EXPANSION OF ANALYTIC FUNCTIONS IN TERMS INVOLVING LUCAS NUMBERS OR SIMILAR NUMBER SEQUENCES

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1. INTRODUCTION

In a previous article [1], certain available results concerning polynomial expansions were applied in order to illustrate a simple general technique for obtaining the coefficients $\beta_n(\alpha)$ in the series

$$(1.1) \qquad f(\alpha) = \sum_{n=0}^{\infty} \beta_n(\alpha) F_{n+1} ,$$

where $f(\alpha)$ is an "arbitrary" analytic function of α, and F_n are Fibonacci numbers. The same method may also be applied to develop series expansions of the form

$$(1.2) \qquad f(\alpha) = \frac{1}{2} A_o(\alpha) L_o + \sum_{n=1}^{\infty} A_n(\alpha) L_n ,$$

where L_n are the Lucas numbers ($L_o = 2$, $L_1 = 1$; $L_{n+2} = L_{n+1} + L_n$ for $n = 0, 1, \ldots$). Such series, which can be derived as special cases of more general expansions, are of use when one desires to make some given function f serve as a generating function* of the Fibonacci or Lucas sequence — two famous sequences whose many number-theoretical properties are of primary concern to this journal.

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*In general, any infinite series of the form

$$f(\alpha) = G\left[\alpha; \{y_n\}\right] = \sum_{n=0}^{\infty} g_n(\alpha)\, y_n$$

is called a generating function of a number sequence $\{y_n\}$ if $g_n(\alpha)$ are linearly independent functions of α. The familiar type, when $g_n(\alpha)$ is taken to be α^n or $\alpha^n/n!$, is a special case of the more general definition.

The main purpose of the present article is to review one technique for finding the coefficients of (1.1) or (1.2), and to give explicit expansions of a variety of transcendental functions in terms involving Lucas numbers. Another objective is to point out how certain extensions might be made to considerations involving similar sequences of integers.

2. EXPANSIONS IN TERMS OF GEGENBAUER POLYNOMIALS

We begin by first seeking a <u>formal</u> series expansion expressed by

$$f(2ax) = D_{o,k}(2a)\, C_{o,k}(x) + \sum_{m=1}^{\infty} D_{m,k}(2a)\, C_{m,k}(x) , \tag{2.1}$$

where the functions* $C_{m,k}(x)$, the well-known <u>Gegenbauer polynomials</u> [2], are given by $C_{o,k}(x) = 1$, and

$$C_{m,k}(x) = \frac{1}{\Gamma(k)} \sum_{r=o}^{[m/2]} (-1)^r \frac{\Gamma(m-r+k)}{\Gamma(m-r+1)} \binom{m-r}{r} (2x)^{m-2r}, \quad \text{(for } k > -1/2, k \neq 0) \tag{2.2}$$

$$= \sum_{r=o}^{[m/2]} \frac{(-1)^r}{m-r} \binom{m-r}{r} (2x)^{m-2r}, \quad \text{(for } k = 0,\ m > 0)$$

These polynomials satisfy the <u>orthogonality relation</u>

$$\int_{-1}^{1} \frac{(1-x^2)^k}{\sqrt{1-x^2}} C_{m,k}(x) C_{p,k}(x)dx = \frac{2\pi\Gamma(2k+m)\delta_{mp}}{4^k (m+k)\Gamma(m+1)[\Gamma(k)]^2} \quad (k \neq 0), \tag{2.3}$$

$$= 2\pi\delta_{mp}/m^2 \ \text{ for } k = 0,\ m \neq 0,$$

with δ_{mp} being equal to 0 when $m \neq p$ and to 1 when $m = p$. If we multiply both sides of (2.1) by $(1 - x^2)^{k-1/2} C_{p,k}(x)$ and then integrate from -1 to 1, we obtain (upon setting $x = \cos \gamma$ and making use of (2.3)) the coefficient formulas

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*We find it convenient to write $C_{m,k}(x)$ instead of using the <u>standard notation</u> $C_m^k(x)$.

