_r(n-k).
\end{aligned}$$

Figure 4 illustrates the coefficients of this recurrence for $1 \le r \le 10$.

$T_r(n)$	$k=1$	2	3	4	5	6	7	8	9	10
$r=1$	1									
2	1	1								
3	2	1	−1							
4	2	3	−1	−1						
5	3	3	−4	−1	1					
6	3	6	−4	−5	1	1				
7	4	6	−10	−5	6	1	−1			
8	4	10	−10	−15	6	7	−1	−1		
9	5	10	−10	−15	21	7	−8	−1	1	
10	5	15	−20	−35	21	28	−8	−9	1	1

FIGURE 4

For r = 1, 2, 3, 4, and 5, these expansions for $T_r(n)$ are the same as those derived by matrix methods in [2]; however, the matrix methods required a separate set of computations for each value of r.

The recurrence in Theorem 2 has an even simpler statement involving binomial coefficients. Noting that

$$S(m, n) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_m \le n} 1 = \binom{n + k - 1}{k},$$

it follows that

$$T\ (n) = \sum_{k=1}^{r} (-1)^{\lfloor (k-1)/2 \rfloor} \binom{\left\lfloor \frac{r-k}{2} \right\rfloor + k}{k} T_r(n - k).$$

Remark: This problem was proposed in a graduate combinatorics class taught by H. W. Gould at West Virginia University.

References

1. L. Moser & M. Wyman. Problem B-6. *Fibonacci Quarterly 1.1* (1963):74.
2. Bjarne Junge & V. E. Hoggatt, Jr. "Polynomials Arising from Reflections Across Multiple Plates." *Fibonacci Quarterly 11.3* (1973):285-291.
3. V. E. Hoggatt, Jr., & Marjorie Bicknell-Johnson. "Reflections Across Two and Three Glass Plates." *Fibonacci Quarterly 17.2* (1979):118-142.

*****