THE FIBONACCI NUMBER F_u WHERE u IS NOT AN INTEGER

ERIC HALSEY
Redlands, California

INTRODUCTION

Fibonacci numbers, like factorials, are not naturally defined for any values except integer values. However the gamma function extends the concept of factorial to numbers that are not integers. Thus we find that $(1/2)! = \sqrt{\pi}/2$. This article develops a function which will give F_n for any integer n but which will furthermore give F_u for any rational number u. The article also defines a quantity $n\not\Delta^m$ and develops a function $f(x, y) = x\not\Delta^y$ where x and y need not be integers.

(1) DEFINITIONS

Let $n\not\Delta^0 = 1$ (Definitions (1) hold for all $n \in N$)

Let

$$n\not\Delta^1 \text{ (read "n cardinal") } = \sum_{k=1}^{n} k\not\Delta^0 = \sum_{k=1}^{n} 1 = n \quad .$$

This gives the cardinal numbers 1, 2, 3, ...

Let

$$n\not\Delta^2 \text{ (read "n triangular") } = \sum_{k=1}^{n} k\not\Delta^1 = \sum_{k=1}^{n} k \quad .$$

This gives the triangular numbers 1, 3, 6, 10, ...

Let

$$n\not\Delta^3 \text{ (read "n tetrahedral") } = \sum_{k=1}^{n} k\not\Delta^2 \quad .$$

This gives the tetrahedral numbers 1, 4, 10, 20, ...

In general, let

$$n\not\Delta^m \text{ (read "n delta-slash m") } = \sum_{k=1}^{n} k\not\Delta^{m-1} \quad .$$

This gives a figurate number series which can be assigned to the m-dimensional analog of the tetrahedron (which is the 3-dimensional analog of the triangle, etc.).

Let us construct an array $(a_{i,j})$, where we assign to each $a_{i,j}$ an appropriate coefficient of Pascal's triangle.

$$(a_{i,j}) = \begin{matrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \vdots & & & & & \end{matrix}$$

It is clear that in this arrangement the usual rule for forming Pascal's triangle is just

$$a_{i,j} = a_{i,j-1} + a_{i-1,j} \quad . \tag{2}$$

But a comparison of this rule with the definitions (1) shows that Pascal's triangle can be written:

$$\begin{matrix} 1\Delta^0 & 1\Delta^1 & 1\Delta^2 & \cdots & 1\Delta^m & \cdots \\ 2\Delta^0 & 2\Delta^1 & 2\Delta^2 & \cdots & 2\Delta^m & \cdots \\ 3\Delta^0 & 3\Delta^1 & 3\Delta^2 & \cdots & 3\Delta^m & \cdots \\ \vdots & & & \cdots & & \cdots \\ n\Delta^0 & n\Delta^1 & n\Delta^2 & \cdots & n\Delta^m & \cdots \\ \vdots & & & & & \end{matrix}$$

where $a_{i,j} = i\Delta^{j-1}$. From the symmetry of Pascal's triangle, $a_{i,j} = a_{j,i}$. Therefore

$$i\Delta^{j-i} = j\Delta^{i-1}; \quad n\Delta^m = (m+1)\Delta^{n-1} \quad . \tag{3}$$

Pascal's triangle is a well-known generator of Fibonacci numbers in the way shown in the following diagram.

1	1	1	1	1	...	$1 = 1 = F_1$
1	2	3	4	5	...	$1 = 1 = F_2$
1	3	6	10	15	...	$1+1 = 2 = F_3$
1	4	10	20	35	...	$1+2 = 3 = F_4$
1	5	15	35	70	...	$1+3+1 = 5 = F_5$
$\vdots$						$\vdots$

We can apply the same course to our abstracted Pascal's triangle.

$1\not\Delta^0$	$1\not\Delta^1$	$1\not\Delta^2$	...	$F_1 = 1\not\Delta^0$
$2\not\Delta^0$	$2\not\Delta^1$	$2\not\Delta^2$	...	$F_2 = 2\not\Delta^0$
$3\not\Delta^0$	$3\not\Delta^1$	$3\not\Delta^2$	...	$F_3 = 3\not\Delta^0 + 1\not\Delta^1$
$\vdots$				$F_4 = 4\not\Delta^0 + 2\not\Delta^1$
				$\vdots$

It is clear that, if we keep forming Fibonacci numbers from Pascal's triangle in this way, $F_n = n\not\Delta^0 + (n-2)\not\Delta^1 + (n-4)\not\Delta^2 + \ldots + (n-2m)\not\Delta^m$, or

$$F_n = \sum_{k=0}^{m} (n-2k)\not\Delta^k \quad , \tag{4}$$

where we require that m be an integer and that $0 < n-2m \le 2$, or in other words that $n/2 - 1 \le m < n/2$. Now let us prove