$$
(2.4)\begin{cases}
D_{m,k}(2a) = \dfrac{4^k(m+k)\,\Gamma(m+1)[\Gamma(k)]^2}{2\,\pi\,(2k+m)}\displaystyle\int_0^{\pi} \sin^{2k}\gamma\, f(2a\cos\gamma)C_{m,k}(\cos\gamma)d\gamma \\
\qquad\qquad (\text{for } m = 0, 1, \ldots;\ k \neq 0) \\[2ex]
D_{m,o}(2a) = \dfrac{m^2}{2\pi}\displaystyle\int_0^{\pi} f(2a\cos\gamma)C_{m,o}(\cos\gamma)d\gamma (\text{for } m \neq 0,\ k = 0)\ , \\[2ex]
D_{o,k}(2a) = \dfrac{\Gamma(k+1)}{\sqrt{\pi}\,\Gamma\left(k+\frac{1}{2}\right)}\displaystyle\int_0^{\pi} \sin^{2k}\gamma\ f(2a\cos\gamma)d\gamma\ .
\end{cases}
$$

Now it will be seen that both (1.1) and (1.2) can easily be obtained as special cases of the more general series (2.1).

3. RELATIONSHIP BETWEEN GEGENBAUER POLYNOMIALS AND THE SO-CALLED FIBONACCI AND LUCAS POLYNOMIALS

The Gegenbauer (ultraspherical) polynomials $C_{m,k}(x)$ of degree m and order k satisfy the recurrence formula, given in reference [2],

$$(3.1)\quad (m+2)C_{m+2,k}(x) = 2(m+1+k)xC_{m+1,k}(x) - (m+2k)C_{m,k}(x)\ ,$$

which reduces to

$$(3.2)\qquad C_{m+2,1}(x) - 2xC_{m+1,1}(x) + C_{m,1}(x) = 0$$

when $k = 1$. Relation (3.2), with conditions $C_{0,1}(x) = 1$ and $C_{1,1}(x) = 2x$, is the well-known recurrence formula defining the Chebyshev polynomials $U_m(x)$ of the second kind, and one may thus write

$$(3.3)\ C_{m,1}(x) = U_m(x) = U_m(\cos\gamma) = \frac{\sin(m+1)\gamma}{\sin\gamma}\ ,\quad (x = \cos\gamma)\ .$$

When $k = 0$, formula (3.1) becomes

$$(3.4)\quad (m+2)C_{m+2,0}(x) - 2(m+1)xC_{m+1,0}(x) - mC_{m,0}(x) = 0\ ,$$

so that, since $C_{o,o}(x) = 1$, $C_{1,o}(x) = 2x$, and

$$(3.5)\quad C_{m,o}(x) = \frac{2}{m}\ T_m(x) = \frac{2}{m}\ T_m(\cos\gamma) = \frac{2\cos m\gamma}{m}\ ,\ (x = \cos\gamma\ ,\ m \neq 0),$$

we have

$$T_{m+2}(x) - 2x\,T_{m+1}(x) + T_m(x) = 0 \quad , \tag{3.6}$$

which is the relation satisfied by Chebyshev polynomials $T_m(x)$ of the first kind.

Now, as pointed out in references [1] and [3], the Fibonacci polynomials $\phi_m(x)$ and the Lucas polynomials $\lambda_m(x)$ are simply modified Chebyshev polynomials having the relationship

$$\phi_{m+1}(x) = (-i)^m U_m(ix), \quad \lambda_m(x) = 2(-i)^m T_m(ix), \quad (i = \sqrt{-1}) \ . \tag{3.7}$$

In view of (3.3) and (3.5), we have

$$\phi_{m+1}(x) = (-i)^m C_{m,1}(ix), \quad \lambda_m(x) = m(-i)^m C_{m,o}(ix), \quad (m \geq 1) \ , \tag{3.8}$$

thus showing that the Fibonacci and Lucas polynomials are related to modified Gegenbauer polynomials for the special cases of $k = 1$ and $k = 0$.

