A POLYNOMIAL FORMULA FOR FIBONACCI NUMBERS

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1. Introduction

A Fibonacci sequence is defined by two initial terms, $F(1)$ and $F(2)$, together with the recursion equation

(1) $\quad F(n + 1) = F(n) + F(n - 1),\ n = 2, 3, 4, \ldots .$

A closed form expression for the n^{th} Fibonacci number is given by

(2) $$F(n) = \frac{1}{\sqrt{5}}\left[\frac{1 + \sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1 - \sqrt{5}}{2}\right]^n,\ n = 1, 2, 3, \ldots .$$

If we let $F(1) = F(2) = 1$ in equation (1), then we get the well-known sequence of Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots .$$

Because $F(n)$ is defined recursively in (1), we must know $F(n)$ and $F(n - 1)$ in order to find $F(n + 1)$. Therefore, to find $F(100)$ for example, we must first compute $F(3), F(4), \ldots, F(98), F(99)$. This becomes a formidable computing task as n gets large. Finding $F(n)$ for large values of n from equation (2) is also a laborious task. Computing time, machine limits, and round-off error are problems that must be considered.

In this paper we assume that m terms of the Fibonacci sequence are known. To construct a formula that generates the m terms, one can use the Lagrangian approach to obtain the collocation polynomial. This method is based on the following theorem from [3].

Theorem: Let (x_k, f_k), $k = 0, 1, 2, \ldots, n$, denote $(n + 1)$ points that would lie on the graph of a function. Then there exists a unique collocation polynomial $p(x) = \sum_{j=0}^{n} a_j x_j$ whose graph passes through the given $(n + 1)$ points.

The Lagrangian method may require sophisticated numerical techniques in order to produce the collocation polynomial. However, the finite differences procedure and the examples presented here are at a level that can appeal to high school teachers with a desire to add interesting exercises involving Fibonacci numbers (or any sequence). Therefore, the emphasis in this paper is not on the derivation of the formula, but on the application of the formula to reproduce the given m Fibonacci numbers. In addition, the formula presented is in a more directly useable form than is usually available, and its purpose is different from equations (1) and (2). In some applications, such a formula may prove to be quite useful.

2. A Polynomial Formula Using Finite Differences

In this section we describe a general method for constructing a polynomial that generates the terms of a sequence. Let $s_1, s_2, \ldots, s_m$ be the terms of a sequence. Form the successive order differences as shown in Table 1.

TABLE 1

n	Sequence Terms	Difference				
		1st	2nd	3rd	4th	5th $\cdots$
1	s_1					
		D_1^1				
2	s_2		D_1^2			
		D_2^1		D_1^3		
3	s_3		D_2^2		D_1^4	
		D_3^1		D_2^3		D_1^5
4	s_4		D_3^2		D_2^4	
		D_4^1		D_3^3		
5	s_5		D_4^2			
		D_5^1				
6	s_6					
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

where

$$\begin{array}{lll}
D_1^1 = s_2 - s_1 & D_1^2 = D_2^1 - D_1^1 & D_1^3 = D_2^2 - D_1^2 \\
D_2^1 = s_3 - s_2 & D_2^2 = D_3^1 - D_2^1 & D_2^3 = D_3^2 - D_2^2 \\
\vdots & \vdots & \vdots \\
D_{m-1}^1 = s_m - s_{m-1} & D_{m-2}^2 = D_{m-1}^1 - D_{m-2}^1 & D_{m-3}^3 = D_{m-2}^2 - D_{m-3}^2 \quad \cdots
\end{array}$$

We assume that some order difference becomes constant. That is, $D_j^i = c$, $j = 1, 2, 3, \ldots, m - i$, for some $i = 1, 2, \ldots, m - 2$. Thus, the next order difference D_j^{i+1} is zero for all j.

Let $k \le m - 1$ be a positive integer such that D_j^k is zero for all $j = 1, 2, \ldots, m - k$. The general term of the original sequence can now be expressed by a polynomial in n. The polynomial formula that generates the sequence is based on the above finite difference table and is given by

$$s_n = s_1 + (n-1)D_1^1 + \frac{(n-1)(n-2)}{2!}D_1^2 + \frac{(n-1)(n-2)(n-3)}{3!}D_1^3 + \cdots + \frac{(n-1)(n-2)\cdots(n-(k-1))}{(k-1)!}D_1^{k-1} \tag{3}$$

Equation (3) is in terms of s_1, the first term of the sequence, and D_1^1, D_1^2, $\ldots$, D_1^{k-1}, the leading first terms of the various order differences. The complete derivation of (3) is given in [1] and [2].