(5) Theorem 1 $$n\not\Delta^m = \binom{n+m-1}{m}$$

Proof: It is sufficient to perform induction on n. Let the theorem be $E(n)$. Then if $n = 1$, $E(1)$ states

$$\binom{n+m-1}{m} = \binom{1+m-1}{m} = \frac{m!}{m!} = 1 \quad .$$

But by definition (1), $(m+1)\Delta^0 = 1$ for any $(m+1) \in N$. Then by equation (3) $1\Delta^m = 1$ for $m = 0, 1, 2, 3, \ldots$ and E(1) is true. Now let us assume that, for arbitrary $m \in N$, E(n) is true. Then

$$n\Delta^m = \binom{n+m-1}{m} .$$

From the definitions (1) it can be seen that

$$1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + n\Delta^{m-1} = n\Delta^m .$$

Therefore the induction hypothesis can be restated

$$(6) \qquad 1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + \binom{n+m-2}{m-1} = \binom{n+m-1}{m} .$$

Add $\binom{n+m-1}{m-1}$ to both sides of equation (6) to obtain

$$(7) \qquad 1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + \binom{n+m-2}{m-1} + \binom{n+m-1}{m-1}$$

$$= \binom{n+m-1}{m} + \binom{n+m-1}{m-1}$$

The right-hand side of equation (7) is $\binom{n+m}{m}$ by the standard identity for combinations, so we have

$$1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + \binom{n+m-2}{m-1} + \binom{n+m-1}{m-1} = \binom{n+m}{m} ,$$

or

$$1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + \binom{n+m-2}{m-1} + \binom{(n+1)+m-2}{m-1}$$

$$= \binom{(n+1)+m-1}{m} ,$$

which is E(n+1). Therefore E(n) implies E(n+1) and Theorem 1 is true by mathematical induction.

Now let us prove

$$(8) \quad \text{Theorem 2} \qquad n\Delta^m = \left[(n+m)\int_0^1 x^{n-1}(1-x)^m dx\right]^{-1}$$

Proof: $\Gamma(n) = (n-1)!$ (gamma function)

$$B(m, n) = B(n, m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{(beta function)}$$

Therefore

$$\frac{1}{B(m, n)} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} ,$$

and

$$\frac{1}{B(m+1, n-m+1)} = \frac{\Gamma(n+2)}{\Gamma(m+1)\Gamma(n-m+1)} = \frac{(n+1)!}{m!(n-m)!}$$

$$= \frac{(n+1)n!}{m!(n-m)!} = (n+1)\binom{n}{m} .$$

Then

$$(9) \qquad \binom{n}{m} = \frac{1}{(n+1)B(m+1, n-m+1)} = \left[(n+1)B(m+1, n-m+1)\right]^{-1} .$$

We can now substitute the right-hand side of equation (5) into equation (9) to obtain

$$n\not{\Delta}^m = \binom{n+m-1}{m} = \left[(n+m)B(m+1, n)\right]^{-1} ,$$

where

$$B(m+1, n) = B(n, m+1) = \int_0^1 x^{n-1}(1-x)^m dx .$$

Therefore

$$n\not{\Delta}^m = \left[(n+m)\int_0^1 x^{n-1}(1-x)^m dx\right]^{-1} .$$

Both equations (5) and (8) assert that $n\not{\Delta}^m = (m+1)\not{\Delta}^{n-1}$. Some interesting special cases of equation (5) are

$$n\not{\Delta}^0 = \binom{n-1}{0} = \frac{(n-1)!}{(n-1)!} = 1 ,$$

$$n\not{\Delta}^1 = \binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n ,$$

and

$$\sum_{k=1}^{n} k = n\not{\Delta}^2 = \binom{n+1}{2} = \frac{(n+1)!}{(n-1)!\,2!} = \frac{(n)(n+1)}{2} .$$

Now we can put equation (8) into equation (4) to obtain

$$(10) \qquad F_n = \sum_{k=0}^{m} \left[(n-k) \int_0^1 x^{n-2k-1}(1-x)^k dx\right]^{-1} ,$$

where m is an integer, $n/2 - 1 \leq m < n/2$. But whereas equations (4) and (5) have meaning only for integer arguments, equations (8) and (10) can be used to find $x \Delta^y$ and F_u, where x, y, and u are any rational numbers.

In particular

$$(11) \qquad F_u = \sum_{k=0}^{m} \left[(u-k) \int_0^1 x^{u-2k-1}(1-x)^k dx\right]^{-1} ,$$

where m is an integer, $u/2 - 1 \leq m < u/2$. The equation (11), and the definite integral in it, are easily programmed for solution on a digital computer. A few values of F_u follow.

u	F_u
4.1000000	3.1550000
4.2000000	3.3200000
4.3000000	3.4950000
4.4000000	3.6800000
4.5000000	3.8750000
4.6000000	4.0800000
4.7000000	4.2950000
4.8000000	4.5200000
4.9000000	4.7550000
5.0000000	5.0000000
5.1000000	5.2550000
5.2000000	5.5200000
5.3000000	5.7950000
5.4000000	6.0800000
5.5000000	6.3750000
5.6000000	6.6800000
5.7000000	6.9950000
5.8000000	7.3200000
5.9000000	7.6550000
6.0000000	8.0000000

u	F_u
0.1	1.0
0.2	1.0
⋮	⋮
2.0	1.0
2.1	1.1
2.2	1.2
⋮	⋮
3.0	2.0
3.1	2.1
⋮	⋮
4.0	3.0

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