Moreover, the Fibonacci and Lucas numbers are particular values of (3.8) when $x \equiv 1/2$; that is

$$\left\{\begin{array}{ll} F_1 = C_{o,1}(i/2) = 1, & F_{m+1} = (-i)^m C_{m,1}(i/2) \\ L_o = 2C_{o,o}(i/2) = 2, & L_m = m(-i)^m C_{m,o}(i/2) \end{array}\right. \quad (m \geq 1) \tag{3.9}$$

With the above relationships, the series expansions (1.1) or (1.2) for a given function f in terms involving Fibonacci or Lucas numbers can be obtained from (2.1) by taking

$$x = i/2 \quad \text{and} \quad 2a = -2i\alpha \ . \tag{3.10}$$

Thus, we have the series

$$\left\{\begin{array}{l} f(\alpha) = \frac{1}{2} D_{o,o}(-2i\alpha) L_o + \sum_{m=1}^{\infty} \frac{i^m}{m} D_{m,o}(-2i\alpha) L_m \ , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = \sqrt{-1}) \\ f(\alpha) = \sum_{m=o}^{\infty} i^m D_{m,1}(-2i\alpha) F_{m+1} \end{array}\right. \tag{3.11}$$

where, from (2.4), (3.3), and (3.5), the coefficients may be expressed by the definite integrals

$$(3.12)\quad \begin{cases} D_{o,o}(-2i\alpha) = \dfrac{1}{\pi}\displaystyle\int_o^\pi f(-2i\alpha\cos\gamma)d\gamma = A_o(\alpha)\ , \\[2ex] D_{m,o}(-2i\alpha) = \dfrac{m}{\pi}\displaystyle\int_o^\pi f(-2i\alpha\cos\gamma)\cos m\gamma d\gamma = (-i)^m m A_m(\alpha), \\ \hfill (m \geq 1) \\[2ex] D_{m,1}(-2i\alpha) = \dfrac{2}{\pi}\displaystyle\int_o^\pi f(-2i\alpha\cos\gamma)\sin\gamma\ \sin(m+1)\gamma d\gamma = (-i)^m \beta_m(\alpha) \\ \hfill (m \geq 0)\ . \end{cases}$$

4. EXAMPLES

Since many specific examples were presented in reference [1] for certain series in terms of Fibonacci numbers, we shall now only give some explicit expansions in terms involving Lucas numbers.

Consider first the function

$$(4.1)\qquad f(\alpha) = e^\alpha\ ,$$

so that from (3.12) we have

$$(4.2)\qquad D_{o,o}(-2i\alpha) = \frac{1}{\pi}\int_o^\pi e^{-2i\alpha\cos\gamma}\,d\gamma = J_o(-2\alpha) = J_o(2\alpha)\ ,$$

and

$$(4.3)\qquad D_{m,o}(-2i\alpha) = \frac{m}{\pi}\int_o^\pi e^{-2i\alpha\cos\gamma}\cos m\gamma\,d\gamma = m(-i)^m J_m(2\alpha)\ ,$$

where J_m are Bessel functions of order m [4]. (Evaluation of the above integrals, as well as others to follow, was made by use of tables and formulas in [2], [4], and [5].) Substituting the values of (4.2) and (4.3) into (3.11) then yields the expansion

$$(4.4)\qquad e^\alpha = \frac{1}{2}J_o(2\alpha)L_o + \sum_{m=1}^\infty J_m(2\alpha)L_m\ ,$$

which converges for $0 \leq |a| < \infty$, since

$$\lim_{m \to \infty} \frac{J_{m+1}(2a) L_{m+1}}{J_m(2a) L_m} = \lim_{m \to \infty} \frac{a}{m+1} \frac{1+\sqrt{5}}{2} = 0 \tag{4.5}$$

for all finite values of a. If application is now made of the familiar relations

$$\left\{ \begin{array}{ll} \cosh a = (e^a + e^{-a})/2, & \sinh a = (e^a - e^{-a})/2, \\ \cos a = (e^{ia} + e^{-ia})/2, & \sin a = i(e^{-ia} - e^{ia})/2, \\ J_m(ia) = i^m I_m(a), & J_m(-a) = (-1)^m J_m(a), \end{array} \right. \tag{4.6}$$