Equation (3) assumes that the order differences, D_j^i, $j = 1, 2, \ldots, m - i$, are zero for some $i = 1, 2, \ldots, m - 1$. However, we have found that this condition is not necessary for the derivation of a generating polynomial. Equation (3) can be extended in order to construct a polynomial that generates the terms of any sequence whether or not the order differences, D_j^i, $j = 1, 2, \ldots, m - 1$, are zero for some $i = 1, 2, \ldots, m - 1$. We use the first term of the sequence, s_1, and the differences $D_1^1, D_1^2, \ldots, D_1^{m-1}$. The general term of the sequence if given by

$$s_n = s_1 + (n-1)D_1^1 + \frac{(n-1)(n-2)}{2!}D_1^2 + \frac{(n-1)(n-2)(n-3)}{3!}D_1^3 + \cdots + \frac{(n-1)(n-2)\cdots 2\ 1}{(m-1)!}D_1^{m-1} \tag{4}$$

3. Examples

In this section we apply equation (4) to several sequences. Consider the first four terms of the Fibonacci sequence, 1, 1, 2, 3. Form the order differences as shown in Table 2.

TABLE 2

n	Sequence	Differences		
1	1			
		0		
2	1		1	
		1		-1
3	2		0	
		1		
4	3			

Thus, $s_1 = 1$, $D_1^1 = 0$, $D_1^2 = 1$, and $D_1^3 = -1$. Substituting these values into (4) yields

$$(5)\qquad s_n = 1 + (n-1)(0) + \frac{(n-1)(n-2)}{2}(1) + \frac{(n-1)(n-2)(n-3)}{6}(-1)$$
$$= \frac{1}{6}(-n^3 + 9n^2 - 20n + 18).$$

For $n = 1, 2, 3, 4$, equation (5) yields the Fibonacci numbers 1, 1, 2, 3. Using (5), it is possible to generate $F(4)$ without having to compute $F(1)$, $F(2)$, $F(3)$ as in the recursion equation (1). Note that (5) does not generate the correct term $F(5) = 5$ for $n = 5$. This procedure produces a polynomial that generates only the terms of the initial sequence.

We do not have to begin the sequence of terms with $F(1)$ in order to apply (4). For example, consider $F(10)$, $F(11)$, $F(12)$, $F(13)$, $F(14)$, namely, 55, 89, 144, 233, 377. Table 3 contains the order differences.

TABLE 3

n	Sequence	Differences			
1	55				
		34			
2	89		21		
		55		13	
3	144		34		8
		89		21	
4	233		55		
		144			
5	377				

Here, $s_1 = 55$, $D_1^1 = 34$, $D_1^2 = 21$, $D_1^3 = 13$, $D_1^4 = 8$, $s_1 = F(10)$, $s_2 = F(11)$, $s_3 = F(12)$, $s_4 = F(13)$, $s_5 = F(14)$. Using (4), we obtain a polynomial that generates the sequence:

$$(6)\qquad s_n = 55 + (n-1)(34) + \frac{(n-1)(n-2)}{2}(21) + \frac{(n-1)(n-2)(n-3)}{6}(13)$$
$$+ \frac{(n-1)(n-2)(n-3)(n-4)}{24}(8)$$
$$= \frac{1}{6}(2n^4 - 7n^3 + 55n^2 + 58n + 222)$$

For $n = 1, 2, 3, 4, 5$, equation (6) yields the Fibonacci numbers

$$s_1 = F(10) = 55, \ldots, s_5 = F(14) = 377.$$

Once again we can generate any single term of the sequence without computing previous terms. For example, in order to generate $F(14) = 377$, we let $n = 5$ in (6). As in the previous example, we do not obtain $F(15) = 610$ by letting $n = 6$ in (6).

Suppose we are given a longer sequence of Fibonacci numbers. To obtain the generating polynomial, the above procedure suggests we must calculate *all* the order differences. Fortunately, this is not the case.

Consider the sequence consisting of the first ten Fibonacci numbers and the order differences given in Table 4.