the following series expansions* can be easily derived from (4.4):

$$\left\{ \begin{array}{l} \sin a = \sum_{m=1}^{\infty} (-1)^m I_{2m-1}(2a) L_{2m-1}, \\ \cos a = I_0(2a) + \sum_{m=1}^{\infty} (-1)^m I_{2m}(2a) L_{2m}, \end{array} \right. \tag{4.7}$$

where I_m are modified Bessel functions of order m, and

$$\left\{ \begin{array}{l} \sinh a = \sum_{m=1}^{\infty} J_{2m-1}(2a) L_{2m-1}, \\ \cosh a = J_0(2a) + \sum_{m=1}^{\infty} J_{2m}(2a) L_{2m}, \end{array} \right. \tag{4.8}$$

The four examples in (4.7) and (4.8) all converge for $0 \leq |a| < \infty$.

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*Although these series are apparently not found in the literature in the specific form we have given for our purposes, they are modified cases of some expansions due to Gegenbauer (e.g., see [4], pp. 368-369). Series for such functions in terms involving certain powers of Lucas numbers may also be obtained and will be presented in a later article.

Next, consider the odd function

$$f(\alpha) = \arctan \alpha . \tag{4.9}$$

Now

$$D_{2m,o}(-2i\alpha) = 0, \quad \text{for} \quad m = 0, 1, \ldots; \tag{4.10}$$

but

$$D_{2m-1,o}(-2i\alpha) = \frac{2m-1}{\pi} \int_0^\pi \arctan(-2i\alpha \cos\gamma)\cos(2m-1)\gamma \, d\gamma , \tag{4.11}$$

which can be integrated by parts to give

$$D_{2m-1,o}(-2i\alpha) = -\frac{2i\alpha}{\pi} \int_0^\pi \frac{\sin(2m-1)\gamma \sin\gamma}{1-4\alpha^2 \cos^2\gamma} \, d\gamma \quad (m = 1, 2, \ldots), \tag{4.12}$$

$$= \frac{i\alpha}{\pi} \int_0^\pi \frac{\cos 2m\gamma - \cos(2m-2)\gamma}{1-4\alpha^2 \cos^2\gamma} \, d\gamma ,$$

or, finally,

$$D_{2m-1,o}(-2i\alpha) = i \left(\frac{\sqrt{1-4\alpha^2} - 1}{2\alpha}\right)^{2m-1} . \tag{4.13}$$

Use of (4.10) and (4.13) in the first equation of (3.11) yields the series expansions

$$\arctan \alpha = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} \left(\frac{1 - \sqrt{1-4\alpha^2}}{2\alpha}\right)^{2m-1} \cdot L_{2m-1} \quad (\alpha \neq 0) , \tag{4.14}$$

which will converge* for $0 \le |\alpha| \le 1/\sqrt{5}$. If we now take $\alpha = \sqrt{2} - 1$, we obtain, since $\arctan(\sqrt{2} - 1) = \pi/8$, the interesting equation

$$\pi = 8 \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} \left[\frac{(\sqrt{2} + 1)(1 - \sqrt{8\sqrt{2} - 11})}{2}\right]^{2m-1} \cdot L_{2m-1} . \tag{4.15}$$

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*For $\alpha = 0$, the right-hand side of (4.14) becomes an indeterminate form, but the correct result is obtained in the limit.

Certain higher transcendental functions such as the Bessel function $J_o(\alpha)$ can also be easily expanded in terms involving Lucas numbers. Thus, if $f(\alpha) = J_o(\alpha)$, we have

$$(4.16)\qquad D_{o,o}(-2i\alpha) = \frac{1}{\pi}\int_o^{\pi} J_o(-2i\alpha\cos\gamma)\,d\gamma = \left[I_o(\alpha)\right]^2 ;$$

and

$$(4.17)\qquad D_{2m,o}(-2i\alpha) = \frac{2m}{\pi}\int_o^{\pi} J_o(-2i\alpha\cos\gamma)\cos 2m\gamma\,d\gamma = 2m\left[I_m(\alpha)\right]^2,$$

and hence from (3.11) we obtain, since for an even function $D_{2m-1,o} = 0$, the series

$$(4.18)\qquad J_o(\alpha) = \frac{1}{2}\left[I_o(\alpha)\right]^2 L_o + \sum_{m=1}^{\infty} (-1)^m \left[I_m(\alpha)\right]^2 L_{2m} .$$

It can be shown in a similar manner that the expansions of Bessel functions for all even orders are given by

$$(4.19)\qquad J_{2n}(\alpha) = \frac{1}{2}\left[I_n(\alpha)\right]^2 L_o + \sum_{m=1}^{\infty} (-1)^{m-n} I_{m+n}(\alpha) I_{m-n}(\alpha) L_{2m} , \qquad (n = 0, 1, 2, \ldots)$$

and are convergent for $0 \le |\alpha| < \infty$.

A proposed problem for the reader is to show that the Bessel function $J_1(\alpha)$ may be expressed in the form

$$(4.20)\qquad J_1(\alpha) = \sum_{m=1}^{\infty} (-1)^m I_m(\alpha) I_{m-1}(\alpha) L_{2m-1} .$$

The reader may also use the last equations in (3.11) and (3.12) to show that, in terms of <u>Fibonacci</u> numbers F_{2m},

$$(4.21)\qquad \arctan\alpha = \sum_{m=1}^{\infty} (-1)^{m-1}\left[\frac{1}{2m-1} - \frac{b^2}{2m+1}\right] b^{2m-1} F_{2m} ,$$

and whence

$$\pi = 8 \sum_{m=1}^{\infty} (-1)^{m-1} \left[\frac{1}{2m-1} - \frac{d^2}{2m+1} \right] d^{2m-1} F_{2m} , \tag{4.22}$$

where

$$b = \frac{1}{2a} \left[1 - \sqrt{1-4a^2} \right], \quad d = \frac{1}{2} \left[(\sqrt{2}+1)(1 - \sqrt{8\sqrt{2} - 11}) \right] . \tag{4.23}$$

The results (4.21) and (4.22) can actually be obtained more readily, as indicated in the following remarks.

5. REMARKS

If one has already found the coefficients $A_n(a)$ in (1.2) for an expansion in terms of Lucas numbers, it is not necessary to carry out the integration in the last equation of (3.12) in order to obtain the coefficients $\beta_n(a)$ in (1.1) for a series in terms involving Fibonacci numbers. For, since $F_o = 0$, $L_o = 2$, and $L_n = F_{n+1} + F_{n-1}$, it is easy to show that

$$\beta_n(a) = A_{n+1}(a) + A_{n-1}(a) , \tag{5.1}$$

and thus that

$$f(a) = \sum_{n=o}^{\infty} \left[A_n(a) + A_{n+2}(a) \right] F_{n+1} . \tag{5.2}$$

Expansions in terms of Fibonacci numbers or of Lucas numbers, however, are not very efficient for computing approximate values of a function. For example, to compute π correctly to 6 places using formula (4.15) requires 36 terms in comparison to 9 terms using the series

$$\pi = 8 \sum_{m=o}^{\infty} \frac{(-1)^m}{2m+1} (\sqrt{2} - 1)^{2m+1} , \tag{5.3}$$

which is based on a slowly convergent Maclaurin expansion. But the series

$$\pi = 16 \sum_{m=o}^{\infty} \frac{(-1)^m}{2m+1} \left[(\sqrt{2}+1)(\sqrt{4-2\sqrt{2}} - 1) \right]^{2m+1} ,$$

based on a more rapidly convergent expansion in terms of Chebyshev polynomials, yields 6-place accuracy with only 5 terms.