TABLE 4

Fibonacci Numbers

$F(1)$	$F(2)$	$F(3)$	$F(4)$	$F(5)$	$F(6)$	$F(7)$	$F(8)$	$F(9)$	$F(10)$
1	1	2	3	5	8	13	21	34	55

Differences for Equation (4)

s_1	D_1^1	D_1^2	D_1^3	D_1^4	D_1^5	D_1^6	D_1^7	D_1^8	D_1^9
1	0	1	-1	2	-3	5	-8	13	-21

There is a definite pattern in the differences given in Table 4. The leading differences alternate in sign beginning with D and the absolute value of these differences yields the first eight Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21. The following examples further illustrate the pattern in the leading differences.

Consider the sixteen Fibonacci numbers beginning with $F(5) = 5$ through $F(2) = 6765$. The Fibonacci numbers and the leading differences are given in Table 5.

TABLE 5

Fibonacci Numbers

$F(5)$	$F(6)$	$F(7)$	$F(8)$	$F(9)$	$F(10)$	$F(11)$	$F(12)$
5	8	13	21	34	55	89	144
$F(13)$	$F(14)$	$F(15)$	$F(16)$	$F(17)$	$F(18)$	$F(19)$	$F(20)$
233	377	610	987	1597	2584	4181	6765

Differences for Equation (4)

s_1	D_1^1	D_1^2	D_1^3	D_1^4	D_1^5	D_1^6	D_1^7	D_1^8	D_1^9	D_1^{10}	D_1^{11}	D_1^{12}	D_1^{13}	D_1^{14}	D_1^{15}
5	3	2	1	1	0	1	-1	2	-3	5	-8	13	-21	34	-55

From Table 5, we see that

$$D_1^1 = F(4),\ D_1^2 = F(3),\ D_1^3 = F(2),\ D_1^4 = F(1).$$

After D_1^5, the differences follow the same pattern of differences as in the previous example. That is, the differences alternate in sign, and the absolute value of the differences yields the first ten Fibonacci numbers.

Therefore, suppose we consider a sequence of sixteen Fibonacci numbers beginning with $F(10) = 55$. Then the differences are found quickly and simply without computation from the patterns in the above examples. The differences for (4) are:

D_1^1	D_1^2	D_1^3	D_1^4	D_1^5	D_1^6	D_1^7	D_1^8	D_1^9	D_1^{10}	D_1^{11}	D_1^{12}	D_1^{13}	D_1^{14}	D_1^{15}
34	21	13	8	5	3	2	1	1	0	1	-1	2	-3	5

Substituting these values into (4), we obtain a polynomial in n which generates the sixteen Fibonacci numbers $F(10) = 55$ through $F(25) = 75025$.

These examples demonstrate a technique for obtaining a polynomial that generates any finite sequence of Fibonacci numbers. The leading order differences must be calculated in order to determine the polynomial, but they follow a discernible pattern. The resulting polynomial generates only those terms in the initial sequence and is useful in some applications.

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References

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3. Allen W. Smith. *Elementary Numerical Analysis.* Englewood Cliffs, NJ: Prentice-Hall, 1986.

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The reviewer has some problems with comments made by the authors. First, the authors could, I believe, have misinterpreted the quote by Schalau and Opolka which is given in the Foreword. The Pythagorean triple problem was completely solved in antiquity if by this statement Schalau and Opolka meant that a method had been developed which totally solved the problem of finding all Pythagorean triples. If Schalau and Opolka meant that no new results could be found, then the authors are correct. I believe that the former is the case.

The authors also claim that there is no technique for systematically generating all Pythagorean triples by the old method. This is, I believe, a matter of opinion. The reviewer happens to believe that the original technique developed by Diophantus is very systematic. That is, (x, y, z) is a Pythagorean triple if and only if $x = u^2 - v^2$, $y = 2uv$, and $z = u^2 + v^2$, where $u > v$. The problem here is the meaning of "systematic." The authors also feel that their method is more time efficient. The reviewer has a problem with this. Finding the greatest common divisor of two integers, even when large, is not a problem for the computer. It does take time but would it take any more time than is needed to go through the contraction method developed by the authors or to find the convergents needed for the continued fraction or to pick and implement the method (class) that gives the correct value of n? I think not.

Overall, I would recommend the book and suggest that those interested in Pythagorean triples or Pellian equations read it.

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