As pointed out by Gould [6], if $f(\alpha)$ has a power series expansion

$$f(\alpha) = \sum_{n=o}^{\infty} \gamma_n \alpha^n , \tag{5.4}$$

then the series

$$G(\alpha) = f(a_1 \alpha) + f(a_2 \alpha) = \sum_{n=o}^{\infty} \gamma_n \alpha^n L_n , \tag{5.5}$$

where

$$a_1 = \frac{1+\sqrt{5}}{2}, \quad a_2 = \frac{1-\sqrt{5}}{2}, \tag{5.6}$$

will furnish a whole special class of generating functions for the Lucas sequence. Similarly, the series

$$H(\alpha) = f(a_1 \alpha) - f(a_2 \alpha) = \sum_{n=o}^{\infty} \sqrt{5}\, \gamma_n \alpha^n F_n \tag{5.7}$$

yields a class of generating functions for the Fibonacci sequence. It is to be noted, however, that this technique of Gould for obtaining generating functions for Lucas or Fibonacci numbers is not intended to accomplish our purpose of making some given function f serve as the generating function by the expansion (1.1) or (1.2). Clearly the functions $G(\alpha)$ and $H(\alpha)$ in (5.5) and (5.7) are not the same as the given function f.

6. CERTAIN EXTENSIONS

In reference [1], we considered the expansion of functions in terms involving numbers (those of Fibonacci) associated with modified

Gegenbauer polynomials for the special case when $k = 1$, and in the present article in terms of numbers (those of Lucas) for the case when $k = 0$. Now from the general class, there appear to be other special cases which may also prove of interest to students "devoted to the study of integers with special properties."

For instance, upon taking $k = 1/2$, one may consider the set of polynomials $R_m(x)$ defined by

$$(6.1)\quad R_m(x) = 4^m(-i)^m C_{m,1/2}(ix) = 4^m(-i)^m P_m(ix), \quad (m = 0, 1, \ldots),$$

where P_m are the <u>Legendre polynomials.</u> From equation (3.1) we then have the recurrence relation

$$(6.2)\qquad (m+2)R_{m+2}(x) = 4(2m+3)xR_{m+1}(x) + 16(m+1)R_m(x)$$

with $R_o(x) = 1$ and $R_1(x) = 4x$; or, more explicitly, from (2.2), we can write

$$(6.3)\qquad R_m(x) = 2^m \sum_{j=o}^{[m/2]} \binom{m}{j}\binom{2m-2j}{m} x^{m-2j},$$

which has a generating function expressed by

$$(6.4)\qquad \frac{1}{\sqrt{1-8xz-16z^2}} = \sum_{n=o}^{\infty} R_n(x)z^n, \quad (|8xz| + |16z^2| < 1).$$

Now let H_m be the sequence of numbers (which we shall call "H-numbers") obtained from $R_m(x)$ by taking $x \equiv 1/2$. Thus

$$(6.5)\qquad H_m = \sum_{j=o}^{[m/2]} \binom{m}{j}\binom{2m-2j}{m} 4^j, \quad (m \geq 0),$$

with $H_o = 1$, $H_1 = 2$, $H_2 = 14$, $H_3 = 68$, ...), and one may investigate what particular properties* these numbers might have.

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*What, for instance, is $\lim_{m\to\infty}(H_m/H_{m+1})$? Are there any interesting identities, etc.?

By the procedure we have illustrated, one may also find expansions which would make a given function serve as a generating function for such numbers. Thus, from (2.1), (2.4), (3.10), and (6.1), we obtain the series

$$(6.6)\qquad f(a) = D_{o,1/2}(-2ia)\,H_o + \sum_{m=1}^{\infty} \frac{i^m}{4^m}\, D_{m,1/2}(-2ia)\,H_m \,,$$

where the coefficients are expressed by

$$(6.7)\quad D_{m,1/2}(-2ia) = \frac{2m+1}{2}\int_o^{\pi} \sin\gamma\; f(-2ia\cos\gamma)P_m(\cos\gamma)d\gamma (m = 0, 1, \ldots).$$

For example, if we take the analytic function

$$(6.8)\qquad f(a) = e^a$$

then

$$(6.9)\quad D_{m,1/2}(-2ia) = \frac{2m+1}{2}\int_o^{\pi} \sin\gamma\, e^{-2ia\cos\gamma}\, P_m(\cos\gamma)d\gamma$$

$$= \frac{2m+1}{2}\int_{-1}^{1} e^{-2iaz}P_m(z)dz = \frac{2m+1}{2}(-i)^m\sqrt{\frac{\pi}{a}}\,J_{m+1/2}(2a),$$

$$(m = 0, 1, 2, \ldots)$$

and hence from (6.6) we have the expansion

$$(6.10)\quad e^a = \frac{1}{2}\sqrt{\frac{\pi}{a}}\left[J_{1/2}(2a)H_o + \sum_{m=1}^{\infty}\frac{2m+1}{4^m}\, J_{m+1/2}(2a)H_m\right],\quad (a \neq 0)$$

where $J_{m+1/2}$ are Bessel functions of order half an odd integer. Other functions f may be expanded in a similar way.

The Lucas numbers, the Fibonacci numbers, and our so-called H-numbers, in terms of which we have expanded a given analytic function, are all seen to be mere special cases of a more general sequence $\{V_{m,k}\}$, where

$$(6.11)\qquad V_{m,k} = \frac{q(m,k)}{\Gamma(k)}\sum_{r=o}^{[m/2]}\frac{\Gamma(m-r+k)}{\Gamma(m-r+1)}\binom{m-r}{r}\qquad (k > -1/2)\ .$$

Our three particular cases of these may be summarized as follows:

$$(6.12)\begin{cases} k = 1, & q(m,k) = 1, & \text{then } V_{m,1} = F_{m+1}, \\ k = 0, & \dfrac{q(m,k)}{\Gamma(k)} = m, & \text{then } V_{m,o} = L_m, \quad (m = 0, 1, 2, \ldots) \\ k = 1/2, & q(m,k) = 4^m, & \text{then } V_{m,1/2} = H_m . \end{cases}$$

In the family* (6.11), however, there may be many other interesting sets of integers worthy of consideration. For example, if k is any integer > 1, then

$$(6.13) \qquad V_{m,k} = \frac{q(m,k)}{(k-1)!} \sum_{r=o}^{[m/2]} \frac{(m-r+k-1)!}{(m-r)!} \binom{m-r}{r}$$

will obviously lead to various sequences of integers whenever $q(m,k)/(k-1)!$ is any arbitrarily chosen function yielding a positive integer. Expansion of a given function $f(\alpha)$ in terms involving the numbers $V_{m,k}$ may easily be made by the familiar procedure already described.

Besides the Gegenbauer polynomials, there are of course other well-known families of orthogonal polynomials which may be modified to furnish still other sources of integer-sequences. A given function could be expanded in terms of such numbers by a technique similar to the one presented in reference [1], or in this article.

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*It can be easily shown that

$$\lim_{m \to \infty} \frac{V_{m,k}}{V_{m+1,k}} = \left(\frac{\sqrt{5}-1}{2}\right) \left[\lim_{m \to \infty} \frac{q(m,k)}{q(m+1,k)}\right],$$

and thus that the value of this limit for all sequences of the general family has the common factor $(\sqrt{5}-1)/2$, which is the classical "golden mean" for the Fibonacci or Lucas sequence. (Of course, an appropriate choice of $q(m,k)$ should be made so that the limit on the right-hand side exists.)

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INFORMATION AND AN OMISSION

Reference related to H-37.

References by R. E. Greenwood: Problem 4047, Amer. Math. Mon. (issue of Feb. 1944, pp. 102-104), proposed by T. R. Running, solved by E. P. Starke. Problem #65, Nat. Math. Mag. (now just Math. Mag.) issue of November 1934, p. 63.

Omission H-37. Also solved by J. A. H. Hunter.

CORRECTION

H-28 Let

$$S_n(r, a, b) = \sum_{j=0}^{\infty} C_j(r, n)\, a^j b^{rn-n-j} = b^{(r-1)n} \sum_{N=0}^{r^{n-1}} \left(\frac{a}{b}\right)^{N_o + N_1 + \ldots N_{n-1}}$$